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SOME RELIABILITY PROPERTIES OF EXTROPY AND ITS RELATED MEASURES USING QUANTILE FUNCTION

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1. Introduction

Let X be a non-negative and absolutely continuous random variable representing the lifetime of a component/system with a probability density function (pdf) $f(.)$. Then the differential extropy of X is defined as (see Lad and Sanfilippo, 2015)

$$J(X) = -\frac{1}{2}\int_0^\infty f^2(x)dx. \tag{1}$$

Extropy was originally introduced in the discrete set up. If X is a discrete random variable taking values $\{x_1, x_2, ..., x_N\}$ with respective probabilities $\mathbf{p}_N = \{p_1, p_2, ..., p_N\}$, then extropy is defined as $J(\mathbf{p}_N) = -\sum_{i=1}^N (1-p_i)\log(1-p_i)$, a complementary dual of Shannon entropy $H(\mathbf{p}_N) = -\sum_{i=1}^N p_i \log p_i$. Similar to Shannon entropy, extropy provides the amount of uncertainty contained in the probability distribution of a random variable. The differential extropy $J(X)$ in (1) is the limiting form of $J(\mathbf{p}_N)$, given by $J(X) = \lim_{\Delta x \to 0}\left\{\frac{J(\mathbf{p}_N)-1}{\Delta x}\right\}$. The notion of extropy has been found applications in various fields. One of the statistical applications of extropy is to score the forecasting distributions using the total log scoring rule. Under the total log scoring rule, the expected score of a forecasting distribution equals the negative sum of the entropy and extropy of this distribution. Another statistical application of extropy is to compare the uncertainties of two random variables. For instance, the uncertainty in X may be measured by the difference in the outcomes from two independently conducted experiments under

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identical conditions. If X_1 and X_2 are two such outcomes, then $X_1 - X_2$ measures the uncertainty in X. If the pdf of X is denoted by f, then the pdf of $X_1 - X_2$ can be defined by $g(u) = \int_{-\infty}^{\infty} f(x)f(x-u)dx$. This implies the probability of $X_1 - X_2$ approximately equals to $g(0) = -2J(X)$. If the extropy of X is less than another random variable Y, then X has more uncertainty than Y (see Jahanshahi *et al.*, 2019; Qiu *et al.*, 2019).

In reliability and life testing, the data are generally truncated and in such cases $J(X)$ is not an appropriate measure. Assume that the component X has survived t units time. Then the variable of interest is the residual lifetime of the component, denoted by $X_t = (X - t|X > t), t > 0$. Based on this idea, Qiu and Jia (2018) proposed the residual extropy and studied its various properties. For a non-negative and absolutely continuous random variable X, differential extropy of the residual random variable X_t is given by

$$J_t(X) = J(X_t) = -\frac{1}{2\left(\bar{F}(t)\right)^2} \int_t^{\infty} f^2(x)dx, t \geq 0. \tag{2}$$

For more recent works on (1) and (2) and its applications, one can also refer to Raqab and Qiu (2018), Yang *et al.* (2018), Alizadeh Noughabi and Jarrahiferiz (2019), Jose and Sathar (2019) and the references therein.

Note that both $J(X)$ and $J_t(X)$ are defined in terms of the distribution function. However, in certain situations we do not have a tractable distribution function while its quantile function exists, where neither $J(X)$ nor $J_t(X)$ is amenable for computing uncertainty. For example, many quantile functions used in applied works such as various forms of lambda distributions, the power-Pareto distributions, Govindarajulu distribution etc., do not have tractable distribution functions. This calls for a separate study on extropy using quantile function. Quantile functions are efficient and equivalent alternatives to the distribution function in modelling and analysis of statistical data (see Gilchrist, 2000; Nair and Sankaran, 2009), defined by

$$Q(u) = F^{-1}(u) = \inf\{x|F(x) \geq u\}, 0 < u < 1. \tag{3}$$

There are certain properties of quantile functions that are not shared by the distribution function. For a detailed and recent study on quantile function and its properties in modelling and analysis we refer to Nair and Vineshkumar (2011), Nair *et al.* (2013), Sreelakshmi *et al.* (2018) and the references therein. Recently, the study of information measures using quantile function has also attracted among researchers. Sunoj and Sankaran (2012) introduced a quantile-based Shannon entropy and its residual form. Quantile versions of the cumulative entropy functions in the residual and past lifetimes are studied by Sankaran and Sunoj (2017). For further study on quantile-based entropy and its related measures we refer to Yu and Wang (2013), Kumar and Rekha (2018), Kayal and Tripathy (2018), Sunoj *et al.* (2018) and Krishnan *et al.* (2019). Motivated with these, the present study extends the domain of application of extropy and its associated measures using quantile functions that lead to useful lifetime models.

The organization of the paper is unfolded as follows. In Section 2 we introduce the quantile-based extropy and its residual version and obtain some characterization results and properties. Section 3 presents extropy of order statistics using quantile function. We extend the measure based on survival function known as cumulative residual extropy and study its properties in Section 4. Finally, Section 5 proposes a non-parametric estimator for quantile-based extropy function and applies the method to a real data set.

2. Quantile-based residual extropy

If $f(\cdot)$ is the pdf of X, then $f(Q(u))$ and $q(u) = \frac{d}{du}Q(u)$ respectively known as the density quantile function and quantile density function (see Parzen, 1979). Using (3), we obtain $F(Q(u)) = u$ and differentiating it with respect to u obtain

$$q(u)f(Q(u)) = 1.$$

An important quantile measure useful in reliability analysis is the hazard quantile function

$$H(u) = h(Q(u)) = \frac{1}{(1-u)q(u)}, \tag{4}$$

where $h(t) = \frac{f(t)}{\bar{F}(t)}$ is the hazard rate of X. Note that $H(u)$ determine $Q(u)$ uniquely (see Nair and Sankaran, 2009). Another important measure useful in quantile-based reliability analysis is the mean residual quantile function (Nair and Sankaran, 2009), given by

$$M(u) = m(Q(u)) = \frac{1}{(1-u)}\int_u^1 (Q(p) - Q(u))dp, \tag{5}$$

where $m(t) = E(X - t|X > t)$ is the mean residual life function (MRLF) of X. $M(u)$ provides the mean remaining life of a unit beyond the $100(1-u)\%$ of the distribution. For more properties and applications of $H(u)$ and $M(u)$, one could refer to Nair *et al.* (2013).

The quantile-based extropy based on (1) is defined by

$$\begin{aligned} L(X) &= -\frac{1}{2}\int_0^1 f^2(Q(p))dQ(p) \\ &= -\frac{1}{2}\int_0^1 (q(p))^{-1}dp \\ &= -\frac{1}{2}\int_0^1 (1-p)H(p)dp. \end{aligned} \tag{6}$$

$L(X)$ provides a quantile version of the extropy, that measures the uncertainty of X, using either quantile density function or hazard quantile function.

In the following example, we consider a quantile model useful in reliability analysis for which no closed form distribution function exists.

EXAMPLE 1. *Consider the quantile function (Midhu et al., 2013)*

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \mu > 0; -\mu \leq c < \mu,$$

corresponding to the linear mean residual quantile function (see Sankaran and Unnikrishnan Nair, 2009)

$$M(u) = cu + \mu,\ \mu > 0,\ -\mu < c < \mu,\ 0 \leq u \leq 1. \tag{7}$$

Then $L(X)$ for (7) is obtained as

$$L(X) = -\frac{1}{2}\left(\frac{-2c - (\mu+c)\log(\mu-c) + (\mu+c)\log(\mu+c)}{4c^2}\right).$$

TABLE 1
Quantile functions and the quantile-based extropy for some distributions.

Distribution	$Q(u)$	$L(X)$
Exponential	$-\frac{\log(1-u)}{\lambda}, \lambda > 0$	$\frac{-\lambda}{4}$
Pareto II	$\gamma((1-u)^{\frac{-1}{c}} - 1), \gamma, c > 0$	$-\frac{c^2}{2(2c+1)\gamma}$
Rescaled Beta	$R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$	$-\frac{c^2}{2(2c-1)R}$
Generalized Pareto	$\frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1], b > 0, a > -1$	$-\frac{(a+1)^2}{2(3a+2)b}$
Power	$\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$	$-\frac{\beta^2}{2(2\beta-1)\gamma}$
Uniform	$a + (b-a)u, -\infty < a < b < \infty$	$\frac{-1}{2(b-a)}$
Davies	$\frac{cu^{\lambda_1}}{(1-u)^{\lambda_2}}, c > 0; \lambda_1, \lambda_2 > 0$	$-\frac{{}_2\tilde{F}_1\left(1,2-\lambda_1;-\lambda_1+\lambda_2+4;1-\frac{\lambda_2}{\lambda_1}\right)\Gamma(2-\lambda_1)\Gamma(\lambda_2+2)}{2(c\lambda_1)}$

Table 1 provides some well known quantile functions and its extropy function. $L(X)$ is not useful for a system that has survived to measure the uncertainty for some units of time t. In such contexts, $J_t(X)$ is employed to measure the uncertainty. As mentioned in Section 1, in life testing experiments truncation is common. The quantile-based residual extropy of X_t is given by

$$\begin{aligned} L_Q(u) &= -\frac{1}{2(1-u)^2}\int_u^1 f^2(Q(u))dQ(u) \\ &= -\frac{1}{2(1-u)^2}\int_u^1 (q(p))^{-1}dp \\ &= -\frac{1}{2(1-u)^2}\int_u^1 (1-p)H(p)dp. \end{aligned} \tag{8}$$

Differentiating (8) with respect to u, we get

$$L'_Q(u) = \frac{H(u)}{2(1-u)} + \frac{2}{(1-u)}\left(\frac{-1}{2(1-u)^2}\int_u^1 (1-p)H(p)dp\right),$$

equivalently

$$q(u) = \left(2(1-u)^2 L'_Q(u) - 4(1-u)L_Q(u)\right)^{-1}. \tag{9}$$

Thus unlike $J_t(X)$ in (2), $L_Q(u)$ uniquely determines the quantile density function using (9). The expressions of $L_Q(u)$ for different quantile functions are given in Table 2.

TABLE 2
Quantile-based residual extropy for some quantile functions.

Distribution	$Q(u)$	$L_Q(u)$
Exponential	$-\frac{\log(1-u)}{\lambda}, \lambda > 0$	$-\frac{\lambda}{4}$
Pareto II	$\gamma((1-u)^{-\frac{1}{c}}-1), \gamma, c > 0$	$-\frac{c^2(1-u)^{\frac{1}{c}}}{2\gamma(2c+1)}$
Rescaled Beta	$R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$	$-\frac{c^2(1-u)^{-\frac{1}{c}}}{2R(2c-1)}$
Generalized Pareto	$\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$	$-\frac{(a+1)^2(1-u)^{\frac{a}{a+1}}}{2b(3a+2)}$
Power	$\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$	$-\frac{\beta^2\left(1-u^{-\frac{1}{\beta}+2}\right)}{2\gamma(2\beta-1)(1-u)^2}$
Uniform	$a+(b-a)u, -\infty < a < b < \infty$	$-\frac{1}{2(b-a)(1-u)}$
Davies	$\frac{cu}{(1-u)}, c > 0.$	$\frac{1-u}{c\left(\frac{u}{(1-u)^2}+\frac{1}{1-u}\right)}$

Note that for some important life distributions where the quantile function $Q(u)$ are of closed form expressions, however, in some cases only the quantile density function $q(u)$ has a closed form expression. Therefore, in the following theorem we prove a characterization of $L_Q(u)$ for a family of distributions that can be represented only through $q(u)$.

THEOREM 2. *The random variable X follows a distribution with quantile density function*

$$q(u) = ku^{\delta}(1-u)^{-(A+\delta)}, k > 0, A, \delta \in R, u \in (0,1). \tag{10}$$

if and only if

$$(1-u)^2 L_Q(u) = -k_1 \bar{B}_u(a,b),$$

where $k_1 > 0$ and $\bar{B}_u(a,b) = \int_u^1 p^{a-1}(1-p)^{b-1}dp$ denote the incomplete beta function.

PROOF. The proof of 'if' part is straightforward. To prove the 'only if' part, assume that $(1-u)^2 L_Q(u) = -k_1 \bar{B}_u(a,b)$ holds. Differentiating both sides with respect to u yields,

$$(1-u)^2 L'_Q(u) - 2(1-u)L_Q(u) = k_1 u^{a-1}(1-u)^{b-1}.$$

Now substituting the above expression in (9) and simplifying we obtain the required quantile model (10). This completes the proof. □

Note that the family of distributions (10) contains some important probability distributions such as, exponential ($\delta = 0, A = 1$), Pareto ($\delta = 0, A < 1$), rescaled beta ($\delta = 0, A > 1$), log-logistic ($\delta = \lambda - 1, A = 2$) and Govindarajulu (Govindarajulu, 1977) ($\delta = \beta - 1, A = -\beta$) (see Nair *et al.*, 2013).

Characterization problems generally identify some unique property possessed by a probability distribution and it helps to obtain an exact model through the physical characteristics of a data. The following theorem provides a characterization to some important lifetime models based on $L_Q(u)$.

THEOREM 3. *Let X be a random variable with quantile function $Q(u)$ and hazard quantile function $H(u)$. The relationship $L_Q(u) = -kH(u)$, where k is a non-negative constant holds for all u if and only if X has*

1. *rescaled beta distribution $Q(u) = R\left(1-(1-u)^{\frac{1}{c}}\right), 0 \le u \le 1, R > 0, c > \frac{1}{2}$ if $k > \frac{1}{4}$*

2. *exponential law $Q(u) = -\frac{\log(1-u)}{\lambda}, 0 \le u \le 1; \lambda > 0$ if $k = \frac{1}{4}$ and*

3. *Pareto II distribution $Q(u) = \gamma((1-u)^{\frac{-1}{c}} - 1), 0 \le u \le 1; c, \alpha > 0$ if $0 < k < \frac{1}{4}$.*

PROOF. Assume that $L_Q(u) = -kH(u)$. Differentiating both sides and using (9), we get

$$H(u) = -2k(1-u)H'(u) + 4kH(u),$$

implies,

$$\frac{H'(u)}{H(u)} = \frac{4k-1}{2k(1-u)}.$$

Now,

$$\log H(u) = \frac{4k-1}{2k}(-\log(1-u)) + \log c,$$

equivalently,

$$H(u) = \frac{c_1}{(1-u)^{\frac{4k-1}{2k}}}.$$

By using (4), $H(u)$ uniquely determines underlying quantile density so that $Q(u) = \int_0^u \frac{1}{(1-p)H(p)} dp$. Thus, when $k = \frac{1}{4}, H(u) = 1$, which characterizes exponential distribution. Similarly, for $k < \frac{1}{4}$ and $k > \frac{1}{4}$ respectively characterizes Pareto II and rescaled beta models. The proof of 'if' part is straightforward. □

DEFINITION 4. *X is said to have increasing (decreasing) quantile-based residual extropy if $L_Q(u)$ is increasing (decreasing) in u.*

THEOREM 5. *Let X be a continuous random variable with quantile function $Q(u)$ and hazard quantile function $H(u)$. If the quantile-based residual extropy is increasing (decreasing) in u, then $L_Q(u) \geq (\leq) \frac{-H(u)}{4}$.*

PROOF. The proof is straightforward from (9). □

DEFINITION 6. *We say that X has less residual quantile extropy than Y, $X \leq_{RQE} Y$ if $L_{Q_X}(u) \leq L_{Q_Y}(u)$ for all $0 < u < 1$.*

DEFINITION 7. *X is said to be smaller than Y in dispersive order, denoted by $X \leq_{disp} Y$, if $Q_Y(u) - Q_X(u)$ is increasing in $u \in (0,1)$.*

The following theorem provides that a system with less reliable lifetime contains less information content in the quantile set up.

THEOREM 8. *If X and Y are two random variables such that $X \leq_{disp} Y$, then $X \leq_{RQE} Y$.*

PROOF. $X \leq_{disp} Y$ implies that $Q_Y(u) - Q_X(u)$ is increasing in u. Using (8),

$$L_{Q_X}(u) = \frac{-1}{2(1-u)^2} \int_u^1 (q_X(p))^{-1} dp \leq \frac{-1}{2(1-u)^2} \int_u^1 (q_Y(p))^{-1} dp,$$

$$L_{Q_X}(u) \leq L_{Q_Y}(u),$$

which completes the proof. □

DEFINITION 9. *A lifetime random variable X is said to have increasing (decreasing) failure rate (IFR (DFR)) if $H(u)$ is increasing (decreasing).*

DEFINITION 10. *X is said to be smaller than Y in convex transform order, denoted by $X \leq_c Y$, if $\frac{q_Y(u)}{q_X(u)}$ is increasing in u, (see Shaked and Shanthikumar, 2007; Nair et al., 2013).*

In the following theorem we prove the closure property quantile-based residual extropy order using the convex transform order

THEOREM 11. *Let X and Y be two non-negative random variables with quantile densities $q_X(\cdot)$ and $q_Y(\cdot)$ respectively such that $q_X(0) \leq q_Y(0)$. If $X \leq_c Y$, then $L_{Q_X}(u) \leq L_{Q_Y}(u)$.*

PROOF. We have $X \leq_c Y$, iff $\frac{q_Y(u)}{q_X(u)}$ is increasing in $0 \leq u \leq 1$. That is, $\frac{q'_X(u)}{q_X(u)} \leq \frac{q'_Y(u)}{q_Y(u)}$, equivalently, $q_X(u) \leq q_Y(u)$ and therefore

$$L_{Q_X}(u) = \frac{-1}{2(1-u)^2}\int_u^1 (q_X(p))^{-1} dp \leq \frac{-1}{2(1-u)^2}\int_u^1 (q_Y(p))^{-1} dp = L_{Q_Y}(u).$$

□

The following theorem gives the upper bound of extropy based on some important ageing classes.

THEOREM 12. *If X is said to have increasing failure rate (IFR (IFRA, NBU)) then $L_{Q_X}(u) \leq L_{Q_Y}(u)$, where Y has exponential distribution.*

PROOF. X is $IFR(IFRA, NBU)$ if and only if $X \leq_c (\leq_*, \leq_{su})Y$, where Y has the exponential distribution with mean $\frac{1}{\lambda}$ (see Shaked and Shanthikumar, 2007). When Y is exponential we have $L_{Q_Y}(u) = \frac{-\lambda}{4}$, we obtain the upper bound of extropy of IFR (IFRA, NBU) classes by Theorem 11. □

In the next theorem, we obtain a simple characterization result that connects quantile-based extropy and mean residual quantile function.

THEOREM 13. *Let X be a non-negative continuous random variable with quantile function $Q(\cdot)$ and mean residual quantile function $M(\cdot)$. Then the relationship*

$$L_Q(u)M(u) = -k,\ k \geq 0 \tag{11}$$

holds if and only if X has generalized Pareto distribution with quantile function $Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right]$, $a > -1, b > 0$.

PROOF. The 'only if' part is straightforward. To prove the 'if' part, assume that (11) holds. From (8), we get

$$\int_u^1 (1-p)H(p)dp = \frac{2k(1-u)^2}{M(u)}. \tag{12}$$

Differentiating (12) with respect to u and simplifying,

$$H(u) = \frac{2k\left(2M(u)+(1-u)M'(u)\right)}{(M(u))^2}.$$

Equivalently,

$$(M(u))^2 H(u) = 2k\left(2M(u)+(1-u)M'(u)\right). \tag{13}$$

Differentiating (5) with respect to u, we have

$$\frac{1}{H(u)} = M(u)-(1-u)M'(u),$$

and substituting it in (13) yield,

$$(M(u))^2 = 2k\left(2(M(u))^2-(1-u)M(u)M'(u)-(1-u)^2\left(M'(u)\right)^2\right),$$

or

$$2k\left(\frac{M'(u)}{M(u)}\right)^2 + \frac{2k}{(1-u)}\left(\frac{M'(u)}{M(u)}\right) + \frac{(1-4k)}{(1-u)^2} = 0,$$

which provides

$$2k\left(\frac{dy}{du}\right)^2 + \frac{2k}{(1-u)}\left(\frac{dy}{du}\right) + \frac{(1-4k)}{(1-u)^2} = 0,$$

where $y = \log M(u)$. The solution of the above differential equation is given by $\frac{M'(u)}{M(u)} = \frac{-2k\pm\sqrt{36k^2-8k}}{4k(1-u)} = \frac{c_1}{(1-u)}$, $c_1 > 0$, or equivalently $M(u) = k_1(1-u)^{-c_1}$, the mean residual quantile function of generalized Pareto model. This completes the proof. □

3. EXTROPY OF ORDER STATISTICS

Suppose $X_1, X_2, ..., X_n$ be a random sample from a population with probability density function $f(.)$ and cumulative distribution function $F(.)$ and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be the order statistics obtained by arranging the preceding random sample in increasing order of magnitude. Then the probability density function of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}(F(x))^{i-1}\left(\bar{F}(x)\right)^{n-i} f(x),$$

where

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx; a,b>0,$$

is the beta function. The corresponding quantile-based density function of $f_{i:n}(x)$ becomes

$$f_{i:n}(u)=f_{i:n}(Q(u))=\frac{u^{i-1}(1-u)^{n-i}}{B(i,n-i+1)q(u)}.$$

Sunoj *et al.* (2017) introduced a quantile-based entropy of order statistics and studied its properties.

Order statistics have a wide variety of applications. It is used in digital image processing, health data, characterization of probability distributions, estimation theory, goodness of fit tests, reliability analysis etc. Order statistics naturally appear in real life whenever we need to arrange observations in ascending order; say for example prices arranged from smallest to largest, scores scored by a player in last ten innings from smallest to largest and so on. The study of order statistics needs special considerations due to their natural dependence. In reliability and life testing studies, the i^{th} order statistic $X_{i:n}$ refers to the lifetime of a $(n-i+1)$ out-of-n system. For properties and applications, see Arnold *et al.* (1992) and David and Nagaraja (2003). A large volume of literature is available on entropy of order statistics (see Wong and Chen, 1990; Baratpour *et al.*, 2008; Zarezadeh and Asadi, 2010; Abbasnejad and Arghami, 2010). In the context of extropy of order statistics and record values, Qiu (2017) obtained some characterization results and bounds to it. However, all these based on the distribution function approach. We consider now the extropy of order statistics using quantile function.

Let $X_{i:n}$ be the i^{th} order statistic. Extropy of the i^{th} order statistic based on (1) is given by

$$J(X_{i:n})=-\frac{1}{2}\int_0^\infty f_{i:n}^2(x)dx. \tag{14}$$

Within the framework of quantile functions, the quantile-based extropy based on (14) is obtained as

$$\begin{aligned} L(X_{i:n}) &= -\frac{1}{2}\int_0^1\left(\frac{1}{B(i,n-i+1)}p^{i-1}(1-p)^{n-i}(q(p))^{-1}\right)^2 dQ(p) \\ &= -\frac{1}{2}\int_0^1\left(\frac{p^{i-1}(1-p)^{n-i}}{B(i,n-i+1)}\right)^2 (q(p))^{-1}dp. \end{aligned} \tag{15}$$

In system reliability, first order statistic represents the lifetime of a series system while the n^{th} order statistic measures the lifetime of a parallel system. For a series system $(i=1)$,

$$L(X_{1:n})=-\frac{1}{2}\int_0^1\left(n(1-p)^{n-1}\right)^2(q(p))^{-1}dp. \tag{16}$$

For a parallel system ($i = n$),

$$L(X_{n:n}) = -\frac{1}{2}\int_0^1 (np^{n-1})^2 (q(p))^{-1} dp. \tag{17}$$

The following theorem provides some interesting properties of quantile-based extropy of order statistics when the pdf of the underlying iid random variables are symmetric.

THEOREM 14. *Let $X_1, X_2, ..., X_n$ be iid samples whose distribution is symmetric about mean μ. Then*

1. $L(X_{i:n}) = L(X_{n-i+1:n})$
2. $\Delta L(X_{i:n}) = -\Delta L(X_{n-i:n}), \forall i = 1, 2, ..., n-1$ *where* $\Delta L(X_{i:n}) = L(X_{i+1:n}) - L(X_{i:n})$.
3. *If* $Y = \frac{X-\mu}{a}$ *then* $L(Y_{i:n}) = aL(X_{i:n})$.

PROOF. For a symmetric random variable $X_{i:n} =^d -X_{n-i+1}$, equivalently $f_{i:n}(\mu + x) = f_{n-i+1:n}(\mu - x)$, and therefore

$$\frac{u^{i-1}(1-u)^{n-i}}{B(i, n-i+1)} = \frac{u^{n-i}(1-u)^{i-1}}{B(n-i+1, i)}.$$

(a) We have

$$\begin{aligned} L(X_{n-i+1:n}) &= -\frac{1}{2}\int_0^1 \left(\frac{p^{n-1}(1-p)^{i-1}}{B(n-i+1, i)}\right)^2 (q(p))^{-1} dp \\ &= -\frac{1}{2}\int_0^1 \left(\frac{p^{i-1}(1-p)^{n-i}}{B(i, n-i+1)}\right)^2 (q(p))^{-1} dp \\ &= L(X_{i:n}). \end{aligned}$$

1. We have $\Delta L(X_{i:n}) = L(X_{i+1:n}) - L(X_{i:n})$ and hence $\Delta L(X_{n-i:n}) = L(X_{n-i+1:n}) - L(X_{n-i}: n) = -\Delta L(X_{i:n})$.

(c) Let $Y = \frac{X-\mu}{a}$.
Then $Q_Y(u) = \frac{Q_X(u)-\mu}{a}$. Now,

$$\begin{aligned} L(Y_{i:n}) &= -\frac{1}{2}\int_0^1 \left(\frac{p^{i-1}(1-p)^{n-i}}{B(i, n-i+1)}\right)^2 \left(\frac{q_X(p)}{a}\right)^{-1} dp \\ &= aL(X_{i:n}). \end{aligned}$$

We now consider the quantile-based residual extropy of the i^{th} order statistic based on the random variable X_t, given by

$$L_{Q_{X_{i:n}}}(u) = \frac{-1}{2(\bar{B_u}(i, n-i+1))^2} \int_u^1 p^{2(i-1)}(1-p)^{2(n-i)}(q(p))^{-1} dp, \tag{18}$$

where $\bar{B_u}(i, n-i+1) = \int_u^1 p^{i-1}(1-p)^{n-i} dp$. For $i = 1, 2$, (18) reduces to the quantile-based residual extropy of series system

$$L_{Q_{X_{1:n}}}(u) = -\frac{n^2}{2(1-u)^{2n}} \int_u^1 (1-p)^{2(n-1)}(q(p))^{-1} dp, \tag{19}$$

and quantile-based residual extropy for parallel system, given by

$$L_{Q_{X_{n:n}}}(u) = -\frac{n^2}{u^{2n}} \int_u^1 p^{2(n-1)}(q(p))^{-1} dp. \tag{20}$$

In the case of series system, differentiating (19) with respect to u, we get

$$L'_{Q_{X_{1:n}}}(u) = \frac{n^2 H(u)}{2(1-u)} + \frac{2n}{(1-u)} L_{X_{1:n}}(u),$$

equivalently, the quantile density function

$$q(u) = \frac{n^2}{2}\left((1-u)^2 L'_{Q_{X_{1:n}}}(u) - 2n L_{X_{1:n}}(u)\right)^{-1}. \tag{21}$$

Thus (21) provides an explicit formula to identify the underlying distribution for different functional forms of quantile-based residual extropy of the first order statistic. Table 3 gives different probability models and quantile-based extropy of the first order statistics. Further if $L_{X_{1:n}}(u)$ is increasing in u, then $L_{Q_{X_{1:n}}}(u) \geq \frac{-nH(u)}{4}$.

THEOREM 15. *Let X be a random variable with hazard quantile function $H(u)$. If $L_{Q_{X_{1:n}}}(u) = -kH(u)$, for all $u \in (0,1)$*

1. *a rescaled beta distribution, if and only if $k > \frac{n}{4}$ and $c > \frac{1}{2n}$;*
2. *an exponential distribution if and only if $k = \frac{n}{4}$;*
3. *a Pareto distribution if and only if $0 < k < \frac{n}{4}$.*

PROOF. The proof is similar to Theorem 3. □

TABLE 3
Quantile functions and the quantile-based extropy of first order statistic.

Distribution	$Q(u)$	$L_{Q_{X_{1:n}}}(u)$
Exponential	$-\frac{\log}{(}1-u)\lambda, \lambda > 0$	$\frac{-n\lambda}{4}$
Pareto II	$\gamma((1-u)^{\frac{-1}{c}}-1), \gamma, c > 0$	$\frac{c^2 n^2 (1-u)^{1/c}}{2\gamma(2cn+1)}$
Rescaled Beta	$R\left(1-(1-u)^{\frac{1}{c}}\right), c, R > 0$	$-\frac{c^2 n^2 (1-u)^{-1/c}}{2R(2cn-1)}$
Generalized Pareto	$\frac{b}{a}[(1-u)^{-\frac{a}{a+1}}-1], b > 0, a > -1$	$-\frac{(a+1)^2 n^2 (1-u)^{\frac{a}{a+1}}}{2b(2(a+1)n+a)}$
Power	$\gamma u^{\frac{1}{\beta}}, \gamma, \beta > 0$	$-\frac{\beta n^2 (1-u)^{-2n}\left(\frac{\Gamma\left(2-\frac{1}{\beta}\right)\Gamma(2n-1)}{\Gamma\left(2n-\frac{1}{\beta}+1\right)}-B_u\left(2-\frac{1}{\beta}, 2n-1\right)\right)}{2\gamma}$
Uniform	$a+(b-a)u, -\infty < a < b < \infty$	$-\frac{n^2}{2(2n-1)(u-1)(a-b)}$
Davies	$\frac{cu}{(1-u)}, c > 0.$	$-\frac{n^2(u-1)^2}{2c(2n-1)(1-u)}$

THEOREM 16. *The relationship $L(X_{i:n}) = nL(X)$ holds, if and only if X is exponential with quantile function $Q(u) = -\frac{1}{\lambda}\log(1-u)$, $\lambda > 0$.*

Next theorem gives the sufficient condition for decreasing quantile-based extropy of first order statistic.

THEOREM 17. *If X has a decreasing density quantile function $f(Q(.))$ then $L_{Q_{X_{1:n}}}(u)$ is decreasing.*

PROOF. For $0 < u_1 < u_2 < 1$, $f(Q(u_2)) \le f(Q(u_1))$. From (19), we write

$$
\begin{aligned}
L_{Q_{X_{1:n}}}(u_1) &= -\frac{n^2}{2(1-u_1)^{2n}}\int_{u_1}^{1}(1-p_1)^{2(n-1)} f(Q(p_1))\,dp_1 \\
&> -\frac{n^2}{2(1-u_2)^{2n}}\int_{u_2}^{1}(1-p_2)^{2(n-1)} f(Q(p_2))\,dp_2 \\
&= L_{Q_{X_{1:n}}}(u_2).
\end{aligned}
$$

□

The following counterexample shows that the above theorem is applicable for the first order statistics only. The theorem violates for $i > 1$.

EXAMPLE 18. *Consider power-Pareto ($c = \lambda_1 = \lambda_2 = 1$) distribution with quantile function $Q(u) = \frac{u}{1-u}$. It is clear that the density quantile function $f(Q(u)) = (1-u)^2$ is* *decreasing in u. Now we obtain*

$$
\begin{aligned}
L_{X_{2:2}}(u) &= -\frac{2}{(1-u^2)^2}\int_u^1 p^2(q(p))^{-1}dp \\
&= -\frac{2}{(1-u^2)^2}\int_u^1 p^2(1-p)^2dp. \qquad (22)
\end{aligned}
$$

From (22), we get $L_{X_{2:2}}(\frac{1}{4}) = -0.068$ and $L_{X_{2:2}}(\frac{1}{2}) = -0.059$. Clearly, $L_{X_{2:2}}(\frac{1}{4}) < L_{X_{2:2}}(\frac{1}{2})$, which is not decreasing in u.

4. Cumulative Residual Extropy

In this section we introduce a cumulative residual extropy using distribution function and quantile approaches. Cumulative extropy of a non-negative continuous random variable X is defined by (see Jahanshahi *et al.*, 2019)

$$\mathscr{C}E(X) = -\frac{1}{2}\int_0^\infty \bar{F}^2(x)dx. \qquad (23)$$

$\mathscr{C}E(X)$ is obtained by replacing the probability density function $f(.)$ in (1) by the survival function $\bar{F}(.)$. Unlike (1), $\mathscr{C}E(X)$ is more stable as the cumulative distribution function $F(.)$ or $\bar{F}(.)$ always exists. For the residual random variable X_t, (23) modified to

$$\mathscr{C}E(X_t) = -\frac{1}{2}\int_t^\infty \left(\frac{\bar{F}^2(x)}{\bar{F}^2(t)}\right)dx, \qquad (24)$$

can be termed as cumulative residual extropy. Based on (23), the quantile-based cumulative residual extropy can be defined as

$$\Phi(X) = -\frac{1}{2}\int_0^1 (1-p)^2 q(p)dp. \qquad (25)$$

The corresponding quantile-based cumulative residual extropy function using (24) becomes

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q(p)dp. \qquad (26)$$

Differentiating both sides of (26) with respect to u, we get

$$q(u) = 2\Phi_Q'(u) - \frac{4\Phi_Q(u)}{(1-u)}. \qquad (27)$$

The identity (27) uniquely determines the quantile density function.

Equation (26) can be also expressed in terms of the hazard quantile function by

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 \frac{(1-p)}{H(p)}dp. \tag{28}$$

Using the interrelationship between hazard quantile function and mean residual quantile function, given by $(H(u))^{-1} = M(u) - (1-u)M'(u)$, (28) becomes

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)M(p)dp + \frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 dM(p). \tag{29}$$

Applying integration by-parts on the second term of (29), yields

$$\Phi_Q(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)M(p)dp - \frac{M(u)}{2}. \tag{30}$$

Equation (30) represents the quantile-based cumulative residual extropy in terms of mean residual quantile function $M(u)$.

EXAMPLE 19. *For proportional hazard quantile function model* $Q_Y(u) = Q_X(1-(1-u)^{\frac{1}{\theta}})$,

$$\Phi_{Q_Y}(u) = -\frac{1}{2\theta(1-u)^2}\int_u^1 (1-p)^{1+\frac{1}{\theta}} q_X(1-(1-p)^{\frac{1}{\theta}})dp.$$

Taking $v = 1-(1-p)^{\frac{1}{\theta}}$, *then*

$$\Phi_{Q_Y}(u) = -\frac{1}{2\theta(1-u)^2}\int_{1-(1-u)^{\frac{1}{\theta}}}^1 (1-v)^{2\theta} q_X(v)dv.$$

THEOREM 20. *For a non-negative continuous random variable* X *with* $\Phi_Q(u) = c$, *where* $c > 0$ *is a constant. Then* $H(u)$ *is a constant, which characterizes exponential distribution.*

PROOF. The proof directly follows from (27). □

DEFINITION 21. *We say that* X *has less cumulative residual quantile extropy than* Y, *denoted by* $X \le_{CRQE} Y$ *if* $\Phi_{Q_X}(u) \le \Phi_{Q_Y}(u)$, *for all* $0 < u < 1$.

The following theorem examines the relationship between two random variables in terms of hazard quantile order and cumulative residual quantile extropy order.

THEOREM 22. *If* $X \le_{HQ} Y$ *then* $X \ge_{CRQE} Y$.

PROOF. Let $X \leq_{HQ} Y$. So that $q_X(u) \leq q_Y(u)$ implies that

$$\int_u^1 (1-p)^2 q_X(p) dp \leq \int_u^1 (1-p)^2 q_Y(p) dp$$

equivalently

$$-\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_X(p) dp \geq -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_Y(p) dp$$

Thus $\Phi_{Q_X}(u) \geq \Phi_{Q_Y}(u)$. □

THEOREM 23. $X \leq_{CRQE} Y \Rightarrow X \leq_{HQ} Y$.

The following counterexample illustrates the above theorem.

EXAMPLE 24. *Let $Q_X(u) = u^2$ and $Q_Y(u) = 2u - u^2$, both do not have a tractable distribution function. We have $\Phi_{Q_X}(u) = \frac{(3u+1)(u-1)}{12}$ and $\Phi_{Q_Y}(u) = -\frac{(u-1)^2}{4}$ holds $X \leq_{CRQE} Y$. But the hazard quantile functions$H_X(u) = \frac{1}{2u(1-u)}$ and $H_Y(u) = \frac{1}{2(1-u)^2}$ has the property $H_X(u) > H_Y(u)$ for $u = \frac{1}{3}$ and $H_X(u) < H_Y(u)$ for $u = \frac{2}{3}$. Thus $X \leq_{CRQE} Y$ does not imply $X \leq_{HQ} Y$.*

THEOREM 25. *If $\frac{\Phi_{Q_Y}(u)}{\Phi_{Q_X}(u)}$ is decreasing in u, then $X \leq_{CRQE} Y \Rightarrow X \leq_{HQ} Y$.*

PROOF.

$$\text{If } \frac{\Phi_{Q_Y}(u)}{\Phi_{Q_X}(u)} \text{ is increasing in } u, \text{ equivalently } \frac{q_Y(u)}{q_X(u)} \geq \frac{\int_u^1 (1-p)^2 q_X(p) dp}{\int_u^1 (1-p)^2 q_Y(p) dp} \leq 1,$$

which implies $H_X(u) \geq H_Y(u)$. Thus $X \leq_{HQ} Y$. □

DEFINITION 26. *X is said to have increasing (decreasing) cumulative residual quantile extropy (ICRQE (DCRQE)) if $\Phi_Q(u)$ is increasing in u.*

Next theorem provides an upper (lower) bound for $\Phi_Q(u)$ based on ICRQE (DCRQE) classes.

THEOREM 27. *If X is ICRQE (DCRQE) then $\Phi_Q(u) \leq (\geq) \frac{1}{4}((1-u)M'(u) - M(u))$.*

For exponential distribution with $Q(u) = -\frac{1}{\lambda}\log(1-u)$, $\Phi_Q(u) = -\frac{1}{4\lambda}$. The exponential distribution is the boundary class of ICRQE and DCRQE classes.

We now prove a characterization theorem connecting cumulative residual quantile extropy and mean residual quantile function for some important lifetime models.

THEOREM 28. *Let X be a random variable with quantile function $Q(u)$ and mean residual quantile function $M(u)$ for all $u \in (0,1)$. The relationship $\Phi_Q(u) = -kM(u)$, where k is a non-negative constant holds for all u if and only if X is distributed as rescaled beta, exponential or Pareto II according as $k \gtreqless \frac{1}{4}$.*

PROOF. Assume that

$$\Phi_Q(u) = -kM(u), \tag{31}$$

holds. Differentiating (31) with respect to u, we get

$$\Phi'_Q(u) = -kM'(u),$$

and using (27), we obtain

$$-2kM'(u) = q(u) + \frac{4\Phi_Q(u)}{1-u}.$$

Substituting $q(u) = \frac{M(u)}{1-u} - M'(u)$ and using (31), yield

$$\frac{M'(u)}{M(u)} = \left(\frac{1-4k}{1-2k}\right)\left(\frac{1}{1-u}\right)$$

When $k = \frac{1}{4}$ implies $M'(u) = 0$, equivalently $M(u) =$ a constant, characterizes exponential distribution, and for $k > (<)\frac{1}{4}$,

$$\frac{d}{du}\log M(u) = \left(\frac{4k-1}{2k-1}\right)\frac{d}{du}(\log(1-u)),$$

implies $M(u) = k_1(1-u)^{1+\frac{2k}{2k-1}}$, where k_1 is the constant of integration and applying $q(u)$ in terms of $M(u)$ provides the required models. □

THEOREM 29. *If $Y = aX + b$, with $a > 0$ and $b > 0$, then $\Phi_{Q_Y}(u) = a\Phi_{Q_X}(u)$.*

PROOF. Let $Y = aX + b$, with $a > 0$ and $b \geq 0$. Then

$$F_Y(y) = P[Y \leq y] = P[aX + b \leq y] = F_X\left(\frac{y-b}{a}\right).$$

By setting $F_X\left(\frac{y-b}{a}\right) = u$, we get $Q_Y(u) = aQ_X(u) + b$, we have

$$\Phi_{Q_Y}(u) = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 q_Y(p)dp = -\frac{1}{2(1-u)^2}\int_u^1 (1-p)^2 a q_X(p)dp = a\Phi_{Q_X}(u).$$

□

REMARK 30. *Theorem 29 implies that $\Phi_Q(u)$ is a shift-independent measure.*

THEOREM 31. *The random variable X follows a distribution with quantile density function (10) if and only if*

$$(1-u)^2\Phi_Q(u) = -k_2\bar{B}_u(c,d),$$

where $k_2, c, d > 0$.

PROOF. The 'if' part is direct. Applying similar steps as in Theorem 2 and using (27), the 'only if' part can be easily proved. □

5. APPLICATIONS OF QUANTILE-BASED EXTROPY

In this section, we construct an empirical estimator for quantile-based extropy and examine its usefulness using simulation and real data analysis.

To propose an empirical estimator for quantile-based extropy function, we consider a random sample $X_1, X_2, ..., X_n$. We compute empirical distribution function $F_{X:n}$. The empirical quantile function is given by (Parzen, 1979)

$$\hat{Q}(u) = n\left(\frac{j}{n} - u\right)X_{(j-1)} + n\left(u - \frac{j-1}{n}\right)X_{(j)}, \tag{32}$$

for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, ..., n$. The corresponding empirical estimator for quantile density function $\hat{q}(u) = \hat{Q}'(u)$ is $\hat{q}(u) = n(X_{(j)} - X_{(j-1)})$ for $\frac{j-1}{n} \leq u \leq \frac{j}{n}$ and $j = 1, 2, ..., n$. From (6), the empirical estimator for quantile-based extropy becomes,

$$\hat{L}(X) = -\frac{1}{2}\int_0^1 (\hat{q}(p))^{-1} dp,$$

where $\hat{q}(p) = n\left(X_{(j)} - X_{(j-1)}\right)$ is the empirical estimator of quantile density function (see Parzen, 1979). Then $\hat{L}(X)$ can be written as

$$\hat{L}(X) = -\frac{1}{2n}\sum_{j=1}^{n}\left(n(X_{(j)} - X_{(j-1)})\right)^{-1}. \tag{33}$$

5.1. Simulation study

To assess the efficiency of the estimator (33), we conduct a Monte Carlo simulation study with various sample sizes $n = 20, 100, 300, 600, 900$ and 1000. The data are generated from Davies (power-Pareto) distribution given in Table 1, with parameters $c = 1, \lambda_1 = 1$

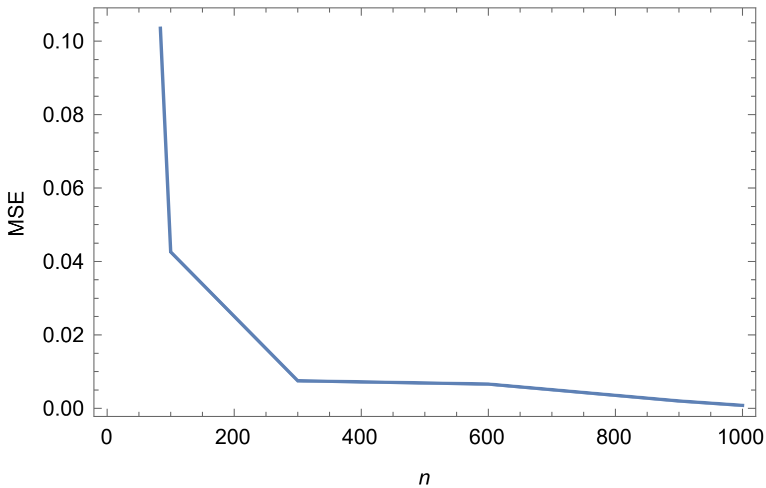

Figure 1 – Mean square error of the estimator.

and $\lambda_2 = 7$. The true value of $L(X)$ for the Davies distribution is -0.038. The bias, MSE and estimated values of $L(X)$ based on (33) are then computed for each of these sample sizes, and given in Table 4. The MSE values are also plotted in Figure 1. It is evident from Table 4 and Figure 1 that both the bias and MSE of $\hat{L}(X)$ decrease with increase in sample size, validates the performance of $\hat{L}(X)$.

TABLE 4
Bias, MSE and extropy estimates of power-Pareto distribution with $c = 1, \lambda_1 = 1$ and $\lambda_2 = 7$.

n	20	100	300	600	900	1000
Bias	0.4782	0.4345	0.4393	0.4288	0.4008	0.3759
MSE	0.3429	0.0426	0.0075	0.0066	0.0020	0.0008
$\hat{L}(X)$	-0.5162	-0.4725	-0.4774	-0.4668	-0.4389	-0.4139

5.2. Data analysis

For establishing the usefulness of the proposed quantile-based extropy, we apply the above empirical estimator $\hat{L}(X)$ in (33) to real-life data set. The data given in Table 5 consist of the times (in months) to first failure of 20 small electric carts, reported in Zimmer *et al.* (1998).

We fit the data using Davies distribution, given in Table 4 with parameters c, λ_1 and λ_2. To estimate the parameters, we use the method of L- moments, which are the

TABLE 5
Time to first failure of 20 electric carts from Zimmer et al. (1998).

Failure order (i)	1	2	3	4	5	6	7	8	9	10
First failure time (in months)	0.9	1.5	2.3	3.2	3.9	5.0	6.2	7.5	8.3	10.4
Failure order (i)	11	12	13	14	15	16	17	18	19	20
First failure time (in months)	11.1	12.6	15.0	16.3	19.3	22.6	24.8	31.5	38.1	53.0

competing alternatives generally used in quantile-based analysis than the conventional moments in distribution function approach (see Hosking, 1992). Since Davies distribution contains three parameters, we take three sample L- moments which are respectively given by

$$l_1 = \binom{n}{1}^{-1} \sum_{i=1}^{n} X_{(i)} = 14.675$$

$$l_2 = \frac{1}{2}\binom{n}{2}^{-1} \sum_{i=1}^{n} \left(\binom{i-1}{1} - \binom{n-i}{1} \right) X_{(i)} = 7.3345$$

$$l_3 = \frac{1}{3}\binom{n}{3}^{-1} \sum_{i=1}^{n} \left(\binom{i-1}{2} - 2\binom{i-1}{1}\binom{n-i}{1} + \binom{n-i}{2} \right) X_{(i)} = 2.4678,$$

where $X_{(i)}$ is the i^{th} order statistic. The corresponding population L— moments based on Davies distribution with parameters c, λ_1 and λ_2 are given by

$$L_1 = \mu = cB(\lambda_1 + 1, 1 - \lambda_2),$$

$$L_2 = \frac{c(\lambda_1 + \lambda_2)}{\lambda_1 - \lambda_2 + 2} B(\lambda_1 + 1, 1 - \lambda_2),$$

and

$$L_3 = \frac{c(\lambda_1^2 + \lambda_2^2 + 4\lambda_1\lambda_2 + \lambda_2 - \lambda_1) B(\lambda_1 + 1, 1 - \lambda_2)}{(\lambda_1 - \lambda_2 + 2)(\lambda_1 - \lambda_2 + 3)}.$$

We equate sample L- moments to population L- moments, given by

$$l_r = L_r, r = 1, 2, 3. \tag{34}$$

Solutions of set of equations (34) give the estimates of c, λ_1 and λ_2, which are obtained as

$$\hat{c} = 18.6139, \hat{\lambda}_1 = 1.1255, \hat{\lambda}_2 = 0.2911.$$

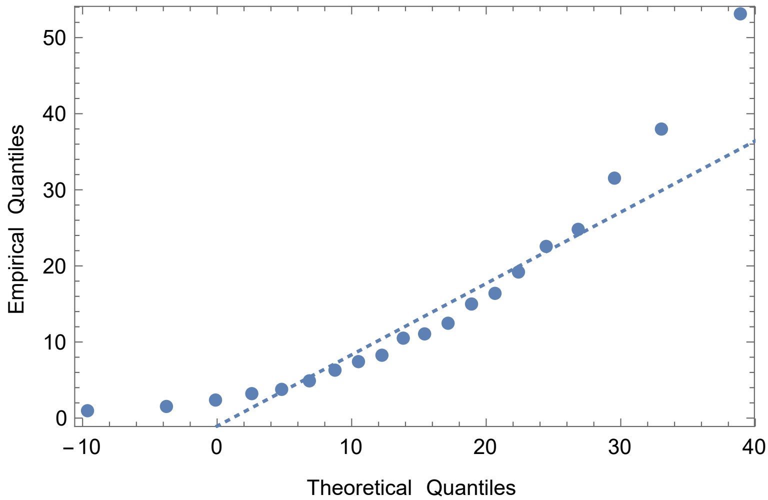

Figure 2 – Q-Q plot for the data set.

Now the empirical estimate of extropy, based on $\hat{L}(X)$ in (33) is -0.0203, while the parametric estimate of the extropy measure $L(X)$ for the same family of distribution with $\hat{c}, \hat{\lambda}_1$ and $\hat{\lambda}_2$ is -0.0174, shows a closeness in values based on the empirical estimator and parameter estimates. Also, based on the given data $\hat{L}(X) = -0.0203$ indicates that the amount of uncertainty contained in the times (in months) to first failure of 20 small electric carts is relatively small.

To check the validity of the Davies distribution quantile model, we use $Q-Q$ plot which are drawn in Figure 2. Even if there exists a slight discrepancy in the tail observations, $Q-Q$ plot shows a reasonably a good fit to the model. We also carried out the chi-square goodness of fit test to check the adequacy of the model. The chi-squared statistic value is 2.9388 with P- value 0.2300, indicates that the proposed model is a reasonably good fit to the given data set .

6. Conclusion

In this paper we have proposed quantile-based extropy and studied some monotone properties, characterizations and ordering relations. We have introduced the quantile-based extropy of order statistics. We have also obtained the cumulative extropy and its residual form based on quantile function and obtained some characteristic results. We have proposed an empirical estimator for quantile-based extropy and illustrated its performance using simulated and real data sets. As an alternative and to make a comparison with the empirical estimator $\hat{L}(X)$, one can also consider other nonparametric methods of estimation such as the quantile-based kernel estimator of $L_Q(X)$, similar to the one proposed by Alizadeh Noughabi and Jarrahiferiz (2019) in the distribution function approach, however, requires a separate study.

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SUMMARY

Extropy is a recent addition to the family of information measures as a complementary dual of Shannon entropy, to measure the uncertainty contained in a probability distribution of a random variable. A probability distribution can be specified either in terms of the distribution function or by the quantile function. In many applied works, there do not have any tractable distribution function but the quantile function exists, where a study on the quantile-based extropy are of importance. The present paper thus focuses on deriving some properties of extropy and its related measures using quantile function. Some ordering relations of quantile-based residual extropy are presented. We also introduce the quantile-based extropy of order statistics and cumulative extropy and studied its properties. Some applications of empirical estimation of quantile-based extropy using simulation and real data analysis are investigated.

Keywords: Extropy; Quantile function; Hazard quantile function; Order statistics; Mean residual quantile function.