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# ON ESTIMATION OF $P(Y < X)$ FOR GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION BASED ON HYBRID CENSORED DATA

Renu Garg [1]
*Department of Statistics, University of Delhi, Delhi-110007, India*
Kapil Kumar
*Department of Statistics, Central University of Haryana, Mahendergarh-123031, India*

## 1. Introduction

In reliability theory, inferences of stress-strength reliability $R = P(Y < X)$, where, $X$ and $Y$ have independent distributions, is a general problem of interest. For example, in mechanical reliability of a system, if $X$ is the strength of a component which is subject to stress $Y$, then $R$ is a measure of system performance. The system fails, if at any time the applied stress exceeds than its strength. The model of stress-strength has found applications in many statistical problems, including quality control, engineering statistics, medical statistics and biostatistics, among others. The problem of the estimation of stress-strength model has received considerable attention in the statistical literature. In connection of classical Mann-Whitney statistic, Birnbaum (1956) introduced stress-strength model. Since then, a lot of work has been done on the estimation of stress-strength model for different distributions from the both frequentist and Bayesian approaches in complete sample case. An excellent monograph by Kotz *et al.* (2003) provides a comprehensive treatment of different stress-strength models. Some recent works on stress-strength model can be found in Kundu and Gupta (2006), Rezaei *et al.* (2010), Babayi *et al.* (2014), Sharma (2018), etc.

Most of the inferences for stress-strength model have been carried out under complete sample case and very little work has been done based on censored data. Specially, stress-strength model is unexplored based on hybrid censored data. For example, Lio and Tsai (2012) studied estimation of stress-strength parameter for Burr XII distribution based on progressively first failure censored samples, Kumar *et al.* (2015) discussed estimation of the stress-strength parameter for Lindley distribution using progressively first failure censoring. Asgharzadeh *et al.* (2017) studied estimation for Weibull distribution

[1] Corresponding Author. E-mail: renustats@gmail.com

based on hybrid censored samples. Some recent work on stress-strength model under different scenario like record, progressive Type-II and progressive first failure censoring schemes are carried out by many scholars like, Krishna *et al.* (2017), Yadav *et al.* (2018), Chaudhary and Tomer (2018), Kohansal and Nadarajah (2019), Krishna *et al.* (2019), Khan and Khatoon (2019), Saini *et al.* (2021), Kazemi and Kohansal (2021), and many others.

In life testing experiments experimenter often does not have complete control on the experiment in hand and items put on test are often lost or removed from the experiment before the completion of the experiment. In this case available data are censored. In literature the most common censoring schemes are Type-I and Type-II censoring schemes which are popularly used in life testing experiments. In Type-I censoring scheme the experiment is terminated after a pre-fixed time and in Type-II censoring scheme experiment is terminated after getting a pre-specified number of failures. A new censoring scheme was introduced by Epstein (1954) which is the mixture of Type-I and Type-II censoring schemes and called it hybrid censoring scheme. In recent years, the hybrid censoring scheme has received considerable attention in the reliability theory and life testing experiments. Some early work on hybrid censoring can be found in Draper and Guttman (1987), Chen and Bhattacharyya (1988) etc. Several interesting results on hybrid censoring can be found in a review work by Balakrishnan and Kundu (2013). Some recent studies on estimation and prediction problems with hybrid censoring scheme can be found in Dey and Pradhan (2014), Tripathi and Rastogi (2016), Valiollahi *et al.* (2017), Kayal *et al.* (2018), Sultana and Tripathi (2020) and references cited therein.

The hybrid censoring scheme can be described as follows: Let $n$ identical units are put on life testing experiment and their lifetimes are assumed to be independently and identically distributed (iid) random variables with probability density function (pdf) $f_X(x)$ and cumulative distribution function (cdf) $F_X(x)$ .

Let $X_{1:n} < X_{2:n} < ... < X_{r:n} < ... < X_{n:n}$ denote the ordered lifetimes of the experimental units. The test is terminated when a pre-specified number $r$ out of $n$ units have failed or a pre-specified time $T$ has been reached. It is also assumed that the failed items are not replaced. In hybrid censoring scheme, the experiment is terminated at $\min(X_{r:n}, T)$ . Thus, under hybrid censoring scheme available data may be in one of the following forms:

**Case I** $x_{1:n}, x_{2:n}, ..., x_{r:n}$, if $x_{r:n} \leq T$;

**Case II** $x_{1:n}, x_{2:n}, ..., x_{m:n}$, if $0 \leq m < r$, $x_{m:n} < T < x_{m+1:n}$,

where, $m$ denotes the number of observed failures that occur before the time point $T$. Note that $x_{m+1:n}, x_{m+2:n}, ..., x_{r:n}$ are not observed in case II. On combining both of the cases, the likelihood function for hybrid censored sample, is given by

$$L(x_{1:n}, x_{2:n}, ..., x_{d:n}) = A \prod_{i=1}^{d} f_X(x_{i:n}) \{1 - F_X(c)\}^{n-d}, \tag{1}$$

where, $A = n(n-1)(n-2)...(n-d+1)$, $c = min(x_{r:n}, T)$ and $d = \sum_{i=1}^{r} I\{x_{i:n} \leq c\}$, here, $I$ is an indicator function.

The pdf and the cdf of the generalized inverted exponential distribution (GIED), respectively, are given by

$$f_X(x) = \frac{\alpha\lambda}{x^2} e^{-\frac{\lambda}{x}} (1 - e^{-\frac{\lambda}{x}})^{\alpha-1}; \quad x > 0, \tag{2}$$

$$F_X(x) = 1 - (1 - e^{-\frac{\lambda}{x}})^{\alpha}; \quad x > 0. \tag{3}$$

where, $\alpha > 0$ and $\lambda >$ are the shape and scale parameters, respectively. Abouammoh and Alshingiti (2009) introduced GIED as a generalization of inverted exponential distribution. They studied many of its distributional properties and reliability characteristics. Krishna and Kumar (2013) discussed reliability estimation in GIED with progressively type II censored sample. They observed that the GIED is a better lifetime model than exponential, inverted exponential, gamma and Weibull distributions in many practical situations. Dey and Pradhan (2014), Tripathi and Rastogi (2016) studied GIED under hybrid censoring. Genç (2017) studied truncated GIED and its properties. Dube *et al.* (2016) discussed progressively first failure GIED. Garg *et al.* (2016) studied randomly censored GIED. Krishna *et al.* (2017) studied stress-strength reliability for GIED based on progressively first failure censored data. Wang *et al.* (2020) obtained inference for confidence sets of GIED under $k-$record values. Mahmoud *et al.* (2021) studied GIED under progressive type- I censoring scheme in presence of competing risks model, and many other. Practically, GIED has several applications in queuing theory, accelerated life testing, horse racing, supermarket queues, sea currents, wind speeds, and others. These studies suggest that GIED is a widely applicable lifetime model.

This article considers the problem of point and interval estimation of the stress-strength reliability $R = P(Y < X)$ under the assumption that $X$ and $Y$ both are independent generalized inverted exponential random variables based on hybrid censored data. Let $X \sim$ GIED$(\alpha, \lambda)$ and $Y \sim$ GIED$(\beta, \lambda)$ be independent random variables, the stress-strength reliability is given by

$$R = P(Y < X) = \int_0^\infty F_Y(x) f_X(x) dx = \frac{\beta}{(\alpha + \beta)}. \tag{4}$$

REMARK 1. *(i) $R$ is independent of $\lambda$, and (ii) when $\alpha = \beta, R = 0.5$, i.e.,in this case $X$ and $Y$ are iid and there is an equal chance that $Y$ is smaller than $X$.*

The rest of the paper is organized as follows: In Section 2, the maximum likelihood estimator of stress-strength reliability is derived. Section 3 deals with the asymptotic and two parametric bootstrap confidence intervals. Bayes estimator of the parameter is presented in section 4. Also, highest posterior density (HPD) credible interval of is constructed using Markov Chain Monte Carlo (MCMC) approach. In section 5, a Monte

Carlo simulation study is performed to compare different estimation procedures and various hybrid censoring schemes. Section 6 deals with real data analysis for illustration purposes. Finally, conclusions and a brief discussion on the paper are given in Section 7.

## 2. Maximum likelihood estimation

Let $(x_1, x_2, \dots, x_{d_1}) = (x_{1:n_1}, x_{2:n_1}, \dots, x_{d_1:n_1})$ be a hybrid censored sample of size $d_1$ from GIED$(\alpha, \lambda)$ with censoring scheme $(r_1, T_1)$ and $(y_1, y_2, \dots, y_{d_2}) = (y_{1:n_2}, y_{2:n_2}, \dots, y_{d_2:n_2})$ be a independent hybrid censored sample of size $d_2$ from GIED$(\beta, \lambda)$ with censoring scheme $(r_2, T_2)$. Then the likelihood function without constant terms is given by

$$L(data, \alpha, \beta, \lambda) \propto \alpha^{d_1} \beta^{d_2} \lambda^{d_1+d_2} \prod_{i=1}^{d_1} \frac{1}{x_i^2} \prod_{j=1}^{d_2} \frac{1}{y_j^2} e^{-\lambda\left(\sum_{i=1}^{d_1} \frac{1}{x_i} + \sum_{j=1}^{d_2} \frac{1}{y_j}\right)}$$
$$\prod_{i=1}^{d_1}\left(1-e^{-\frac{\lambda}{x_i}}\right)^{\alpha-1} \prod_{j=1}^{d_2}\left(1-e^{-\frac{\lambda}{y_j}}\right)^{\beta-1}$$
$$\left(1-e^{-\frac{\lambda}{c_1}}\right)^{\alpha(n_1-d_1)} \left(1-e^{-\frac{\lambda}{c_2}}\right)^{\beta(n_2-d_2)}, \tag{5}$$

where $c_k = min(r_k, T_k)$, $d_k = \sum_{i=1}^{r_k} I\{x_{i:n \le c_k}\} and k = 1, 2$. Since, MLEs do not exist for $d_1 = d_2 = 0$, therefore $d_1$ and $d_2$ both are assumed greater than zero. Therefore, the log likelihood function becomes

$$l(\alpha, \beta, \lambda) = d_1 \log \alpha + d_2 \log \beta + (d_1 + d_2) \log \lambda - 2S_1 - 2S_2 - \lambda(S_3 + S_4)$$
$$+(\alpha - 1)\sum_{i=1}^{d_1} \log(1 - e^{-\frac{\lambda}{x_i}}) + (\beta - 1)\sum_{j=1}^{d_2} \log(1 - e^{-\frac{\lambda}{y_j}})$$
$$+\alpha(n_1 - d_1)\log(1 - e^{-\frac{\lambda}{c_1}}) + \beta(n_2 - d_2)\log(1 - e^{-\frac{\lambda}{c_2}}), \tag{6}$$

where $S_1 = 2\sum_{i=1}^{d_1} \log x_i$, $S_2 = 2\sum_{j=1}^{d_2} \log y_j$, $S_3 = \sum_{i=1}^{d_1} \frac{1}{x_i}$ and $S_4 = \sum_{j=1}^{d_2} \frac{1}{y_j}$. MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of the parameters $\alpha$, $\beta$ and $\lambda$, respectively, are the solutions of the following non-linear equations:

$$\frac{\partial l}{\partial \alpha} = \frac{d_1}{\alpha} + \sum_{i=1}^{d=1} \log(1 - e^{-\frac{\lambda}{x_i}}) + (n_1 - d_1)\log(1 - e^{-\frac{\lambda}{c_1}}) = 0; \tag{7}$$

$$\frac{\partial l}{\partial \beta} = \frac{d_2}{\beta} + \sum_{j=1}^{d=2} \log(1 - e^{-\frac{\lambda}{y_j}}) + (n_2 - d_2)\log(1 - e^{-\frac{\lambda}{c_2}}) = 0; \tag{8}$$

$$\frac{\partial l}{\partial \lambda} = \frac{d_1 + d_2}{\lambda} - (S_3 + S_4) + (\alpha - 1)\sum_{i=1}^{d_1} \frac{e^{-\frac{\lambda}{x_i}}}{x_i(1 - e^{-\frac{\lambda}{x_i}})} + (\beta - 1)$$

$$\sum_{j=1}^{d_2} \frac{e^{-\frac{\lambda}{y_j}}}{y_j(1 - e^{-\frac{\lambda}{y_j}})} + \frac{\alpha(n_1 - d_1)e^{-\frac{\lambda}{c_1}}}{c_1(1 - e^{-\frac{\lambda}{c_1}})} + \frac{\beta(n_2 - d_2)e^{-\frac{\lambda}{c_2}}}{c_2(1 - e^{-\frac{\lambda}{c_2}})} = 0. \tag{9}$$

From equations (7) and (8) we obtain

$$\hat{\alpha}(\lambda) = -d_1 \left[ \sum_{i=1}^{d=1} \log(1 - e^{-\frac{\lambda}{x_i}}) + (n_1 - d_1)\log(1 - e^{-\frac{\lambda}{c_1}}) \right]^{-1} \tag{10}$$

and

$$\hat{\beta}(\lambda) = -d_2 \left[ \sum_{j=1}^{d=2} \log(1 - e^{-\frac{\lambda}{y_j}}) + (n_2 - d_2)\log(1 - e^{-\frac{\lambda}{c_2}}) \right]^{-1}. \tag{11}$$

The parameter $\lambda$ can be estimated by maximizing the profile log-likelihood function $l\left(\hat{\alpha}(\lambda), \hat{\beta}(\lambda), \lambda\right)$ with respect to $\lambda$. Now, substituting $\hat{\alpha}(\lambda)$ and $\hat{\beta}(\lambda)$ in in equation (9), $\hat{\lambda}$ can be obtained as a solution of the following non-linear equation:

$$\lambda = h(\lambda), \tag{12}$$

where,

$$h(\lambda) = (d_1 + d_2)\Bigg[(S_3 + S_4) - (\hat{\alpha}(\lambda) - 1)\sum_{i=1}^{d_1} \frac{e^{-\frac{\lambda}{x_i}}}{x_i(1 - e^{-\frac{\lambda}{x_i}})} - (\hat{\beta}(\lambda) - 1)$$

$$\times \sum_{j=1}^{d_2} \frac{e^{-\frac{\lambda}{y_j}}}{y_j(1 - e^{-\frac{\lambda}{y_j}})} - \frac{\hat{\alpha}(\lambda)(n_1 - d_1)e^{-\frac{\lambda}{c_1}}}{c_1(1 - e^{-\frac{\lambda}{c_1}})} - \frac{\hat{\beta}(\lambda)(n_2 - d_2)e^{-\frac{\lambda}{c_2}}}{c_2(1 - e^{-\frac{\lambda}{c_2}})}\Bigg]^{-1}.$$

A simple iterative procedure can be considered to solve equation (12). Start with an initial value of $\lambda$, say $\lambda^{(0)}$, and obtain $\lambda^{(1)}$ from $h(\lambda^{(0)})$, $\lambda^{(2)}$ from $h(\lambda^{(1)}), \ldots, \lambda^{(l+1)}$ from $h(\lambda^{(l)})$. Stop the process when $|\lambda^{(l+1)} - \lambda^{(l)}| < \epsilon$ is satisfied, where, $\epsilon$ is a pre-specified tolerance limit. Once we obtain MLE $\hat{\lambda}$ of $\lambda$ then MLEs of $\alpha$ and $\beta$ can be deduced from (10) and (11) as $\hat{\alpha} = \hat{\alpha}(\hat{\lambda})$ and $\beta = \hat{\beta}(\hat{\lambda})$ , respectively. Therefore, the MLE of $R$ can be obtained using invariance property of MLEs as

$$\hat{R} = \frac{\hat{\beta}}{\left(\hat{\alpha} + \hat{\beta}\right)}. \tag{13}$$

## 3. Different confidence intervals

In this Section, asymptotic confidence interval of $R$ is constructed based on the asymptotic distribution of $\hat{R}$. Also,the use of two parametric bootstrap confidence intervals of $R$ are proposed.

### 3.1. *Asymptotic confidence interval*

In this subsection, we derive the asymptotic confidence interval of $R$ based on the approximate asymptotic variance-covariance matrix, which is given by

$$I^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 l}{\partial \alpha^2} & -\frac{\partial^2 l}{\partial \alpha \partial \beta} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 l}{\partial \beta \partial \alpha} & -\frac{\partial^2 l}{\partial \beta^2} & -\frac{\partial^2 l}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l}{\partial \lambda \partial \beta} & -\frac{\partial^2 l}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\alpha,\beta,\lambda)=(\hat{\alpha},\hat{\beta},\hat{\lambda})},$$

where

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{d_1}{\alpha^2}, \quad \frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{\partial^2 l}{\partial \beta \partial \alpha} = 0, \quad \frac{\partial^2 l}{\partial \beta^2} = -\frac{d_2}{\beta^2},$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \frac{\partial^2 l}{\partial \lambda \partial \alpha} = \sum_{i=1}^{d_1} \frac{e^{-\frac{\lambda}{x_i}}}{x_i(1-e^{-\frac{\lambda}{x_i}})} + \frac{(n_1-d_1)e^{-\frac{\lambda}{c_1}}}{c_1(1-e^{-\frac{\lambda}{c_1}})},$$

$$\frac{\partial^2 l}{\partial \beta \partial \lambda} = \frac{\partial^2 l}{\partial \lambda \partial \beta} = \sum_{j=1}^{d_2} \frac{e^{-\frac{\lambda}{y_j}}}{y_j(1-e^{-\frac{\lambda}{y_j}})} + \frac{(n_2-d_2)e^{-\frac{\lambda}{c_2}}}{c_2(1-e^{-\frac{\lambda}{c_2}})},$$

$$\frac{\partial^2 l}{\partial \lambda^2} = -\frac{(d_1+d_2)}{\lambda^2} - (\alpha-1)\sum_{i=1}^{d_1} \frac{e^{-\frac{\lambda}{x_i}}}{x_i^2(1-e^{-\frac{\lambda}{x_i}})^2} - (\beta-1)\sum_{j=1}^{d_2} \frac{e^{-\frac{\lambda}{y_j}}}{y_j^2(1-e^{-\frac{\lambda}{y_j}})^2}$$
$$\frac{\alpha(n_1-d_1)e^{-\frac{\lambda}{c_1}}}{c_1^2(1-e^{-\frac{\lambda}{c_1}})^2} + \frac{\beta(n_2-d_2)e^{-\frac{\lambda}{c_2}}}{c_2^2(1-e^{-\frac{\lambda}{c_2}})^2}.$$

Now, we find the approximate estimate of the variance of $\hat{R}$, using the delta method, see, Greene (2003). Let define $G = \left(\frac{\partial R}{\partial \alpha}\ \frac{\partial R}{\partial \beta}\ \frac{\partial R}{\partial \lambda}\right)^T = \frac{1}{(\alpha+\beta)^2}(-\beta\ \ \alpha\ \ 0)^T$. Thus, an approximate estimate of $\text{Var}(\hat{R})$ is given by

$$\hat{\text{V}}\text{ar}(\hat{R}) = \left[GI^{-1}G^T\right]_{(\alpha,\beta,\lambda)=(\hat{\alpha},\hat{\beta},\hat{\lambda})}.$$

Now, using the asymptotic normality property of MLEs, the MLE $\hat{R}$ is asymptotically normal distributed with mean $R$ and variance $\hat{\text{V}}\text{ar}(\hat{R})$. Therefore, the asymptotic $(1-\gamma)\%$ confidence interval for $R$ is given by

$$\left(\hat{R}-z_{\gamma/2}\sqrt{\hat{\text{V}}\text{ar}(\hat{R})},\ \hat{R}+z_{\gamma/2}\sqrt{\hat{\text{V}}\text{ar}(\hat{R})}\right), \tag{14}$$

where $z_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution.

Since, $0 < R < 1$, a better confidence interval may be obtained using transformed confidence interval. Here, we use the logit transformation for the confidence interval estimation as suggested by Krishnamoorthy and Lin (2010). Let $\hat{\theta} = \ln(\hat{R}/(1-\hat{R}))$ be the MLE of $\theta = \ln(R/(1-R))$, using the asymptotic normality property of MLEs and the delta method, the asymptotic $(1-\gamma)\%$ confidence interval for $\theta$ is given by $(\theta_L, \theta_U)$, where, $\theta_L = \ln\left(\frac{\hat{R}}{1-\hat{R}}\right) - z_{\gamma/2}\frac{\sqrt{\hat{\text{V}}\text{ar}(\hat{R})}}{\hat{R}(1-\hat{R})}$ and $\theta_U = \ln\left(\frac{\hat{R}}{1-\hat{R}}\right) + z_{\gamma/2}\frac{\sqrt{\hat{\text{V}}\text{ar}(\hat{R})}}{\hat{R}(1-\hat{R})}$. Thus, the two sided equal tail asymptotic $100(1-\gamma)\%$ confidence interval for $R$ is obtained as

$$\left(\frac{\exp(\theta_L)}{1+\exp(\theta_L)}, \frac{\exp(\theta_U)}{1+\exp(\theta_U)}\right). \tag{15}$$

### 3.2. *Bootstrap confidence intervals*

Here, we propose the use of two parametric bootstrap confidence intervals. The two bootstrap methods that are widely used in practice are (i) the percentile bootstrap (boot-p) method proposed by Efron (1982), and (ii) the bootstrap-t (boot-t) method proposed by Hall (1988). The boot-t confidence interval is developed based on a Studentized 'pivot' and requires an estimator of the variance of the MLE of $R$ . We use the following algorithms for two parametric bootstrap confidence intervals for the stress-strength reliability $R$.

**Boot-p method.**

**Step 1.** Generate a hybrid censored sample $\underset{\sim}{x} = (x_1, x_2, \dots, x_{d_1})$ with pre-fixed censoring scheme $(r_1, T_1)$ of size $d_1$ from GIED$(\alpha, \lambda)$ and generate another hybrid censored sample $\underset{\sim}{y} = (y_1, y_2, \dots, y_{d_2})$ with censoring scheme $(r_2, T_2)$ of size $d_2$ from GIED$(\beta, \lambda)$. Compute the MLEs $\hat{\alpha}, \hat{\beta}, \hat{\lambda}$ of the parameters $\alpha, \beta, \lambda$ .

**Step 2.** Generate a bootstrap sample $\underset{\sim}{x}^* = (x_1^*, x_2^*, \dots, x_{d_1}^*)$ with pre-fixed censoring scheme $(r_1, T_1)$ of size $d_1$ from GIED$(\hat{\alpha}, \hat{\lambda})$ and generate a bootstrap sample $\underset{\sim}{y}^* = (y_1^*, y_2^*, \dots, y_{d_2}^*)$ with censoring scheme $(r_2, T_2)$ of size $d_2$ from GIED$(\hat{\beta}, \hat{\lambda})$. Compute the MLEs $\hat{R}^*$ of $R$ using equation (13).

**Step 3.** Repeat step 2, NBOOT times.

**Step 4.** Let $G(x) = P(R^* \leq x)$ be the cdf of $\hat{R}^*$. Define $\hat{R}_{BP}(x) = G^{-1}(x)$ for a given $x$. Now, the approximate $100(1-\gamma)\%$ boot-p confidence interval of $R$ is given by

$$\left(\hat{R}_{BP}(\gamma/2), \hat{R}_{BP}(1-\gamma/2)\right).$$

**Boot-t method.**

**Step 1.** Same as in boot-p method.

**Step 2.** Same as in boot-p method.

**Step 3.** Compute the following statistic $T^* = \sqrt{d_1}(\hat{R}^* - \hat{R})/\sqrt{\hat{\text{V}}\text{ar}(\hat{R}^*)}$. Compute $\hat{\text{V}}\text{ar}(\hat{R}^*) = \hat{B}^*/d_1$ as in theorem 3.2 in Section 3.1.

**Step 4.** Repeat steps 2 and 3, NBOOT times.

**Step 5.** Let $H(x) = P(T^* < x)$ be the cdf of $T^*$. Define $\hat{R}_{Bt}(x) = \hat{R} + H^{-1}(x)\sqrt{\frac{\hat{\text{V}}\text{ar}(\hat{R})}{d_1}}$ for a given $x$. The approximate $100(1-\gamma)\%$ boot-t confidence interval of $R$ is given by

$$\left(\hat{R}_{Bt}(\gamma/2), \hat{R}_{Bt}(1-\gamma/2)\right).$$

## 4. Bayesian estimation

In this Section, we derive the Bayes estimator of $R$ under the assumption that the shape parameters $\alpha$, $\beta$ and scale parameter $\lambda$ are random variables. It is quite natural to assume independent gamma priors of the shape and scale parameters having, respective, pdfs

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)}\alpha^{a_1-1}e^{-b_1\alpha} \;; \alpha,\ a_1, b_1 > 0,$$
$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta} \;; \beta,\ a_2, b_2 > 0,$$
$$g_3(\lambda) = \frac{b_3^{a_3}}{\Gamma(a_3)}\lambda^{a_3-1}e^{-b_3\lambda} \;; \lambda,\ a_3, b_3 > 0, \text{ respectively.}$$

where, $(a_1, b_1)$, $(a_2, b_2)$ and $(a_3, b_3)$ are known hyper-parameters and chosen to reflect prior knowledge about unknown parameters $\alpha$, $\beta$ and $\lambda$, respectively. The choice of gamma priors is due to their flexibilities. They accommodate a variety of shapes depending on hyper-parameters. In literature, many authors have been used gamma priors for parameters of GIED, see, Dey and Pradhan (2014), Dube *et al.* (2016), etc. Also, it is noted that when $a_i = b_i = 0.0001; i = 1,2,3$ the gamma priors become non-informative priors. Moreover, it is assumed that $\alpha$, $\beta$ and $\lambda$ are *a-priori* independent. The joint prior distribution of $\alpha$, $\beta$ and $\lambda$ can be written as

$$g(\alpha,\beta,\lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)}, \tag{16}$$

and

$$\pi(\alpha,\beta,\lambda|\text{data}) = \frac{L(\text{data}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\text{data}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda},$$

$$\begin{aligned}\pi(\alpha,\beta,\lambda|\text{data}) = {} & \alpha^{d_1+a_1-1}\beta^{d_2+a_2-1}\lambda^{d_1+d_2+a_3-1}e^{-\lambda(b_3+S_3+S_4)}e^{-(b_1\alpha+b_2\beta)} \\ & e^{(\alpha-1)\sum_{i=1}^{d_1}\ln(1-e^{-\frac{\lambda}{x_i}})}e^{\alpha(n_1-d_1)\sum_{i=1}^{d_1}\ln(1-e^{-\frac{\lambda}{c_1}})}e^{(\beta-1)\sum_{j=1}^{d_2}\ln(1-e^{-\frac{\lambda}{y_j}})} \\ & e^{\beta(n_2-d_2)\sum_{j=1}^{d_2}\ln(1-e^{-\frac{\lambda}{c_2}})}.\end{aligned} \tag{17}$$

Since, the above posterior distribution cannot be obtained analytically, we adopt Lindley's approximation method and MCMC techniques to compute Bayes estimate and the corresponding HPD credible interval of $R$ .

### 4.1. *Lindley's approximation method*

The Lindley approximation method to approximate the ratio of two integrals was proposed by Lindley (1980). According to this method the approximate Bayes estimator of $R$ under squared error loss function (SELF) is given by

$$\begin{aligned}\hat{R}_{LB} = \hat{R} + \frac{1}{2}\Big[ & \left(\hat{R}_{11}+2\hat{R}_1\hat{\rho}_1\right)\hat{\sigma}_{11} + \left(\hat{R}_{12}+2\hat{R}_1\hat{\rho}_2\right)\hat{\sigma}_{12} + \left(\hat{R}_{21}+2\hat{R}_2\hat{\rho}_1\right)\hat{\sigma}_{21} \\ & + \left(\hat{R}_{22}+2\hat{R}_2\hat{\rho}_2\right)\hat{\sigma}_{22} + \left(\hat{R}_{11}\hat{\sigma}_{11}+\hat{R}_2\hat{\sigma}_{12}\right)\left(\hat{l}_{30}\hat{\sigma}_{11}+2\hat{l}_{21}\hat{\sigma}_{12}+\hat{l}_{12}\hat{\sigma}_{22}\right) \\ & + \left(\hat{R}_1\hat{\sigma}_{21}+\hat{R}_2\hat{\sigma}_{22}\right)\left(\hat{l}_{21}\hat{\sigma}_{11}+2\hat{l}_{12}\hat{\sigma}_{21}+\hat{l}_{03}\hat{\sigma}_{22}\right)\Big],\end{aligned} \tag{18}$$

$$R_1 = \frac{\partial R}{\partial \alpha} = \frac{-\beta}{(\alpha+\beta)^2}, \quad R_{11} = \frac{\partial^2 R}{\partial \alpha^2} = \frac{2\beta}{(\alpha+\beta)^3}, \quad R_2 = \frac{\partial R}{\partial \beta} = \frac{\alpha}{(\alpha+\beta)^2},$$

$$R_{22} = \frac{\partial^2 R}{\partial \beta^2} = 0, \quad R_{12} = R_{21} = \frac{\partial^2 R}{\partial \alpha \partial \beta} = 0, \quad \rho_1 = \frac{\partial \rho}{\partial \alpha} = \left(\frac{(a_1-1)}{\alpha} - b_1\right),$$

$$\rho_2 = \frac{\partial \rho}{\partial \beta} = \left(\frac{(a_2-1)}{\beta} - b_2\right), \quad l_{30} = \frac{\partial^3 l}{\partial \alpha^3} = \frac{2d_1}{\alpha^3}, \quad l_{12} = \frac{\partial^3 l}{\partial \alpha \partial \beta^2} = 0,$$

$$l_{21}=\frac{\partial^3 l}{\partial\alpha^2\partial\beta}=0,\quad l_{03}=\frac{\partial^3 l}{\partial\beta^3}=\frac{2d_2}{\beta^3}$$

and $\hat{\sigma_{ij}}=(i,j)$-th element of the observed variance-covariance matrix $I^{-1}(\hat{\theta})$. Thus, the Bayes estimate of $R$ under SELF is given by

$$\begin{aligned}\hat{R}_{LB}=\hat{R}+\frac{1}{2}\Big[&\left(\hat{R}_{11}+2\hat{R}_1\hat{\rho}_1\right)\hat{\sigma}_{11}+\left(\hat{R}_{12}+2\hat{R}_1\hat{\rho}_2\right)\hat{\sigma}_{12}+\left(\hat{R}_{21}+2\hat{R}_2\hat{\rho}_1\right)\hat{\sigma}_{21}\\&+\left(\hat{R}_{22}+2\hat{R}_2\hat{\rho}_2\right)\hat{\sigma}_{22}+\left(\hat{R}_1\hat{\sigma}_{11}^2+\hat{R}_2\hat{\sigma}_{11}\hat{\sigma}_{12}\right)\hat{l}_{30}+\left(\hat{R}_1\hat{\sigma}_{21}\hat{\sigma}_{22}+\hat{R}_2\hat{\sigma}_{22}^2\right)\hat{l}_{30}\Big].\end{aligned}\tag{19}$$

All the values at the right hand sides in (18) and (19) are to be computed at MLEs $(\hat{\alpha},\hat{\beta})$ of $(\alpha,\beta)$. Although, using Lindley's approximation methods, the Bayes estimates of the unknown parameters can be obtained easily but we cannot construct the HPD credible intervals. For this purpose we use MCMC method to compute Bayes estimate as well as HPD credible interval of $R$.

### *4.2. MCMC Method*

Gibbs sampler generates a sequence of samples from the full conditional probability distributions. The Gibbs sampler can be efficient when the full conditional distributions are easy to sample from. The Metropolis-Hastings (MH) algorithm can be used to obtain random samples from any arbitrarily complicated target distribution of any dimension that is known up to a normalizing constant. It was first developed by Metropolis *et al.* (1953) and later extended by Hastings (1970). In fact, Gibbs sampler is a special case of MH algorithm. The full posterior conditional distributions of $\alpha$, $\beta$ and $\lambda$ respectively, are obtained as

$$\alpha|(\lambda,data)\sim f_{GA}\left((d_1+a_1),\ \left(b_1-\sum_{i=1}^{d_1}\ln(1-e^{-\frac{\lambda}{x_i}})-(n_1-d_1)\ln(1-e^{-\frac{\lambda}{c_1}})\right)\right),\tag{20}$$

$$\beta|(\lambda,data)\sim f_{GA}\left((d_2+a_2),\ \left(b_2-\sum_{j=1}^{d_2}\ln(1-e^{-\frac{\lambda}{y_j}})-(n_2-d_2)\ln(1-e^{-\frac{\lambda}{c_2}})\right)\right)\tag{21}$$

and

$$\begin{aligned}\pi(\lambda|\alpha,\beta,\mathbf{data})\propto\lambda^{d_1+d_2+a_3-1}&e^{-\lambda(b_3+S_3+S_4)}e^{(\alpha-1)\sum_{i=1}^{d_1}\ln(1-e^{-\frac{\lambda}{x_i}})}\\&e^{\alpha(n_1-d_1)\sum_{i=1}^{d_1}\ln(1-e^{-\frac{\lambda}{c_1}})}e^{(\beta-1)\sum_{j=1}^{d_2}\ln(1-e^{-\frac{\lambda}{y_j}})}\\&e^{\beta(n_2-d_2)\sum_{j=1}^{d_2}\ln(1-e^{-\frac{\lambda}{c_2}})},\end{aligned}\tag{22}$$

where $f_{GA}(x;a,b) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}$ ; $x,\ a,b > 0$ is gamma distribution with shape and scale parameters $a$ and $b$, respectively. The posterior conditional distribution of $\lambda$ is not in a well known form and therefore random numbers from this distribution can be generated by using MH algorithm. Here we consider normal distribution as proposal density. Therefore, Gibbs sampling has following steps for simulation process:

**Step 1.** Start with an initial guess $(\alpha^{(0)}, \beta^{(0)}, \lambda^{(0)})$.

**Step 2.** Set $k = 1$.

**Step 3.** Generate $\lambda^{(k)}$ from $\pi_\lambda$ using MH algorithm with following steps:
(i) Generate a candidate point $\lambda_c^{(k)}$ from the proposal density $N(\mu, \sigma^2)$.
(ii) Generate $u$ form $U(0,1)$.
(iii) Calculate $\eta\left(\lambda_c^{(k)}|\lambda^{(k-1)}\right) = min\left\{\frac{\pi_\lambda(\lambda_c^{(k)}|\text{data})}{\pi_\lambda(\lambda^{(k-1)}|\text{data})},\ 1\right\}$.
(iv) If $\mu \le \eta$ set $\lambda^{(k)} = \lambda_c^{(k)}$ with acceptance probability $\eta$ otherwise $\lambda^{(k)} = \lambda^{(k-1)}$ .

**Step 4.** Generate $\alpha^{(k)}$ from
$f_{GA}\Big((d_1 + a_1),\ \Big(b_1 - \sum_{i=1}^{d_1} \ln(1 - e^{-\lambda^{(k)}/x_i}) - (n_1 - d_1)ln(1 - e^{-\lambda^{(k)}/c_1})\Big)\Big)$.

**Step 5.** Generate $\beta^{(k)}$ from
$f_{GA}\Big((d_2 + a_2),\ \Big(b_2 - \sum_{j=1}^{d_2} \ln(1 - e^{-\lambda^{(k)}/y_j}) - (n_2 - d_2)ln(1 - e^{-\lambda^{(k)}/c_2})\Big)\Big)$.

**Step 6.** Compute $R^{(k)} = \frac{\beta^{(k)}}{\left(\alpha^{(k)}+\beta^{(k)}\right)}$.

**Step 7.** Set $k = k + 1$.

**Step 7.** Repeat steps 3-7, $M$ times.

The selection of appropriate initial values and the normal distribution as proposal density is an important issue in the MH algorithm. The rapid convergence is facilitated by selecting appropriate initial values and proposal normal distributions. The MLEs of parameters may be considered as the initial values in step 1. Here, $\mu = \hat{\lambda}_{post}$ and $\sigma^2 = 5.8 \times \text{Var}(\hat{\lambda}_{post})$ from posterior (17), see, Ntzoufras (2009) (pp 44-45). Now, the Bayes estimate of $R$ under SELF is a posterior mean and is obtained as

$$\hat{R}_{Bayes} = E(R|\text{data}) = \frac{1}{M - M_0} \sum_{k=M_0+1}^{M} R^{(k)}, \tag{23}$$

where, $M_0$ is the burn-in-period i.e. we discard first $R^{(1)}, R^{(2)}, \ldots, R^{(M_0)}$ observations and work with the remaining $M_1 = (M - M_0)$ observations, which are viewed as being an independent sample from the stationary distribution of the Markov chain which is typically the posterior distribution.

### 4.3. HPD credible interval

Once we have desired posterior sample, the HPD credible interval for $R$ can be constructed by using the algorithm proposed by Chen and Shao (1999). Let $R_1 \leq R_2 \leq \ldots, \leq R_{(M-M_O)}$ denote the ordered values of $R^{(M_0+1)}, R^{(M_0+2)}, \ldots, R^{(M-M_O)}$, the $100(1-\gamma)\%$ HPD credible interval for $R$ is given by

$$\left(R_{(k)}, R_{(k+[1-\gamma]M)}\right),$$

where $k$ is chosen such that

$$\left(R_{k+[(1-\gamma)(M-M_0)]} - R_{(k)}\right) = min_{1\leq j\leq \gamma M}\left(R_{j+[(1-\gamma)(M-M_0)]} - R_{(j)}\right);$$

$k = 1, 2, \ldots, (M - M_0)$, $[x]$ being the integer part of $x$.

## 5. Monte Carlo simulation study

This Section deals with the Monte Carlo simulation study to compare the performance of different estimation procedures under various hybrid censoring schemes. The ML and Bayes estimates under SELF using gamma informative and non-informative priors in terms of average estimate (AE) and mean squared errors (MSE) are compared. Also, the asymptotic, two types of bootstrap confidence and HPD credible intervals using informative as well as non-informative priors in terms of average lengths and coverage probabilities are compared. Different parameter values, various censoring schemes and different sample sizes are considered. In Bayes estimation, non-informative and informative priors are denoted by Prior 0 and Prior 1 , respectively. For non-informative prior hyper-parameters are taken as $a_i = b_i = 0.0001$; $i = 1, 2, 3$ in (16). For Prior 1, hyper-parameters are so chosen that prior means are exactly equal to the true values of the parameters. Two sets of true values of parameters $\alpha = 1.5$, $\beta = 2$, $\lambda = 1$ so that $R = 0.5714$ with corresponding informative hyper-parameters $a_1 = 3$, $b_1 = 2$, $a_2 = 4$, $b_2 = 2$, $a_3 = 2$, $b_3 = 2$ and $\alpha = 0.75$, $\beta = 3$, $\lambda = 1$ so that $R = 0.80$ with corresponding informative hyper-parameters $a_1 = 3$, $b_1 = 4$, $a_2 = 6$, $b_2 = 2$, $a_3 = 2$, $b_3 = 2$ are taken. In ML estimation the tolerance limit $\epsilon = 10^{-6}$ is considered for iterative process. Also, eight hybrid censoring schemes are considered and given in Table 1. The AEs and MSEs of ML and Bayes estimators are obtained over 1,000 pairs of hybrid censored samples generated from GIED. All calculations are performed on the statistical software R. The average length of 95% asymptotic confidence interval based on logit scale, boot-p, boot-t confidence and HPD credible intervals of stress-strength parameter $R$ are obtained. Here, $NBOOT = 1,000$ for bootstrap methods and $M = 10,000$ with burn-in-period $M_0 = 2,000$ for MCMC technique are considered.

The results of the Monte Carlo simulation study are presented in Tables 2, 3, 4 and 5, respectively. Tables 2 and 4 show that the MLE compares very well with the Bayes estimator in terms of AEs and MSEs. On comparing the two Bayes estimators based on informative and non-informative priors, it can be seen that the Bayes estimator based

*TABLE 1*
*Censoring schemes for Monte Carlo simulation study.*

| $(n,r,T)$ | (30,20,1.5) | (30,25,1.5) | (30,20,2.5) | (30,25,2.5) | (50,30,2) | (50,40,2) | (50,30,3) | (50,40,3) |
|---|---|---|---|---|---|---|---|---|
| CS | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $S_6$ | $S_7$ | $S_8$ |

*TABLE 2*
*The AE and MSE of the ML and Bayes estimates of R when $\alpha = 1.5, \beta = 4.5, \lambda = 1$ and $R = 0.5714$.*

| | | | $\hat{R}_{Bayes}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | LB | | | | MCMC | | | |
| | $\hat{R}_{MLE}$ | | Prior 0 | | Prior 1 | | Prior 0 | | Prior 1 | |
| CS | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE |
| $(S_1,S_2)$ | 0.5733 | 0.0068 | 0.5735 | 0.0073 | 0.5694 | 0.0042 | 0.5698 | 0.0057 | 0.5683 | 0.0042 |
| $(S_1,S_3)$ | 0.5792 | 0.0070 | 0.5846 | 0.0083 | 0.5744 | 0.0044 | 0.5741 | 0.0057 | 0.5687 | 0.0042 |
| $(S_1,S_4)$ | 0.5698 | 0.0062 | 0.5685 | 0.0068 | 0.5707 | 0.0042 | 0.5743 | 0.0054 | 0.5709 | 0.0040 |
| $(S_2,S_1)$ | 0.5789 | 0.0064 | 0.5855 | 0.0075 | 0.5750 | 0.0044 | 0.5802 | 0.0058 | 0.5716 | 0.0042 |
| $(S_2,S_3)$ | 0.5788 | 0.0065 | 0.5854 | 0.0076 | 0.5726 | 0.0061 | 0.5746 | 0.0051 | 0.5695 | 0.0041 |
| $(S_2,S_4)$ | 0.5705 | 0.0055 | 0.5705 | 0.0058 | 0.5714 | 0.0038 | 0.5743 | 0.0051 | 0.5775 | 0.0040 |
| $(S_3,S_1)$ | 0.5729 | 0.0068 | 0.5785 | 0.0081 | 0.5691 | 0.0040 | 0.5630 | 0.0058 | 0.5688 | 0.0042 |
| $(S_3,S_2)$ | 0.5623 | 0.0058 | 0.5627 | 0.0063 | 0.5680 | 0.0041 | 0.5631 | 0.0053 | 0.5676 | 0.0042 |
| $(S_3,S_4)$ | 0.5699 | 0.0058 | 0.5692 | 0.0065 | 0.5638 | 0.0041 | 0.5719 | 0.0046 | 0.5678 | 0.0036 |
| $(S_4,S_1)$ | 0.5796 | 0.0066 | 0.5878 | 0.0077 | 0.5702 | 0.0036 | 0.5693 | 0.0054 | 0.5686 | 0.0041 |
| $(S_4,S_2)$ | 0.5719 | 0.0055 | 0.5751 | 0.0058 | 0.5720 | 0.0040 | 0.5641 | 0.0051 | 0.5703 | 0.0038 |
| $(S_4,S_3)$ | 0.5821 | 0.0062 | 0.5903 | 0.0074 | 0.5737 | 0.0038 | 0.5709 | 0.0052 | 0.5677 | 0.0041 |
| $(S_5,S_6)$ | 0.5668 | 0.0038 | 0.5657 | 0.0041 | 0.5677 | 0.0032 | 0.5678 | 0.0038 | 0.5677 | 0.0027 |
| $(S_5,S_7)$ | 0.5708 | 0.0043 | 0.5749 | 0.0048 | 0.5736 | 0.0033 | 0.5721 | 0.0036 | 0.5723 | 0.0029 |
| $(S_5,S_8)$ | 0.5684 | 0.0038 | 0.5673 | 0.0041 | 0.5682 | 0.0029 | 0.5685 | 0.0035 | 0.5701 | 0.0027 |
| $(S_6,S_5)$ | 0.5784 | 0.0039 | 0.5845 | 0.0044 | 0.5795 | 0.0033 | 0.5706 | 0.0034 | 0.5715 | 0.0028 |
| $(S_6,S_7)$ | 0.5779 | 0.0040 | 0.5843 | 0.0046 | 0.5764 | 0.0028 | 0.5687 | 0.0029 | 0.5686 | 0.0027 |
| $(S_6,S_8)$ | 0.5745 | 0.0033 | 0.5757 | 0.0035 | 0.5744 | 0.0027 | 0.5685 | 0.0036 | 0.5721 | 0.0025 |
| $(S_7,S_5)$ | 0.5753 | 0.0046 | 0.5797 | 0.0052 | 0.5723 | 0.0033 | 0.5693 | 0.0036 | 0.5697 | 0.0033 |
| $(S_7,S_6)$ | 0.5656 | 0.0037 | 0.5645 | 0.0040 | 0.5672 | 0.0029 | 0.5721 | 0.0031 | 0.5695 | 0.0028 |
| $(S_7,S_8)$ | 0.5650 | 0.0039 | 0.5638 | 0.0042 | 0.5707 | 0.0031 | 0.5658 | 0.0035 | 0.5704 | 0.0028 |
| $(S_8,S_5)$ | 0.5794 | 0.0039 | 0.5858 | 0.0044 | 0.5767 | 0.0029 | 0.5694 | 0.0028 | 0.5697 | 0.0028 |
| $(S_8,S_6)$ | 0.5726 | 0.0034 | 0.5741 | 0.0036 | 0.5738 | 0.0027 | 0.5686 | 0.0032 | 0.5675 | 0.0024 |
| $(S_8,S_7)$ | 0.5734 | 0.0037 | 0.5795 | 0.0041 | 0.5776 | 0.0029 | 0.5777 | 0.0030 | 0.5665 | 0.0027 |

*TABLE 3*
*The AL and CP of 95% asymptotic, bootstrap confidence/HPD credible intervals of R when* $\alpha = 1.5, \beta = 4.5, \lambda = 1$ *and* $R = 0.5714$.

| | | | | | | | $\hat{R}_{Bayes}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{R}_{MLE}$ | | $\hat{R}_{Bp}$ | | $\hat{R}_{Bt}$ | | Prior 0 | | Prior 1 | |
| CS | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| $(S_1, S_2)$ | 0.2842 | 0.925 | 0.3086 | 0.923 | 0.3314 | 0.910 | 0.2862 | 0.938 | 0.2678 | 0.961 |
| $(S_1, S_3)$ | 0.2931 | 0.934 | 0.3154 | 0.925 | 0.3458 | 0.908 | 0.2943 | 0.943 | 0.2736 | 0.964 |
| $(S_1, S_4)$ | 0.2806 | 0.941 | 0.3028 | 0.914 | 0.3244 | 0.905 | 0.2805 | 0.938 | 0.2630 | 0.953 |
| $(S_2, S_1)$ | 0.2916 | 0.938 | 0.3080 | 0.935 | 0.3385 | 0.923 | 0.2910 | 0.929 | 0.2713 | 0.961 |
| $(S_2, S_3)$ | 0.2907 | 0.944 | 0.3041 | 0.928 | 0.3363 | 0.913 | 0.2923 | 0.943 | 0.2714 | 0.968 |
| $(S_2, S_4)$ | 0.2772 | 0.955 | 0.2889 | 0.927 | 0.3135 | 0.915 | 0.2777 | 0.941 | 0.2595 | 0.959 |
| $(S_3, S_1)$ | 0.2915 | 0.935 | 0.3158 | 0.926 | 0.3433 | 0.909 | 0.2924 | 0.933 | 0.2710 | 0.957 |
| $(S_3, S_2)$ | 0.2836 | 0.946 | 0.3023 | 0.935 | 0.3250 | 0.922 | 0.2839 | 0.943 | 0.2640 | 0.947 |
| $(S_3, S_4)$ | 0.2767 | 0.944 | 0.2965 | 0.928 | 0.3175 | 0.914 | 0.2772 | 0.955 | 0.2600 | 0.968 |
| $(S_4, S_1)$ | 0.2830 | 0.930 | 0.3030 | 0.924 | 0.3295 | 0.902 | 0.2818 | 0.945 | 0.2618 | 0.962 |
| $(S_4, S_2)$ | 0.2718 | 0.949 | 0.2886 | 0.926 | 0.3097 | 0.912 | 0.2732 | 0.942 | 0.2548 | 0.955 |
| $(S_4, S_3)$ | 0.2823 | 0.937 | 0.2988 | 0.930 | 0.3267 | 0.915 | 0.2808 | 0.946 | 0.2615 | 0.950 |
| $(S_5, S_6)$ | 0.2266 | 0.949 | 0.2378 | 0.937 | 0.2472 | 0.930 | 0.2410 | 0.948 | 0.2166 | 0.965 |
| $(S_5, S_7)$ | 0.2418 | 0.942 | 0.2540 | 0.932 | 0.2680 | 0.922 | 0.2252 | 0.935 | 0.2284 | 0.962 |
| $(S_5, S_8)$ | 0.2260 | 0.942 | 0.2369 | 0.951 | 0.2463 | 0.944 | 0.2302 | 0.943 | 0.2156 | 0.964 |
| $(S_6, S_5)$ | 0.2333 | 0.951 | 0.2418 | 0.931 | 0.2558 | 0.922 | 0.2302 | 0.945 | 0.2189 | 0.948 |
| $(S_6, S_7)$ | 0.2333 | 0.940 | 0.2420 | 0.940 | 0.2560 | 0.926 | 0.2135 | 0.955 | 0.2190 | 0.959 |
| $(S_6, S_8)$ | 0.2135 | 0.941 | 0.2214 | 0.941 | 0.2308 | 0.931 | 0.2410 | 0.946 | 0.2047 | 0.958 |
| $(S_7, S_5)$ | 0.2412 | 0.932 | 0.2538 | 0.939 | 0.2680 | 0.925 | 0.2257 | 0.929 | 0.2284 | 0.944 |
| $(S_7, S_6)$ | 0.2267 | 0.947 | 0.2379 | 0.951 | 0.2473 | 0.942 | 0.2256 | 0.952 | 0.2161 | 0.953 |
| $(S_7, S_8)$ | 0.2265 | 0.935 | 0.2367 | 0.945 | 0.2465 | 0.932 | 0.2278 | 0.943 | 0.2155 | 0.957 |
| $(S_8, S_5)$ | 0.2316 | 0.947 | 0.2410 | 0.940 | 0.2541 | 0.925 | 0.2105 | 0.950 | 0.2163 | 0.956 |
| $(S_8, S_6)$ | 0.2111 | 0.941 | 0.2206 | 0.942 | 0.2289 | 0.932 | 0.2277 | 0.954 | 0.2025 | 0.962 |
| $(S_8, S_7)$ | 0.2320 | 0.950 | 0.2410 | 0.936 | 0.2539 | 0.928 | 0.2180 | 0.942 | 0.2165 | 0.953 |

*TABLE 4*
*The AE and MSE of the ML and Bayes estimates of R when $\alpha = 0.75, \beta = 3, \lambda = 1$ and $R = 0.80$.*

| | | | $\hat{R}_{Bayes}$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | LB | | | | MCMC | | | |
| | $\hat{R}_{MLE}$ | | Prior 0 | | Prior 1 | | Prior 0 | | Prior 1 | |
| CS | AE | MSE | AE | MSE | AE | MSE | AE | MSE | AE | MSE |
| $(S_1, S_2)$ | 0.8046 | 0.0032 | 0.8115 | 0.0035 | 0.8013 | 0.0020 | 0.7934 | 0.0032 | 0.7945 | 0.0020 |
| $(S_1, S_3)$ | 0.8063 | 0.0038 | 0.8255 | 0.0054 | 0.8019 | 0.0024 | 0.7922 | 0.0032 | 0.7955 | 0.0021 |
| $(S_1, S_4)$ | 0.8064 | 0.0030 | 0.8132 | 0.0034 | 0.8038 | 0.0019 | 0.7957 | 0.0027 | 0.7986 | 0.0020 |
| $(S_2, S_1)$ | 0.8052 | 0.0039 | 0.8241 | 0.0055 | 0.8043 | 0.0022 | 0.7960 | 0.0031 | 0.7943 | 0.0022 |
| $(S_2, S_3)$ | 0.8065 | 0.0035 | 0.8256 | 0.0050 | 0.8038 | 0.0022 | 0.7920 | 0.0033 | 0.7972 | 0.0020 |
| $(S_2, S_4)$ | 0.8052 | 0.0032 | 0.8121 | 0.0036 | 0.8032 | 0.0017 | 0.7987 | 0.0031 | 0.7981 | 0.0020 |
| $(S_3, S_1)$ | 0.8053 | 0.0033 | 0.8256 | 0.0048 | 0.8008 | 0.0022 | 0.7925 | 0.0029 | 0.7972 | 0.0019 |
| $(S_3, S_2)$ | 0.8039 | 0.0030 | 0.8128 | 0.0034 | 0.7997 | 0.0018 | 0.7922 | 0.0025 | 0.7965 | 0.0018 |
| $(S_3, S_4)$ | 0.8052 | 0.0029 | 0.8137 | 0.0034 | 0.8008 | 0.0018 | 0.7903 | 0.0027 | 0.7976 | 0.0017 |
| $(S_4, S_1)$ | 0.8078 | 0.0034 | 0.8283 | 0.0050 | 0.8022 | 0.0022 | 0.7939 | 0.0029 | 0.7933 | 0.0018 |
| $(S_4, S_2)$ | 0.8071 | 0.0030 | 0.8159 | 0.0035 | 0.8020 | 0.0015 | 0.7973 | 0.0025 | 0.7944 | 0.0017 |
| $(S_4, S_3)$ | 0.8090 | 0.0033 | 0.8295 | 0.0050 | 0.8047 | 0.0020 | 0.7917 | 0.0027 | 0.7972 | 0.0018 |
| $(S_5, S_6)$ | 0.8042 | 0.0020 | 0.8095 | 0.0023 | 0.8036 | 0.0012 | 0.7953 | 0.0017 | 0.7962 | 0.0013 |
| $(S_5, S_7)$ | 0.8050 | 0.0024 | 0.8205 | 0.0032 | 0.8092 | 0.0015 | 0.7957 | 0.0019 | 0.7960 | 0.0013 |
| $(S_5, S_8)$ | 0.8023 | 0.0019 | 0.8077 | 0.0021 | 0.8027 | 0.0014 | 0.7960 | 0.0016 | 0.7982 | 0.0013 |
| $(S_6, S_5)$ | 0.8057 | 0.0023 | 0.8212 | 0.0031 | 0.8116 | 0.0015 | 0.7985 | 0.0018 | 0.7955 | 0.0015 |
| $(S_6, S_7)$ | 0.8048 | 0.0024 | 0.8201 | 0.0033 | 0.8110 | 0.0014 | 0.7961 | 0.0018 | 0.7965 | 0.0015 |
| $(S_6, S_8)$ | 0.8031 | 0.0018 | 0.8084 | 0.0020 | 0.8033 | 0.0013 | 0.7967 | 0.0016 | 0.7983 | 0.0012 |
| $(S_7, S_5)$ | 0.8047 | 0.0023 | 0.8203 | 0.0031 | 0.8074 | 0.0013 | 0.7923 | 0.0017 | 0.7942 | 0.0013 |
| $(S_7, S_6)$ | 0.8000 | 0.0019 | 0.8056 | 0.0021 | 0.8030 | 0.0014 | 0.7944 | 0.0016 | 0.7956 | 0.0013 |
| $(S_7, S_8)$ | 0.8013 | 0.0020 | 0.8070 | 0.0022 | 0.8020 | 0.0014 | 0.7934 | 0.0016 | 0.7956 | 0.0013 |
| $(S_8, S_5)$ | 0.8066 | 0.0019 | 0.8220 | 0.0028 | 0.8099 | 0.0012 | 0.7952 | 0.0016 | 0.7980 | 0.0012 |
| $(S_8, S_6)$ | 0.8013 | 0.0017 | 0.8070 | 0.0019 | 0.8047 | 0.0011 | 0.7958 | 0.0014 | 0.7977 | 0.0011 |
| $(S_8, S_7)$ | 0.8061 | 0.0022 | 0.8215 | 0.0031 | 0.8098 | 0.0014 | 0.7939 | 0.0017 | 0.7936 | 0.0013 |

*TABLE 5*
*The AL and CP of 95% asymptotic, bootstrap confidence/HPD credible intervals of R when* $\alpha = 0.75, \beta = 3, \lambda = 1$ *and* $R = 0.80$.

| | | | | | | | $\hat{R}_{Bayes}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{R}_{MLE}$ | | $\hat{R}_{Bp}$ | | $\hat{R}_{Bt}$ | | Prior 0 | | Prior 1 | |
| CS | AL | CP | AL | CP | AL | CP | AL | CP | AL | CP |
| $(S_1,S_2)$ | 0.2155 | 0.955 | 0.2321 | 0.920 | 0.2653 | 0.870 | 0.2123 | 0.937 | 0.1920 | 0.968 |
| $(S_1,S_3)$ | 0.2291 | 0.943 | 0.2201 | 0.960 | 0.2429 | 0.920 | 0.2223 | 0.946 | 0.1977 | 0.963 |
| $(S_1,S_4)$ | 0.2144 | 0.951 | 0.2276 | 0.840 | 0.2642 | 0.780 | 0.2109 | 0.953 | 0.1895 | 0.958 |
| $(S_2,S_1)$ | 0.2293 | 0.942 | 0.2237 | 0.930 | 0.2587 | 0.870 | 0.2203 | 0.942 | 0.1980 | 0.965 |
| $(S_2,S_3)$ | 0.2292 | 0.944 | 0.2244 | 0.940 | 0.2581 | 0.830 | 0.2226 | 0.945 | 0.1966 | 0.965 |
| $(S_2,S_4)$ | 0.2146 | 0.948 | 0.2076 | 0.940 | 0.2231 | 0.930 | 0.2092 | 0.933 | 0.1897 | 0.968 |
| $(S_3,S_1)$ | 0.2206 | 0.949 | 0.2081 | 0.910 | 0.2286 | 0.880 | 0.2084 | 0.948 | 0.1845 | 0.957 |
| $(S_3,S_2)$ | 0.2024 | 0.931 | 0.2042 | 0.950 | 0.2214 | 0.930 | 0.1979 | 0.946 | 0.1783 | 0.957 |
| $(S_3,S_4)$ | 0.2013 | 0.939 | 0.2068 | 0.928 | 0.2362 | 0.887 | 0.1981 | 0.949 | 0.1775 | 0.966 |
| $(S_4,S_1)$ | 0.2189 | 0.935 | 0.1746 | 0.900 | 0.1809 | 0.880 | 0.2071 | 0.931 | 0.1864 | 0.974 |
| $(S_4,S_2)$ | 0.2005 | 0.934 | 0.1909 | 0.940 | 0.2049 | 0.920 | 0.1943 | 0.953 | 0.1789 | 0.966 |
| $(S_4,S_3)$ | 0.2182 | 0.948 | 0.1859 | 0.900 | 0.2014 | 0.860 | 0.2082 | 0.946 | 0.1844 | 0.968 |
| $(S_5,S_6)$ | 0.1629 | 0.939 | 0.1850 | 0.930 | 0.2019 | 0.870 | 0.1595 | 0.941 | 0.1495 | 0.957 |
| $(S_5,S_7)$ | 0.1820 | 0.939 | 0.1825 | 0.929 | 0.2062 | 0.884 | 0.1706 | 0.953 | 0.1580 | 0.973 |
| $(S_5,S_8)$ | 0.1639 | 0.939 | 0.1658 | 0.947 | 0.1797 | 0.912 | 0.1591 | 0.949 | 0.1488 | 0.955 |
| $(S_6,S_5)$ | 0.1818 | 0.947 | 0.2198 | 0.920 | 0.2419 | 0.880 | 0.1691 | 0.946 | 0.1579 | 0.957 |
| $(S_6,S_7)$ | 0.1822 | 0.935 | 0.2191 | 0.940 | 0.2446 | 0.930 | 0.1702 | 0.948 | 0.1573 | 0.951 |
| $(S_6,S_8)$ | 0.1634 | 0.952 | 0.2143 | 0.936 | 0.2316 | 0.887 | 0.1584 | 0.953 | 0.1484 | 0.965 |
| $(S_7,S_5)$ | 0.1801 | 0.944 | 0.1801 | 0.930 | 0.2022 | 0.890 | 0.1677 | 0.963 | 0.1544 | 0.966 |
| $(S_7,S_6)$ | 0.1608 | 0.935 | 0.1649 | 0.925 | 0.1764 | 0.901 | 0.1542 | 0.951 | 0.1448 | 0.965 |
| $(S_7,S_8)$ | 0.1602 | 0.942 | 0.1650 | 0.940 | 0.1766 | 0.911 | 0.1547 | 0.954 | 0.1448 | 0.958 |
| $(S_8,S_5)$ | 0.1789 | 0.969 | 0.1769 | 0.935 | 0.2008 | 0.893 | 0.1646 | 0.958 | 0.1510 | 0.967 |
| $(S_8,S_6)$ | 0.1590 | 0.942 | 0.1575 | 0.917 | 0.1721 | 0.888 | 0.1515 | 0.946 | 0.1422 | 0.958 |
| $(S_8,S_7)$ | 0.1788 | 0.941 | 0.1760 | 0.924 | 0.2000 | 0.876 | 0.1650 | 0.956 | 0.1530 | 0.965 |

on gamma informative priors outperforms based on non-informative priors in terms of both AEs and MSEs. For both the ML and Bayes estimation procedures, as the effective sample size increases the AEs become close to their true values and MSEs decrease. Again, as the true value of $R$ increases AEs depart from their true values and the MSEs decrease. Also, the Bayes estimate outperforms the MLE in terms of AEs and MSEs. In general, the combination of censoring schemes $S_6$ and $S_8$ give best results in comparison to other censoring schemes in terms of AEs, MSEs, average length of different confidence/credible intervals.

Tables 3 and 5 present the average confidence and credible lengths with corresponding coverage probabilities. The nominal level for the confidence and the credible intervals is 0.95 in each case. These Tables show that the average length of asymptotic, bootstrap confidence and HPD credible intervals narrow down as effective sample sizes increase. Also, the boot-t confidence intervals are wider than the asymptotic, boot-p confidence and HPD credible intervals. The HPD credible intervals provide the smallest average credible lengths for different censoring schemes and for different parameter values. The asymptotic confidence interval based on MLE is the second best confidence interval. Also, it is evident that the HPD credible intervals provide the highest coverage probabilities in most cases considered.

Thus, the Bayesian estimation procedure is recommended when prior information about parameters is available. For quick and easy results ML estimation procedure may be considered.

## 6. Real data analysis

In this Section, the analysis of a pair of real data sets is presented for illustrative purposes. The strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows are presented in Tables 6 and 7, respectively. Single fibers were tested under tension at gauge lengths of 10 mm and 20 mm. These data sets were originally reported by Bader and Priest (1982). These data sets have also been used by Kundu and Gupta (2006) and many others.

*TABLE 6*
*Data set 1 (X: gauge length 10 mm).*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.901 | 2.132 | 2.203 | 2.228 | 2.257 | 2.350 | 2.361 | 2.396 | 2.397 | 2.445 |
| 2.454 | 2.474 | 2.518 | 2.522 | 2.525 | 2.532 | 2.575 | 2.614 | 2.616 | 2.618 |
| 2.624 | 2.659 | 2.675 | 2.738 | 2.740 | 2.856 | 2.917 | 2.928 | 2.937 | 2.937 |
| 2.977 | 2.996 | 3.030 | 3.125 | 3.139 | 3.145 | 3.220 | 3.223 | 3.235 | 3.243 |
| 3.264 | 3.272 | 3.294 | 3.332 | 3.346 | 3.377 | 3.408 | 3.435 | 3.493 | 3.501 |
| 3.537 | 3.554 | 3.562 | 3.628 | 3.852 | 3.871 | 3.886 | 3.971 | 4.024 | 4.027 |
| 4.225 | 4.395 | 5.020 | | | | | | | |

*TABLE 7*
*Data set 2 (Y: gauge length 20 mm).*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.312 | 1.314 | 1.479 | 1.552 | 1.700 | 1.803 | 1.861 | 1.865 | 1.944 | 1.958 |
| 1.966 | 1.997 | 2.006 | 2.021 | 2.027 | 2.055 | 2.063 | 2.098 | 2.140 | 2.179 |
| 2.224 | 2.240 | 2.253 | 2.270 | 2.272 | 2.274 | 2.301 | 2.301 | 2.359 | 2.382 |
| 2.382 | 2.426 | 2.434 | 2.435 | 2.478 | 2.490 | 2.511 | 2.514 | 2.535 | 2.554 |
| 2.566 | 2.570 | 2.586 | 2.629 | 2.633 | 2.642 | 2.648 | 2.684 | 2.697 | 2.726 |
| 2.770 | 2.773 | 2.800 | 2.809 | 2.818 | 2.821 | 2.848 | 2.880 | 2.954 | 3.012 |
| 3.067 | 3.084 | 3.090 | 3.096 | 3.128 | 3.233 | 3.433 | 3.585 | 3.585 | |

*TABLE 8*
*MLEs, K-S and A-D tests for real data sets assuming different scale parameters.*

| Data Set | Shape Parameter | Scale Parameter | K-S Test | | A-D Test | |
|---|---|---|---|---|---|---|
| | | | Statistic | p-value | Statistic | p-value |
| Data Set 1 (X: gauge length 10 mm) | 175.2867 | 16.811 | 0.086 | 0.7399 | 0.4213 | 0.8268 |
| Data Set 2 (Y: gauge length 20 mm) | 205.8832 | 13.8826 | 0.0414 | 0.9998 | 0.1939 | 0.992 |

First of all, the GIED is fitted to the two data sets separately. The estimated parameters, Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) statistics with corresponding $p$-values are presented in Table 8. This table shows that the K-S as well A-D tests suggest that the null hypothesis that each data set is drawn from GIED may be accepted at 5% level of significance. Also, we draw various diagnostic plots like empirical & fitted pdfs, empirical & fitted cdfs, probability-probability (P-P) plots and quantile-quantile (Q-Q) plots for both complete data sets and presented in Figure 1 and Figure 2, respectively. These Figures also support above conclusions.

Now, we assume that $X \sim \text{GIED}(\alpha, \lambda_1)$ and $Y \sim \text{GIED}(\beta, \lambda_2)$. The MLEs of the unknown parameters are as follows: $\hat{\alpha} = 175.2867$, $\hat{\lambda}_1 = 16.8110$, $\hat{\beta} = 205.8832$, $\hat{\lambda}_2 = 13.8826$, and the associated log-likelihood value is $L_1 = -106.8932$. Also, suppose that $X \sim \text{GIED}(\alpha, \lambda)$ and $Y \sim \text{GIED}(\beta, \lambda)$. The MLEs of the unknown parameters are as follows: $\hat{\alpha} = 109.6088$, $\hat{\beta} = 323.8219$, $\hat{\lambda} = 15.1607$, and the associated log-likelihood value is $L_0 = -107.8660$. Following testing of hypothesis is considered:

$$H_0 : \lambda_1 = \lambda_2 \ \text{ against } \ H_1 : \lambda_1 \neq \lambda_2,$$

and the test statistic $\chi^2 = -2(L_0 - L_1)$ follows the chi-square distribution with one degree of freedom. Thus, the value of $\chi^2$ and the corresponding p-value can be evaluated to accept or reject $H_0$. Here, $\chi^2 = 1.9456$ with corresponding $p$-value 0.1631. Hence at 5% level of significance the null hypothesis cannot be rejected. Therefore, in this case the assumption of $\lambda_1 = \lambda_2$ is justified. Also, in this case, the estimated parameters, K-S and A-D statistics with corresponding p-values are reported in Table 9. Table 9 supports the choice of GIEDs with different shapes and common scale parameters.

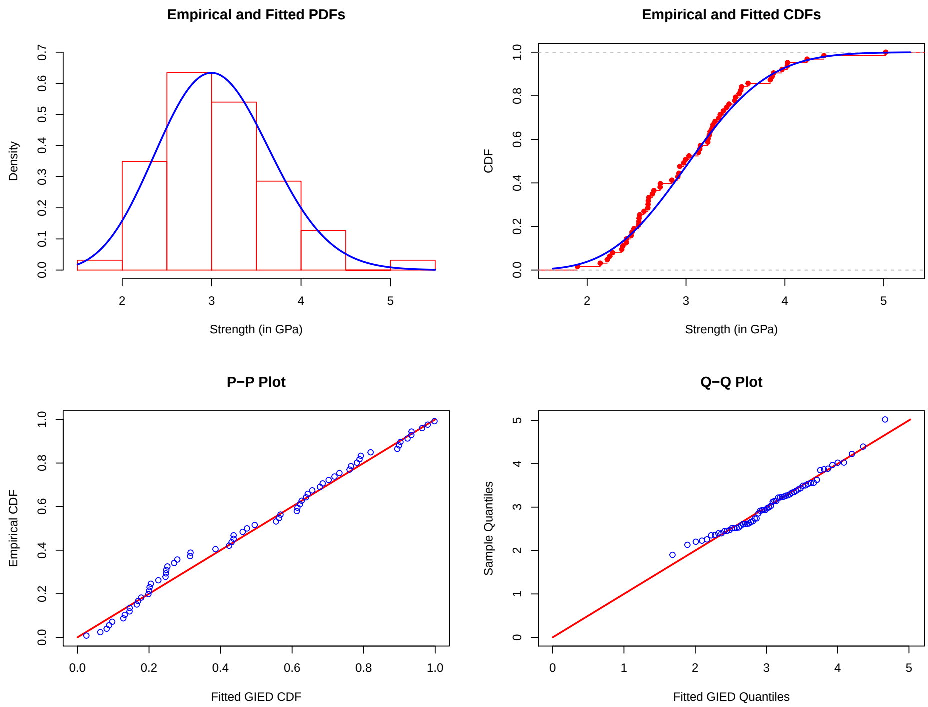


*Figure 1* – Diagnostic plots for the fitted GIED for data set 1 ($X$ : gauge length 10 mm).

*TABLE 9*
*MLEs, K-S and A-D tests for real data sets assuming common scale parameter.*

| Data Set | Shape Parameter | Scale Parameter | K-S Test | | A-D Test | |
|---|---|---|---|---|---|---|
| | | | Statistic | p-value | Statistic | p-value |
| Data Set 1 (X: gauge length 10 mm) | 109.6088 | 15.1607 | 0.0796 | 0.8194 | 0.5333 | 0.7125 |
| Data Set 2 (Y: gauge length 20 mm) | 323.8219 | 15.1607 | 0.0583 | 0.9733 | 0.3089 | 0.9311 |

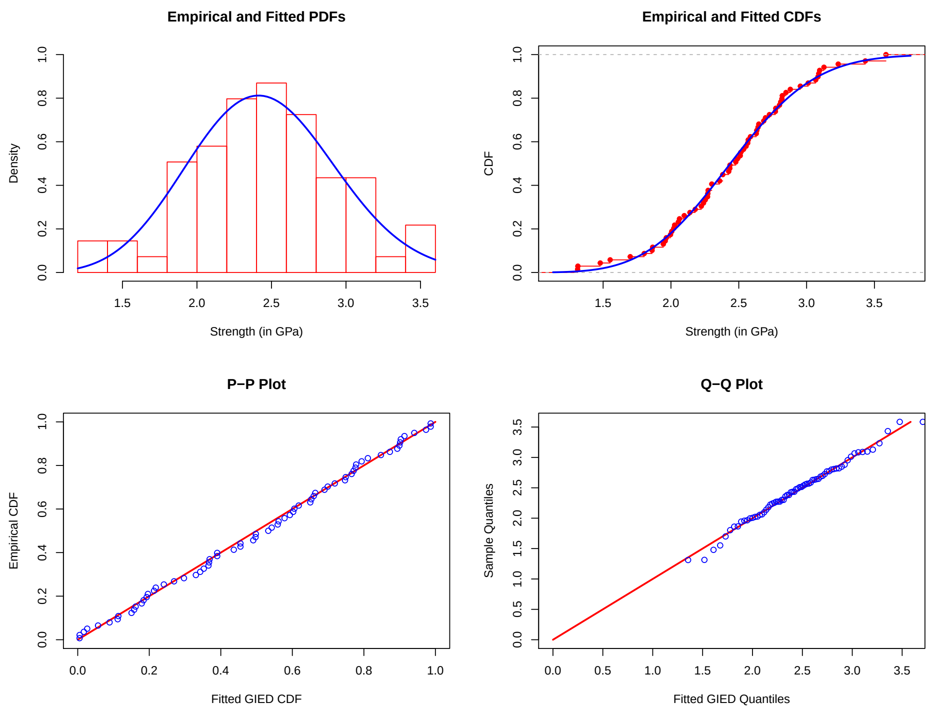


*Figure 2* – Diagnostic plots for the fitted GIED for data set 2 ($Y$ : gauge length 20 mm).

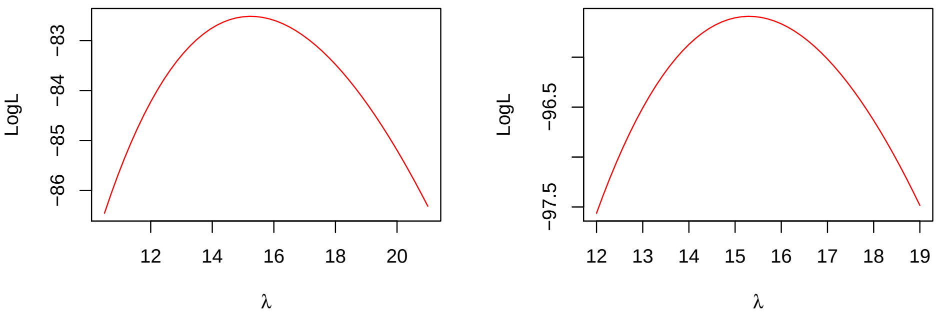


*Figure 3* – The plots of profile log-likelihood of $\lambda$ for data sets corresponding to censoring schemes $(CS_1, CS_2)$ and $(CS_3, CS_4)$, respectively.

Further, two pairs of hybrid censored samples are generated from the above data sets using the two sets of hybrid censoring schemes. The hybrid censoring schemes and the corresponding generated data sets are reported in Table 10. Based on these data sets the ML and Bayes estimates under SELF of $R$ are computed. The plots of profile log-likelihood of $\lambda$ for both the censoring schemes are plotted in Figure 3. This Figure shows that the likelihood equation for $\lambda$ have a unique solution. The Bayes estimates of the stress-strength parameter $R$ for both censoring schemes are computed using Lindley's approximation and MCMC methods. Also, the convergence of generated sequences of $R$ for its stationary distribution using graphical diagnostic tools like trace plots, ACF plots and histograms are checked. In MCMC method we use thining and burn-in period. Here, $M = 20,000$, thining = every $5^{th}$ observation and burn-in period $M_0 = 800$ are considered for MCMC method. Figures 4 and 5 show the trace plots, ACF plots and

*TABLE 10*

*The generated hybrid censored samples corresponding to real data sets.*

| Censoring Scheme | | Data sets |
|---|---|---|
| $CS_1 = (r_1 = 50, T_1 = 2.60)$ | X: | 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454,<br>2.474, 2.518, 2.522, 2.525, 2.532, 2.575 |
| $CS_2 = (r_2 = 40, T_1 = 3.50)$ | Y: | 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966,<br>1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240,<br>2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434,<br>2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554 |
| $CS_3 = (r_1 = 40, T_1 = 3)$ | X: | 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454,<br>2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659,<br>2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996 |
| $CS_4 = (r_2 = 30, T_2 = 2.70)$ | Y: | 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966,<br>1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240,<br>2.253 |

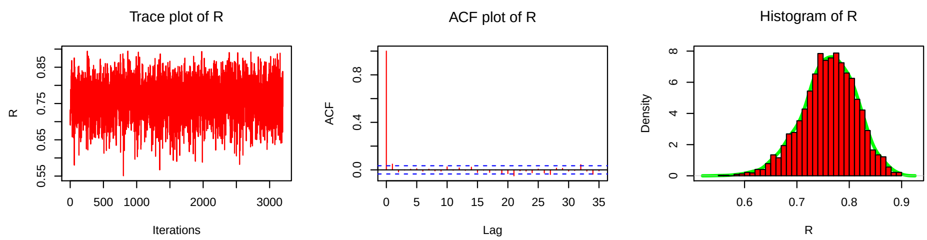


*Figure 4* – The trace, ACF and histogram with Gaussian density plots of the generated chain for parameter $R$ corresponding to censoring scheme $(CS_1, CS_2)$.

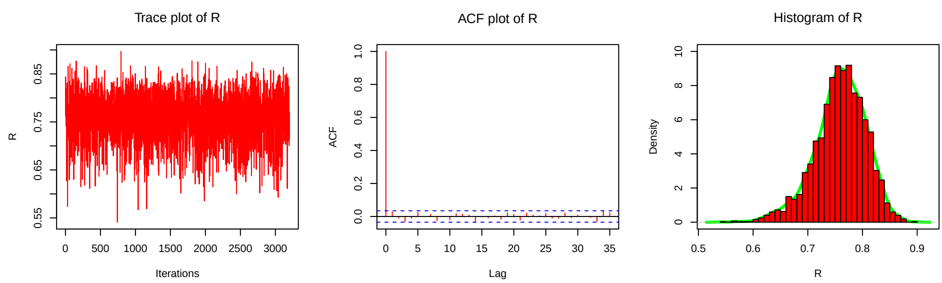


*Figure 5* – The trace, ACF and histogram with Gaussian density plots of the generated chain for parameter $R$ corresponding to censoring scheme $(CS_3, CS_4)$.

histograms with Gaussian kernel density estimates of $R$ for both censoring schemes, respectively. The trace plots of the generated chains indicate a random scatter and show the convergence of chains for parameter $R$. ACF plots clearly show that chains are not at all auto-correlated, so the generated samples may be considered as independent samples from the target posterior distributions. The histogram with Gaussian kernel density shows almost symmetrical and unimodal distribution of $R$.

Based on the censoring schemes $(CS_1, CS_2)$, the ML and Bayes estimates using Lindley's approximation and MCMC methods are 0.7640, 0.7635, 0.7613, respectively. Also, the 95% asymptotic confidence interval on logit scale, boot-p, boot-t confidence and HPD credible intervals are (0.6466, 0.8514), (0.6511, 0.8691), (0.6647, 0.8997), (0.6515, 0.8555), respectively. Similarly, based on censoring scheme $(CS_3, CS_4)$, the ML and Bayes estimates using Lindley's approximation and MCMC methods are 0.7628, 0.7601, 0.7593, respectively. Also, the 95% asymptotic confidence interval based on logit scale, boot-p, boot-t confidence and HPD credible intervals are (0.6426, 0.8518), (0.6559, 0.8729), (0.6694, 0.9130), and (0.6609, 0.8419), respectively.

## 7. Conclusions

In this article, the problem of classical and Bayesian estimation of stress-strength reliability $R = P(Y < X)$ for generalized inverted exponential distribution using hybrid censored samples was considered. The hybrid censoring scheme is an operational censoring scheme and very useful in real life applications. Different estimation methods for estimating the stress-strength reliability in the case of different shapes and common scale unknown parameters of GIED were considered. The MLE of $R$ and its asymptotic distribution was computed. A simple iterative procedure was provided for computation of MLEs of the unknown parameters and $R$. Also, two parametric bootstrap confidence intervals were proposed and it was observed that the asymptotic confidence interval works the best even for small effective sample sizes.

The Bayes estimator of $R$ under squared error loss function using non-informative and gamma informative priors was constructed. Bayes estimate did not come out in closed form, Lindley's approximation and MCMC methods were used for computation of the Bayes estimate and associated HPD credible interval. The performance of the point and interval estimates of $R$ is examined by extensive simulations. Simulation results suggested that the performance of Bayes estimator in point estimation and HPD credible interval based on gamma informative priors work very well and these can be used for all practical purposes. A real data example is also discussed for illustration purposes.

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## SUMMARY

Based on the hybrid censored samples, this article deals with the problem of point and interval estimation of the stress-strength reliability $R = P(Y < X)$ when $X$ and $Y$ both have independent generalized inverted exponential distributions with different shape and common scale parameters. The maximum likelihood estimation, Bayes estimation and parametric bootstrap methods are used for estimating $R$. Also, asymptotic confidence interval of $R$ is derived based on asymptotic distribution of $R$. Bayesian estimation procedure is carried out using Lindley approximation and

Markov Chain Monte Carlo methods. Bayes estimate and the HPD credible interval of $R$ are obtained using non-informative and gamma informative priors. A Monte Carlo simulation study is carried out for comparing the different proposed estimation methods. Finally, a pair of real data sets is analyzed for illustration purposes.

*Keywords*: Stress-strength reliability; Generalized inverted exponential distribution; Maximum likelihood estimation; Bootstrap confidence interval; Bayes estimation; MCMC method; HPD credible interval.