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BIVARIATE DISCRETE MODIFIED WEIBULL (BDMW) DISTRIBUTION

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1. Introduction

Study on bivariate distributions attracted the attention of researchers over a long period of time. These distributions are important in modeling data in many areas. Construction of discrete bivariate data is quite common in practice. Discrete bivariate data arise in many real life situations and are often highly correlated. Several bivariate discrete distributions are available in the literature. Different discrete bivariate distributions are given in Kocherlakota and Kocherlakota (1992). Even if the samples are taken from continuous distributions, the observed values are actually discrete because they are measured to only a finite number of decimal places and cannot really constitute all points in a continuum. The observations taken on a continuous scale may be recorded in a way making discrete model more appropriate. So deriving discrete analogues of continuous distribution has drawn attention of researchers (Chakraborty, 2015). Different discrete distributions have been derived by discretizing known continuous distributions.

The Weibull distribution is a versatile family of life distribution. It is one of the most popular and widely used distributions for failure time in reliability theory. Also it has been extensively used in the analysis of manufactured data due to its physical interpretation and flexibility for empirical fit.

Modified Weibull distribution was proposed by Lai *et al.* (2003) is capable of designing a bathtub shaped hazard rate function. Noughabi *et al.* (2011) introduced a discrete analogue for modified Weibull distribution.

Analyzing discrete bivariate data is quite common in practice. Discrete bivariate data arise quite naturally in many real life situations. Kundu and Nekoukhou (2018) introduced a new bivariate discrete Weibull distribution which can be taken as a discrete analogue of Marshall and Olkin bivariate Weibull distribution (see Kundu and Gupta,

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2013). The main aim of the present paper is to discretize the new bivariate modified Weibull distribution proposed by El-Bassiouny *et al.* (2018).
The rest of this study is organized as follows. Section 2 provides the univariate modified Weibull distribution, bivariate modified Weibull distribution, discrete modified Weibull distribution and generalized likelihood ratio test. Several properties of bivariate discrete modified Weibull distribution are discussed in Section 3. Section 4 discusses parameter estimation of the distribution. Section 5 involves data analysis using a real life data set. Section 6 provides the mixture of bivariate discrete modified Weibull distributions and its properties and finally we conclude the paper in Section 7.

2. Preliminaries

2.1. *Univariate Modified Weibull (MW) Distribution*

Here we are considering univariate modified Weibull distribution having parameters β, γ, and λ with distribution function

$$F_{\mathrm{MW}}(x;\beta,\gamma,\lambda) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}, \quad x>0, \beta>0, \quad \gamma>0, \quad \lambda>0. \tag{1}$$

2.2. *Bivariate Modified Weibull (BMW) Distribution*

2.2.1. *The Joint Survival Function of Bivariate Modified Weibull Distribution*

Suppose that U_i, $i=1,2,3$, are univariate modified Weibull (MW) distributions and $U_1 \sim \mathrm{MW}(\beta_1,\gamma,\lambda)$, $U_2 \sim \mathrm{MW}(\beta_2,\gamma,\lambda)$ and $U_3 \sim \mathrm{MW}(\beta_3,\gamma,\lambda)$ are independent random variables. Let $X_1 = \min(U_1, U_3)$ and $X_2 = \min(U_2, U_3)$. Then, we get the bivariate vector (X_1, X_2) is a bivariate modified Weibull (BMW) distribution, $(X_1, X_2) \sim \mathrm{BMW}(\beta_1, \beta_2, \beta_3, \gamma, \lambda)$.

The joint survival function of X_1 and X_2, $S_{\mathrm{BMW}}(x_1, x_2)$, is given by

$$\begin{aligned} S_{\mathrm{BMW}}(x_1,x_2) &= P(X_1 > x_1, X_2 > x_2) \\ &= P\big(\min(U_1,U_3) > x_1 \min(U_2,U_3) > x_2\big) \\ &= P\big(U_1 > x_1, U_2 > x_2, U_3 > \max(x_1,x_2)\big), \end{aligned} \tag{2}$$

where U_i, $i=1,2,3$, are independent random variables. Therefore, we get the survival function as

$$S_{\mathrm{BMW}}(x_1,x_2) = e^{-\beta_1 x_1^{\gamma} e^{\lambda x_1}} e^{-\beta_2 x_2^{\gamma} e^{\lambda x_2}} e^{-\beta_3 z^{\gamma} e^{\lambda z}}, \quad z = \max(x_1,x_2), \tag{3}$$

where $x_1, x_2 > 0$, $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$, $\gamma > 0$, $\lambda > 0$ (see El-Bassiouny *et al.*, 2018).

2.3. *Discrete Modified Weibull (DMW) Distribution*

Discrete modified Weibull distribution was proposed by Noughabi *et al.* (2011), discrete analogue of the modified Weibull distribution of Lai *et al.* (2003).

Survival function of discrete modified Weibull distribution, $S_{\text{DMW}}(x;q,\theta,c)$, is

$$S_{\text{DMW}}(x;q,\theta,c)=q^{x^{\theta}c^{x}},\quad x=0,1,2,\ldots. \tag{4}$$

The corresponding probability mass function, $f_{\text{DMW}}(x;q,\theta,c)$, is

$$f_{\text{DMW}}(x;q,\theta,c)=q^{x^{\theta}c^{x}}-q^{(x+1)^{\theta}c^{x+1}},\quad x=0,1,2,\ldots,\quad 0<q<1,\quad \theta>0,\quad c\geq 1. \tag{5}$$

2.4. *Generalized likelihood ratio test*

Let $X_1,X_2,\ldots,X_n$ be n independent observations from a distribution with probability mass function $f(x,\underline{\theta})$ where $\underline{\theta}$ is the parameter vector assuming $\underline{\theta}=(\theta_1,\theta_2,\ldots,\theta_p)$. If the likelihood function is denoted by $L(\underline{x};\underline{\theta})$, then

$$\log L(\underline{x};\underline{\theta})=\sum_{i=1}^{n}\log\big(f(x_i,\underline{\theta})\big). \tag{6}$$

Here we test the null hypothesis

$$H_0:\underline{X}\sim f_{\theta}(\underline{x}),\quad \theta\in\Theta_0, \tag{7}$$

against the alternative

$$H_1:\underline{X}\sim f_{\theta}(\underline{x}),\quad \theta\in\Theta_1. \tag{8}$$

Under the null hypothesis let $\underline{\theta}$ satisfy a set of k restrictions.

Then, the Generalized likelihood ratio test procedure is based on the statistic

$$\Lambda=\frac{\sup_{\underline{\theta}\in\Theta_0}L(\underline{x};\underline{\theta})}{\sup_{\underline{\theta}\in\Theta}L(\underline{x};\underline{\theta})}, \tag{9}$$

where Θ is the parametric space. The supremum in the numerator is evaluated under the set of restrictions imposed by H_0, while that in the denominator is unrestricted.

The test statistic is

$$-2\log\Lambda=\log L(\underline{x};\hat{\theta})-\log L(\underline{x};\hat{\theta}'), \tag{10}$$

where $\hat{\theta}$ is the maximum likelihood estimator of $\underline{\theta}$ with no restrictions, while $\hat{\theta}'$ is the maximum likelihood estimator under the null hypothesis H_0. The test statistic (10) is asymptotically distributed as χ^2 with k degrees of freedom.

2.5. *Mixture of bivariate distributions*

Finite mixtures of distributions have provided a mathematical based approach in statistical modeling of a wide variety of random phenomena. As an extremely flexible method of modeling, finite mixture models have received an increasing attention over the years from practical and theoretical point of view. Consequently, the extent and the potential of the applications of finite mixture models have widened considerably in the last decade. Application of mixture models spread over astronomy, biology, genetics, medicine, psychiatry, economics, engineering, marketing and other fields in the biological, physical and social sciences. For details see McLachlan and Peel (2000). In many of these applications, finite mixture models support a variety of techniques in major areas of statistics including cluster and latent class analysis, discriminant analysis, image analysis and survival analysis. There is a vast literature available on finite mixture models. For example, see Everitt and Hande (1981) and Titterington *et al.* (1985) and review article of Titterington (1990). Here, first we present some general idea about mixture distribution.

Let (X_1, X_2) be a bivariate discrete distribution with joint probability mass function

$$f(x_1, x_2) = \sum_{i=1}^{g} \alpha_i f_i(x_1, x_2), \tag{11}$$

where for each $i = 1, 2, \ldots, g$, $\alpha_i > 0$ with $\sum_{i=1}^{g} \alpha_i = 1$ and $f_i(x_1, x_2) \geq 0$ such that $\sum_{x_j} \sum_{x_k} f(x_j, x_k) = 1$. Then, we say that (X_1, X_2) has a mixture distribution and $f(x_1, x_2)$ is a finite mixture distribution. The parameters $\alpha_1, \alpha_2, \ldots, \alpha_g$ are known as mixing weights and $f_i(x_1, x_2)$, the components of the mixture for $i = 1, 2, \ldots, g$. We denote Θ as the collection of all distinct parameters occurring in the components and Ψ as the complete collection of all distinct parameters occurring in the mixture model.

Let $\Delta = \{F(x_1, x_2), x_1, x_2 \in R\}$ be the class of distribution functions from which mixtures are to be formed. We identify the class of finite mixtures of Δ with the appropriate class of distribution functions, defined by

$$\hat{H} = \{H(x) : H(x) = \sum_{i=1}^{g} \alpha_i F_i(x_1, x_2), \quad \alpha_i > 0, \quad F_i(.,.) \in \Delta, \quad i = 1, 2, \ldots, g\}, \tag{12}$$

so that $\hat{H}$ is the convex hull of Δ. We denote $F_i(x_1, x_2)$ by F_i and the mixture by $H = \sum_{i=1}^{g} F_i$.

3. Bivariate discrete modified Weibull distribution

Consider three independent discrete modified Weibull(DMW) distributions, $U_i \sim \text{DMW}(q_i, \theta, c)$, $i = 0, 1, 2$. Let $X_1 = \min(U_1, U_0)$, $X_2 = \min(U_2, U_0)$, then (X_1, X_2) has a bivariate discrete modified Weibull (BDMW) distribution, $(X_1, X_2) \sim \text{BDMW}(q_0, q_1, q_2, \theta, c)$, where $x_1, x_2 = 0, 1, 2, \ldots$, $0 < q_0 < 1$, $0 < q_1 < 1$, $0 < q_2 < 1$, $\theta > 0$, $c \geq 1$. As a special case, if $c = 1$, then bivariate discrete Weibull (BDW) distribution of Kundu and Nekoukhou (2018) is achieved.

3.1. Joint Survival Function

If (X_1, X_2) has a bivariate discrete modified Weibull (BDMW) distribution, $(X_1, X_2) \sim \text{BDMW}(q_0, q_1, q_2, \theta, c)$, the joint survival function, $S_{\text{BDMW}}(x_1, x_2)$, is given by

$$S_{\text{BDMW}}(x_1, x_2) = \begin{cases} S_{\text{DMW}}(x_1; q_1, \theta, c) S_{\text{DMW}}(x_2; q_0 q_2, \theta, c) & \text{if } x_1 < x_2 \\ S_{\text{DMW}}(x_1; q_0 q_1, \theta, c) S_{\text{DMW}}(x_2; q_2, \theta, c) & \text{if } x_2 < x_1 \\ S_{\text{DMW}}(x; q_0 q_1 q_2, \theta, c) & \text{if } x_1 = x_2 = x. \end{cases} \tag{13}$$

PROOF. See in the Appendix. □

3.2. Joint Probability Mass Function

The joint probability mass function $f_{\text{BDMW}}(x_1, x_2)$ of bivariate discrete modified Weibull (BDMW) distribution with parameters q_0, q_1, q_2, θ and c is

$$f_{\text{BDMW}}(x_1, x_2) = \begin{cases} f_1(x_1, x_2; q_0, q_1, q_2, \theta, c) & \text{if } x_1 < x_2 \\ f_2(x_1, x_2; q_0, q_1, q_2, \theta, c) & \text{if } x_2 < x_1 \\ f_0(x; q_0, q_1, q_2, \theta, c) & \text{if } x_1 = x_2 = x, \end{cases} \tag{14}$$

where

$$\begin{aligned} f_1(x_1, x_2; q_0, q_1, q_2, \theta, c) &= f_{\text{DMW}}(x_1; q_1, \theta, c) f_{\text{DMW}}(x_2; q_0 q_2, \theta, c) \\ f_2(x_1, x_2; q_0, q_1, q_2, \theta, c) &= f_{\text{DMW}}(x_1; q_0 q_1, \theta, c) f_{\text{DMW}}(x_2; q_2, \theta, c) \\ f_0(x; q_0, q_1, q_2, \theta, c) &= q_1^{x^\theta c^x} f_{\text{DMW}}(x; q_0 q_2, \theta, c) - (q_0 q_1)^{(x+1)^\theta c^{x+1}} f_{\text{DMW}}(x; q_2, \theta, c). \end{aligned} \tag{15}$$

PROOF. See in the Appendix. □

Now we present some applications of BDMW in reliability through the following results.

3.3. Application of BDMW in Risk Model

Suppose a system has two components, say component 1 and component 2. First component fails due to two different causes, say Cause A and Cause C. Second component fails due to two causes, say Cause B and Cause C. Let the failure time of the components are discrete, say X_1 for component 1 and X_2 for component 2. If U_0, U_1 and U_2 denote the lifetime of Components due to Cause C, Cause A and Cause B, respectively. Then, $X_1 = \min(U_0, U_1)$ and $X_2 = \min(U_0, U_2)$ denote the lifetime of Component 1 and Component 2, respectively. Therefore, if $U_1 \sim \text{DMW}(q_1, \theta, c)$, $U_2 \sim \text{DMW}(q_2, \theta, c)$ and $U_0 \sim \text{DMW}(q_0, \theta, c)$, then $(X_1, X_2) \sim \text{BDMW}(q_0, q_1, q_2, \theta, c)$.

3.4. Application of BDMW in Shock Model

Suppose a system has two components, say component A and component B. It is assumed that the system receive shocks from three different sources. System receives shocks randomly and independently of other shocks at discrete times from the sources, say Source 0, Source 1 and Source 2. Also Component A receives shocks from Source 0 and Source 1. Similarly, Component B receives shocks from Source 0 and Source 2.

Let U_0, U_1 and U_2 denote the discrete times at which shocks appear from Source 0, Source 1 and Source 2, respectively. Any component fails as soon as it receives the first shock. Here $X_1 = \min(U_0, U_1)$ and $X_2 = \min(U_0, U_2)$ denote the lifetime of Component A and Component B, respectively. Therefore, if $U_1 \sim \text{DMW}(q_1, \theta, c)$, $U_2 \sim \text{DMW}(q_2, \theta, c)$ and $U_0 \sim \text{DMW}(q_0, \theta, c)$, then $(X_1, X_2) \sim \text{BDMW}(q_0, q_1, q_2, \theta, c)$.

In Figures 1 and 2 the joint probability mass function (PMF) is plotted for two different values of the parameters.

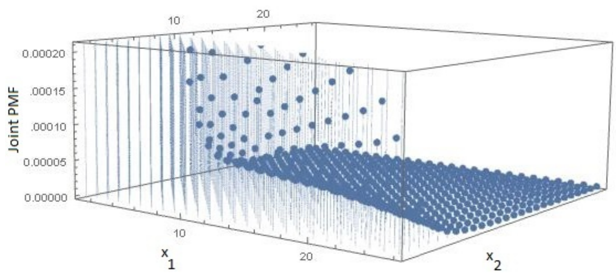

Figure 1 – The joint PMF of a BDMW distribution with parameters $q_0 = 0.95$, $q_1 = 0.5$, $q_2 = 0.8$, $\theta = 0.9$ and $c = 1.018$.

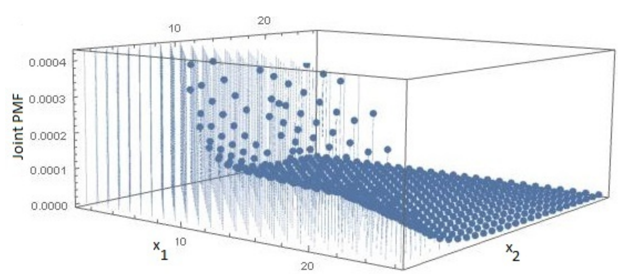

Figure 2 – The joint PMF of a BDMW distribution with parameters $q_0 = 0.9$, $q_1 = 0.6$, $q_2 = 0.85$, $\theta = 0.9$ and $c = 1.018$.

3.5. Joint Cumulative Distribution Function

If (X_1, X_2) is a bivariate discrete modified Weibull distribution with parameters q_0, q_1, q_2, θ and c, then the cumulative distribution function of (X_1, X_2), $F_{\text{BDMW}}(x_1, x_2)$, is given by

$$F_{\text{BDMW}}(x_1, x_2) = \begin{cases} f_{\text{DMW}}(x_2; q_2, \theta, c) - q_1^{x_1^{\theta} c^{x_1}} f_{\text{DMW}}(x_2; q_0 q_2, \theta, c) \text{ if } x_1 < x_2 \\ f_{\text{DMW}}(x_1; q_1, \theta, c) - q_2^{x_2^{\theta} c^{x_2}} f_{\text{DMW}}(x_1; q_0 q_1, \theta, c) \text{ if } x_2 \leq x_1. \end{cases} \tag{16}$$

PROOF. See in the Appendix. □

3.6. Marginal Survival Function

If (X_1, X_2) is bivariate discrete modified Weibull distribution, then the marginal distributions of X_i, $i = 1, 2$, are discrete modified Weibull with marginal survival function

$$S_{X_i}(x_i) = S_{\text{DMW}}(x_i; q_i q_0, \theta, c), \; i = 1, 2. \tag{17}$$

PROOF. The marginal survival function of X_i, $i = 1, 2$, $(S_{X_i}(x_i))$ can be obtained as

$$\begin{aligned} S_{X_i}(x_i) &= P(X_i \geq x_i) \\ &= P\big(\min(U_i, U_0) \geq x_i\big) \\ &= P(U_i \geq x_i, U_0 \geq x_i) \\ &= P(U_i \geq x_i) P(U_0 \geq x_i) \\ &= (q_i q_0)^{x_i^{\theta} c^{x_i}} \\ &= S_{\text{DMW}}(x_i; q_i q_0, \theta, c), \quad i = 1, 2. \end{aligned}$$

If $(X_1, X_2) \sim \text{BDMW}(q_0, q_1, q_2, \theta, c)$, then the marginals are DMW distributions. Also $X_1 \sim \text{DMW}(q_0 q_1, \theta, c)$ and $X_2 \sim \text{DMW}(q_0 q_2, \theta, c)$. □

3.7. Conditional Probability Mass Function of X_1 given $X_2 = x_2$

The conditional probability mass function of a bivariate discrete modified Weibull distribution is given by

$$f_{X_1|X_2=x_2}(x_1|x_2) = \begin{cases} f_{\text{DMW}}(x_1; q_1, \theta, c) & \text{if } x_1 < x_2 \\ \frac{f_{\text{DMW}}(x_1; q_0 q_1, \theta, c) f_{\text{DMW}}(x_2; q_2, \theta, c)}{f_{\text{DMW}}(x_2; q_0 q_2, \theta, c)} & \text{if } x_2 < x_1 \\ q_1^{x^{\theta} c^{x}} - (q_0 q_1)^{(x+1)^{\theta} c^{x+1}} \frac{f_{\text{DMW}}(x; q_2, \theta, c)}{f_{\text{DMW}}(x; q_0 q_2, \theta, c)} & \text{if } x_1 = x_2 = x. \end{cases} \tag{18}$$

PROOF.

$$
\begin{aligned}
f_{X_1|X_2=x_2}(x_1|x_2) &= \frac{f_{\mathrm{BDMW}}(x_1,x_2)}{f_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c)} \\
&= \begin{cases} f_{\mathrm{DMW}}(x_1;q_1,\theta,c) & \text{if } x_1 < x_2 \\ \frac{f_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)f_{\mathrm{DMW}}(x_2;q_2,\theta,c)}{f_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c)} & \text{if } x_2 < x_1 \\ q_1^{x^{\theta}c^{x}} - (q_0q_1)^{(x+1)^{\theta}c^{x+1}} \frac{f_{\mathrm{DMW}}(x;q_2,\theta,c)}{f_{\mathrm{DMW}}(x;q_0q_2,\theta,c)} & \text{if } x_1 = x_2 = x. \end{cases}
\end{aligned}
$$

□

3.8. *The conditional survival function of X_1 given $X_2 \geq x_2$*

If (X_1, X_2) is a bivariate discrete modified Weibull distribution with parameters q_0, q_1, q_2, θ and c, then

$$
S_{X_1|X_2\geq x_2}(x_1) = \begin{cases} S_{\mathrm{DMW}}(x_1;q_1,\theta,c) & \text{if } x_1 < x_2 \\ \dfrac{S_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)}{S_{\mathrm{DMW}}(x_2;q_0,\theta,c)} & \text{if } x_2 < x_1 \\ S_{\mathrm{DMW}}(x;q_1,\theta,c) & \text{if } x_1 = x_1 = x. \end{cases} \tag{19}
$$

PROOF. Using the definition

$$
\begin{aligned}
S_{X_1|X_2\geq x_2}(x_1) &= P(X_1 \geq x_1 | X_2 \geq x_2) \\
&= \frac{P(X_1 \geq x_1, X_2 \geq x_2)}{P(X_2 \geq x_2)} \\
&= \begin{cases} \dfrac{q_1^{x_1^{\theta}c^{x_1}}(q_0q_2)^{x_2^{\theta}c^{x_2}}}{(q_0q_2)^{x_2^{\theta}c^{x_2}}} & \text{if } x_1 < x_2 \\ \dfrac{(q_0q_1)^{x_1^{\theta}c^{x_1}}q_2^{x_2^{\theta}c^{x_2}}}{(q_0q_2)^{x_2^{\theta}c^{x_2}}} & \text{if } x_2 < x_1 \\ \dfrac{(q_0q_1q_2)^{x^{\theta}c^{x}}}{(q_0q_2)^{x^{\theta}c^{x}}} & \text{if } x_1 = x_2 = x \end{cases} \\
&= \begin{cases} S_{\mathrm{DMW}}(x_1;q_1,\theta,c) & \text{if } x_1 < x_2 \\ \dfrac{S_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)}{S_{\mathrm{DMW}}(x_2;q_0,\theta,c)} & \text{if } x_2 < x_1 \\ S_{\mathrm{DMW}}(x;q_1,\theta,c) & \text{if } x_1 = x_2 = x. \end{cases}
\end{aligned}
$$

□

THEOREM 1. *If* $(Y_1, Y_2) \sim BMW(\beta_1, \beta_2, \beta_3, \theta, \lambda)$ *as proposed by El-Bassiouny et al. (2018), then* $(X_1, X_2) \sim BDMW(q_0, q_1, q_2, \theta, c)$*, where* $X_1 = [Y_1]$, $X_2 = [Y_2]$, $q_0 = e^{-\beta_1}$, $q_1 = e^{-\beta_2}$, $q_2 = e^{-\beta_3}$ *and* $c = e^{\lambda}$*. Therefore, the proposed modified bivariate discrete Weibull distribution can be considered as a discrete analogue of the continuous bivariate modified Weibull distribution.*

THEOREM 2. *If* $X_1 = \min(U_1, U_0)$ *and* $X_2 = \min(U_2, U_0)$ *where* $U_i \sim DMW(q_i, \theta, c)$*,* $i = 0, 1, 2$*, then* X_1 *and* X_2 *are Positive Quadrant Dependent (PQD).*

PROOF. If $x_1 \geq x_2$, then

$$\begin{aligned} S_{\text{BDMW}}(x_1, x_2) &= S_{\text{DMW}}(x_1; q_1, \theta, c) S_{\text{DMW}}(x_2; q_2, \theta, c) S_{\text{DMW}}(z; q_0, \theta, c) \\ &= P(X_1 \geq x_1, X_2 \geq x_2), \end{aligned}$$

where $z = \max(y_1, y_2)$.

$$\begin{aligned} P(X_1 \geq x_1) P(X_2 \geq x_2) &= S_{\text{DMW}}(x_1; q_0 q_1, \theta, c) S_{\text{DMW}}(x_2; q_0 q_2, \theta, c) \\ &= S_{\text{DMW}}(x_1; q_0, \theta, c) S_{\text{DMW}}(x_1; q_1, \theta, c) \times \\ &\quad \times S_{\text{DMW}}(x_2; q_0, \theta, c) S_{\text{DMW}}(x_2; q_2, \theta, c). \end{aligned}$$

Since $0 \leq q_i \leq 1$, $i = 0, 1, 2$, then

$$P(X_1 \geq x_1, X_2 \geq x_2) \geq P(X_1 \geq x_1) P(X_2 \geq x_2).$$

Similarly, we can prove that

$$P(X_1 \geq x_1, X_2 \geq x_2) \geq P(X_1 \geq x_1) P(X_2 \geq x_2) \text{ for } x_1 < x_2.$$

□

THEOREM 3. *Suppose* $(X_{i1}, X_{i2}) \sim BDMW(q_{i0}, q_{i1}, q_{i2}, \theta, c)$ *for* $i = 1, 2, \ldots, n$ *and they are independently distributed. If* $Y_1 = \min(X_{11}, X_{21}, \ldots X_{n1})$ *and* $Y_2 = \min(X_{12}, X_{22}, \ldots X_{n2})$*, then*

$$(Y_1, Y_2) \sim BDMW\left(\prod_{i=1}^{n} q_{i0}, \prod_{i=1}^{n} q_{i1}, \prod_{i=1}^{n} q_{i2}, \theta, c \right). \tag{20}$$

PROOF. Let $(X_{i1}, X_{i2}) \sim \text{BDMW}(q_{i0}, q_{i1}, q_{i2}, \theta, c)$ for $i = 1, 2, \ldots, n$, where $X_{i1} = \min(U_{i0}, U_{i1})$ and $X_{i2} = \min(U_{i0}, U_{i2})$. U_{i0}, U_{i1} and U_{i2}, $i = 1, 2, \ldots, n$ are independent discrete modified Weibull distribution ($U_{ij} \sim \text{DMW}(q_{ij}, \theta, c)$, $j = 0, 1, 2$).

The joint survival function of (Y_1, Y_2) is

$$\begin{aligned}
P(Y_1 \geq y_1, Y_2 \geq y_2) &= P\big(\min(X_{i1}, i=1,2,\ldots,n \geq y_1, X_{i2}, i=1,2,\ldots,n \geq y_2)\big)\\
&= P(X_{i1} \geq y_1, X_{i2} \geq y_2, i=1,2,\ldots,n)\\
&= P\big(\min(U_{i0}, U_{i1}) \geq y_1, \min(U_{i0}, U_{i2}) \geq y_2\big)\\
&= P\big(U_{i1} \geq y_1, U_{i2} \geq y_2, U_{i0} \geq \max(y_1, y_2), i=1,2,\ldots,n\big)\\
&= \prod_{i=1}^{n} P(U_{i1} \geq y_1) \prod_{i=1}^{n} P(U_{i2} \geq y_2) \prod_{i=1}^{n} P\big(U_{i0} \geq \max(y_1, y_2)\big)\\
&= \prod_{i=1}^{n} q_{i1}^{y_1^{\theta} c^{y_1}} \prod_{i=1}^{n} q_{i2}^{y_2^{\theta} c^{y_2}} \prod_{i=1}^{n} q_{i0}^{z^{\theta} c^{z}},\ z = \max(y_1, y_2).
\end{aligned}$$

Therefore,

$$(Y_1, Y_2) \sim \mathrm{BDMW}\left(\prod_{i=1}^{n} q_{i0}, \prod_{i=1}^{n} q_{i1}, \prod_{i=1}^{n} q_{i2}, \theta, c\right).$$

□

THEOREM 4. *If discrete random variables X_0, X_1 and X_2 are independent geometric distributions with probability mass function $P(X_i = x) = q_i^x p_i$, $x = 0, 1, \ldots$, $0 < p_i, q_i \leq 1$, $p_i + q_i = 1$, $i = 0, 1, 2$ and $U_i = (\frac{X_i}{c})^{\frac{1}{\theta+1}}$, $c = \log k$, then $(\min(U_0, U_1),\ \min(U_0, U_2))$ follows a bivariate discrete modified Weibull distribution with parameters q_0, q_1, q_2, θ and c.*

PROOF. The probability function of X_i be given by $P(X_i = x) = q_i^x p_i$, $x = 0, 1, \ldots$, $0 < p_i, q_i \leq 1$, $p_i + q_i = 1$, $i = 0, 1, 2$ and $U_i = (\frac{X_i}{c})^{\frac{1}{\theta+1}}$, $c = \log k$. Then, if the survival function of U_i is given by $S_{U_i}(u)$, we have

$$\begin{aligned}
S_{U_i}(u) &= P\left(\left(\frac{X_i}{\log k}\right)^{\frac{1}{\theta+1}} \geq u\right)\\
&= P(X_i \geq \log k\, u^{\theta+1})\\
&= P(X_i \geq u^{\theta} c^{u})\\
&= q_i^{u^{\theta} c^{u}} \quad \text{(Roy, 2002)}.
\end{aligned}$$

Since X_is are independent, U_is are also independent. Consider $Y_1 = \min(U_0, U_1)$ and

$Y_2 = \min(U_0, U_2)$. The joint survival function of Y_1 and Y_2 is

$$\begin{aligned}S_{Y_1,Y_2}(y_1, y_2) &= P(Y_1 \geq y_1, Y_2 \geq y_2)\\ &= P\big(\min(U_0, U_1) \geq y_1, \min(U_0, U_2) \geq y_2\big)\\ &= P\big(U_1 \geq y_1, U_2 \geq y_2, U_0 \geq \max(y_1, y_2)\big)\\ &= q_1^{y_1^{\theta} c^{y_1}} q_2^{y_2^{\theta} c^{y_2}} q_0^{z^{\theta} c^{z}}, \quad z = \max(y_1, y_2),\end{aligned}$$

that is the survival function of bivariate discrete modified Weibull distribution with parameters q_0, q_1, q_2, θ and c. □

3.9. *Joint Hazard Rate Function (HRF)*

The joint hazard rate function $h_{\text{BDMW}}(x_1, x_2)$ of bivariate discrete modified Weibull distribution is given by

$$h_{\text{BDMW}}(x_1,x_2) = \begin{cases} \left(1 - \frac{S_{\text{DMW}}(x_1+1;q_1,\theta,c)}{S_{\text{DMW}}(x_1;q_1,\theta,c)}\right)\left(1 - \frac{S_{\text{DMW}}(x_2+1;q_0q_2,\theta,c)}{S_{\text{DMW}}(x_2;q_0q_2,\theta,c)}\right) & \text{if } x_1 < x_2 \\ \left(1 - \frac{S_{\text{DMW}}(x_1+1;q_0q_1,\theta,c)}{S_{\text{DMW}}(x_1;q_0q_1,\theta,c)}\right)\left(1 - \frac{S_{\text{DMW}}(x_2+1;q_2,\theta,c)}{S_{\text{DMW}}(x_2;q_2,\theta,c)}\right) & \text{if } x_2 < x_1 \\ \left(1 - \frac{S_{\text{DMW}}(x+1;q_0q_2,\theta,c)}{S_{\text{DMW}}(x;q_0q_2,\theta,c)}\right) - \frac{S_{\text{DMW}}(x+1;q_0q_1,\theta,c)}{S_{\text{DMW}}(x;q_0q_1,\theta,c)}\left(1 - \frac{S_{\text{DMW}}(x+1;q_2,\theta,c)}{S_{\text{DMW}}(x;q_2,\theta,c)}\right) & \\ & \text{if } x_1 = x_2 = x. \end{cases} \tag{21}$$

PROOF. See in the Appendix. □

Also the joint hazard rate function (HRF) is plotted in the Figures 3 and 4.

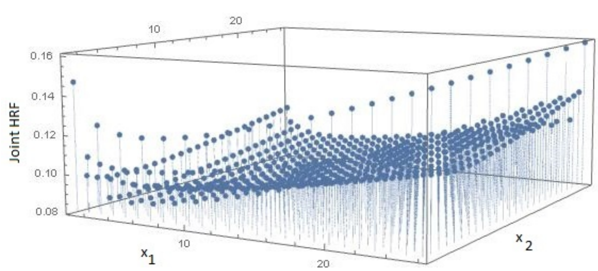

Figure 3 – The joint HRF of a BDMW distribution with parameters $q_0 = 0.95$, $q_1 = 0.5$, $q_2 = 0.8$, $\theta = 0.9$ and $c = 1.018$.

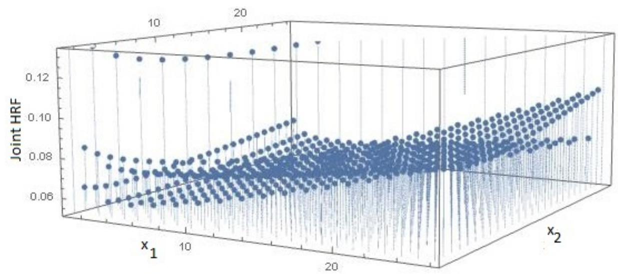

Figure 4 – The joint HRF of a BDMW disribution with parameters $q_0 = 0.9$, $q_1 = 0.6$, $q_2 = 0.85$, $\theta = 0.9$ and $c = 1.018$.

3.10. *Joint Probability Generating Function*

The joint probability generating function (PGF) of X_1 and X_2, $G_{\text{BDMW}}(x_1, x_2)$, for $|Z_1| < 1$ and $|Z_2| < 1$, can be written as infinite mixtures.

$$\begin{aligned}G_{\text{BDMW}}(x_1,x_2) = &\sum_{i=0}^{\infty}\sum_{j=0}^{i-1}\left((q_0q_1)^{i^\theta c^i} - (q_0q_1)^{(i+1)^\theta c^{i+1}}\right)\left(q_2^{i^\theta c^i} - q_2^{(i+1)^\theta c^{i+1}}\right) z_1^i z_2^j \\ &+ \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty}\left(q_1^{i^\theta c^i} - q_1^{(i+1)^\theta c^{i+1}}\right)\left((q_0q_2)^{i^\theta c^i} - (q_0q_2)^{(i+1)^\theta c^{i+1}}\right) z_1^i z_2^j \\ &+ \sum_{i=0}^{\infty} q_1^{i^\theta c^i}\left((q_0q_2)^{i^\theta c^i} - (q_0q_2)^{(i+1)^\theta c^{i+1}}\right) z_1^i z_2^i \\ &- \sum_{i=0}^{\infty}(q_0q_1)^{(i+1)^\theta c^{i+1}}\left(q_2^{i^\theta c^i} - q_2^{(i+1)^\theta c^{i+1}}\right) z_1^i z_2^i. \end{aligned} \tag{22}$$

PROOF. See in the Appendix. □

4. ESTIMATION

4.1. *Maximum Likelihood Estimation*

To estimate unknown parameters q_0, q_1, q_2, θ and c of bivariate discrete Weibull distribution using maximum likelihood estimation, consider a sample of size n $\{(x_{11}, x_{21}), (x_{12}, x_{22}), \ldots, (x_{1n}, x_{2n})\}$ from bivariate discrete modified Weibull distribution. $I_1 = \{i; x_{1i} < x_{2i}\}$, $I_2 = \{i; x_{1i} > x_{2i}\}$ and $I_0 = \{i; x_{1i} = x_{2i} = x_i\}$ and n_j are the number of elements in the set I_j, $j = 0, 1, 2$.

The likelihood function $L(\underline{x};q_0,q_1,q_2,\theta,c)$ is given by

$$L(\underline{x};q_0,q_1,q_2,\theta,c)=\prod_{j=1}^{n_1}f(x_{1j},x_{2j})\prod_{j=1}^{n_2}f(x_{1j},x_{2j})\prod_{j=1}^{n_0}f(x_{1j},x_{2j}). \tag{23}$$

Thus, the log likelihood function $l=\log L(\underline{x};q_0,q_1,q_2,\theta,c)$ is

$$\begin{aligned}
l=&\sum_{j=1}^{n_1}\log\left(q_1^{x_{1j}^{\theta}c^{x_{1j}}}-q_1^{(x_{1j}+1)^{\theta}c^{x_{1j}+1}}\right)\left((q_0q_2)^{x_{2j}^{\theta}c^{x_{2j}}}-(q_0q_2)^{(x_{2j}+1)^{\theta}c^{x_{2j}+1}}\right)\\
&+\sum_{j=1}^{n_2}\log\left((q_0q_1)^{x_{1j}^{\theta}c^{x_{1j}}}-(q_0q_1)^{(x_{1j}+1)^{\theta}c^{x_{1j}+1}}\right)\left(q_2^{x_{2j}^{\theta}c^{x_{2j}}}-q_2^{(x_{2j}+1)^{\theta}c^{x_{2j}+1}}\right)\\
&+\sum_{j=1}^{n_2}\log\left(q_1^{x_j^{\theta}c^{x_j}}\left((q_0q_2)^{x_j^{\theta}c^{x_j}}-(q_0q_2)^{(x_j+1)^{\theta}c^{x_j+1}}\right)\right.\\
&\left.-(q_0q_1)^{(x_j+1)^{\theta}c^{x_j+1}}\left(q_2^{x_j^{\theta}c^{x_j}}-q_2^{(x_j+1)^{\theta}c^{x_j+1}}\right)\right)\\
=&\sum_{j=1}^{n_1}\log\left(q_1^{x_{1j}^{\theta}c^{x_{1j}}}-q_1^{(x_{1j}+1)^{\theta}c^{x_{1j}+1}}\right)+\sum_{j=1}^{n_1}\log\left((q_0q_2)^{x_{2j}^{\theta}c^{x_{2j}}}-(q_0q_2)^{(x_{2j}+1)^{\theta}c^{x_{2j}+1}}\right)\\
&+\sum_{j=1}^{n_2}\log\left((q_0q_1)^{x_{1j}^{\theta}c^{x_{1j}}}-(q_0q_1)^{(x_{1j}+1)^{\theta}c^{x_{1j}+1}}\right)+\sum_{j=1}^{n_2}\log\left(q_2^{x_{2j}^{\theta}c^{x_{2j}}}-q_2^{(x_{2j}+1)^{\theta}c^{x_{2j}+1}}\right)\\
&+\sum_{j=1}^{n_2}\log\left(q_1^{x_j^{\theta}c^{x_j}}\left((q_0q_2)^{x_j^{\theta}c^{x_j}}-(q_0q_2)^{(x_j+1)^{\theta}c^{x_j+1}}\right)\right.\\
&\left.-(q_0q_1)^{(x_j+1)^{\theta}c^{x_j+1}}\left(q_2^{x_j^{\theta}c^{x_j}}-q_2^{(x_j+1)^{\theta}c^{x_j+1}}\right)\right).
\end{aligned} \tag{24}$$

Then, we have

$$
\begin{aligned}
\frac{\partial l}{\partial \theta} = & \sum_{j=1}^{n_1} \frac{g_1(x_{1j}, q_1, \theta, c) - g_1(x_{1j}+1, q_1, \theta, c)}{f_{\mathrm{DMW}}(x_{1j}; q_1, \theta, c)} \\
& + \sum_{j=1}^{n_1} \frac{g_1(x_{2j}, q_0 q_2, \theta, c) - g_1(x_{2j}+1, q_0 q_2, \theta, c)}{f_{\mathrm{DMW}}(x_{2j}; q_0 q_2, \theta, c)} \\
& + \sum_{j=1}^{n_2} \frac{g_1(x_{1j}, q_0 q_1, \theta, c) - g_1(x_{1j}+1, q_0 q_1, \theta, c)}{f_{\mathrm{DMW}}(x_{1j}; q_0 q_1, \theta, c)} \\
& + \sum_{j=1}^{n_2} \frac{g_1(x_{2j}, q_2, \theta, c) - g_1(x_{2j}+1, q_2, \theta, c)}{f_{\mathrm{DMW}}(x_{2j}; q_2, \theta, c)} \\
& + \sum_{j=1}^{n_0} \frac{g_1(x_j, q_0 q_1 q_2, \theta, c) + g_1(x_j+1, q_0 q_1 q_2, \theta, c)}{q_1^{x_j^\theta c^{x_j}} f_{\mathrm{DMW}}(x_j; q_0 q_2, \theta, c) - (q_0 q_1)^{(x_j+1)^\theta c^{x_j+1}} f_{\mathrm{DMW}}(x_j; q_2, \theta, c)} \\
& - \sum_{j=1}^{n_0} \frac{g_2(x_j, q_1, q_0 q_2, \theta, c) + g_2(x_j, q_2, q_0 q_1, \theta, c)}{q_1^{x_j^\theta c^{x_j}} f_{\mathrm{DMW}}(x_j; q_0 q_2, \theta, c) - (q_0 q_1)^{(x_j+1)^\theta c^{x_j+1}} f_{\mathrm{DMW}}(x_j; q_2, \theta, c)},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial c} = & \sum_{j=1}^{n_1} \frac{g_3(x_{1j}, q_1, \theta, c) - g_3(x_{1j}+1, q_1, \theta, c)}{f_{\mathrm{DMW}}(x_{1j}; q_1, \theta, c)} \\
& + \sum_{j=1}^{n_1} \frac{g_3(x_{2j}, q_0 q_2, \theta, c) - g_3(x_{2j}+1, q_0 q_2, \theta, c)}{f_{\mathrm{DMW}}(x_{2j}; q_0 q_2, \theta, c)} \\
& + \sum_{j=1}^{n_2} \frac{g_3(x_{1j}, q_0 q_1, \theta, c) - g_3(x_{1j}+1, q_0 q_1, \theta, c)}{f_{\mathrm{DMW}}(x_{1j}; q_0 q_1, \theta, c)} \\
& + \sum_{j=1}^{n_2} \frac{g_3(x_{2j}, q_2, \theta, c) - g_3(x_{2j}+1, q_2, \theta, c)}{f_{\mathrm{DMW}}(x_{2j}; q_2, \theta, c)} \\
& + \sum_{j=1}^{n_0} \frac{g_3(x_j, q_0 q_1 q_2, \theta, c) + g_3(x_j+1, q_0 q_1 q_2, \theta, c)}{q_1^{x_j^\theta c^{x_j}} f_{\mathrm{DMW}}(x_j; q_0 q_2, \theta, c) - (q_0 q_1)^{(x_j+1)^\theta c^{x_j+1}} f_{\mathrm{DMW}}(x_j; q_2, \theta, c)} \\
& - \sum_{j=1}^{n_0} \frac{g_4(x_j, q_1, q_0 q_2, \theta, c) + g_4(x_j, q_2, q_0 q_1, \theta, c)}{q_1^{x_j^\theta c^{x_j}} f_{\mathrm{DMW}}(x_j; q_0 q_2, \theta, c) - (q_0 q_1)^{(x_j+1)^\theta c^{x_j+1}} f_{\mathrm{DMW}}(x_j; q_2, \theta, c)},
\end{aligned} \tag{26}
$$

$$\frac{\partial l}{\partial q_0} = \frac{1}{q_0}\Bigg[\sum_{j=1}^{n_1}\frac{g_5(x_{2j},q_0q_2,\theta,c)}{f_{\mathrm{DMW}}(x_{2j};q_0q_2,\theta,c)} + \sum_{j=1}^{n_2}\frac{g_5(x_{1j},q_0q_1,\theta,c)}{f_{\mathrm{DMW}}(x_{1j};q_0q_1,\theta,c)}$$
$$+\sum_{j=1}^{n_0}\frac{q_1^{x_j^{\theta}c^{x_j}} g_5(x_j,q_0q_2,\theta,c) - g_6(x_j+1,q_0q_1,\theta,c) f_{\mathrm{DMW}}(x_j,q_2,\theta,c)}{q_1^{x_j^{\theta}c^{x_j}} f_{\mathrm{DMW}}(x_j;q_0q_2,\theta,c) - (q_0q_1)^{(x_j+1)^{\theta}c^{x_j+1}} f_{\mathrm{DMW}}(x_j;q_2,\theta,c)}\Bigg], \tag{27}$$

$$\frac{\partial l}{\partial q_1} = \frac{1}{q_1}\Bigg[\sum_{j=1}^{n_1}\frac{g_5(x_{1j},q_1,\theta,c)}{f_{\mathrm{DMW}}(x_{1j};q_1,\theta,c)} + \sum_{j=1}^{n_2}\frac{g_5(x_{1j},q_0q_1,\theta,c)}{f_{\mathrm{DMW}}(x_{1j};q_0q_1,\theta,c)}$$
$$+\sum_{j=1}^{n_0}\frac{f_{\mathrm{DMW}}(x_j,q_0q_2,\theta,c) g_6(x_j,q_1,\theta,c) - g_6(x_j+1,q_0q_1,\theta,c) f_{\mathrm{DMW}}(x_j,q_2,\theta,c)}{q_1^{x_j^{\theta}c^{x_j}} f_{\mathrm{DMW}}(x_j;q_0q_2,\theta,c) - (q_0q_1)^{(x_j+1)^{\theta}c^{x_j+1}} f_{\mathrm{DMW}}(x_j;q_2,\theta,c)}\Bigg] \tag{28}$$

and

$$\frac{\partial l}{\partial q_2} = \frac{1}{q_2}\Bigg[\sum_{j=1}^{n_1}\frac{g_5(x_{2j},q_0q_2,\theta,c)}{f_{\mathrm{DMW}}(x_{2j};q_0q_2,\theta,c)} + \sum_{j=1}^{n_2}\frac{g_5(x_{2j},q_2,\theta,c)}{f_{\mathrm{DMW}}(x_{2j};q_2,\theta,c)}$$
$$+\sum_{j=1}^{n_0}\frac{q_1^{x_j^{\theta}c^{x_j}} g_5(x_j,q_0q_2,\theta,c) - (q_0q_1)^{x_j^{\theta}c^{x_j}} g_5(x_j,q_2,\theta,c)}{q_1^{x_j^{\theta}c^{x_j}} f_{\mathrm{DMW}}(x_j;q_0q_2,\theta,c) - (q_0q_1)^{(x_j+1)^{\theta}c^{x_j+1}} f_{\mathrm{DMW}}(x_j;q_2,\theta,c)}\Bigg], \tag{29}$$

where:

- $g_1(x,q,\theta,c) = x^{\theta}c^{x}q^{x^{\theta}c^{x}}\log(x+q)$,
- $g_2(x,q_1,q_2,\theta,c) = q_1^{x^{\theta}c^{x}} q_2^{(x+1)^{\theta}c^{(x+1)}}(x^{\theta}c^{x}\log(x+q_1)+(x+1)^{\theta}c^{x+1}\log(x+q_2+1))$,
- $g_3(x,q,\theta,c) = x^{\theta+1}c^{x-1}q^{x^{\theta}c^{x}}\log(q)$,
- $g_4(x,q_1,q_2,\theta,c) = q_1^{x^{\theta}c^{x}} q_2^{(x+1)^{\theta}c^{(x+1)}}(x^{\theta+1}c^{x-1}\log(q_1)+(x+1)^{\theta+1}c^{x}\log(q_2))$,
- $g_5(x,q,\theta,c) = x^{\theta}c^{x}q^{x^{\theta}c^{x}} - (x+1)^{\theta}c^{x+1}q^{(x+1)^{\theta}c^{x+1}}$ and $g_6(x,q,\theta,c) = x^{\theta}c^{x}q^{x^{\theta}c^{x}-1}$.

The maximum likelihood estimators of the parameters q_0, q_1, q_2, θ and c can be obtained by solving these five non linear equations. The solution of these equations are not easy to solve. We need a numerical technique to get the maximum likelihood estimators.

5. Data Analysis

In this Section we discuss maximum likelihood estimators of the parameters of BDMW distribution using a real life data and compare the bivariate discrete modified Weibull distribution with the BDW distribution proposed by Kundu and Nekoukhou (2018) using maximized Log-Likelihood (-L), Akaike information criterion (AIC), corrected Akaike information criterion (AICc), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQC). Here the maximum likelihood estimates of parameters are obtained by limited memory quasi Newton algorithm (Byrd *et al.*, 1995).

The data set given in Table 1 consists of a football match scored in Italian football match (Series A) during 1996 to 2011, between ACF Fiorentina (X_1) and Juventus (X_2).

The values for -L, AIC, AICc, BIC and HQC of bivariate discrete modified Weibull distribution less than the values corresponding to the discrete Weibull distribution. Values are provided in Table 2. Thus, we can say that the bivariate discrete modified Weibull distribution provide better fit than the bivariate discrete Weibull distribution.

TABLE 1
The Score data between ACF Fiorentina(X_1) and Juventus (X_2)

Obs.	Match Date	X_1	X_2	Obs.	Match Date	X_1	X_2.
1	25/10/2011	1	2	14	16/02/2002	1	2
2	17/04/2011	0	0	15	19/12/2001	1	1
3	27/11/2010	1	1	16	12/05/2001	1	3
4	06/03/2010	1	2	17	06/01/2001	3	3
5	17/10/2009	1	1	18	21/04/2000	0	1
6	24/01/2009	0	1	19	18/12/1999	1	1
7	31/08/2008	1	1	20	24/04/1999	1	2
8	02/03/2008	3	2	21	12/12/1998	1	0
9	07/10/2007	1	1	22	21/02/1998	3	0
10	09/04/2006	1	1	23	04/10/1997	1	2
11	04/12/2005	1	2	24	22/02/1997	1	1
12	09/04/2005	3	3	25	28/09/1996	0	1
13	10/11/2004	0	1	26	23/03/1996	0	1

TABLE 2
The MLEs,-L,AIC,AICc,BIC and HQIC values for the data.

	BDW distribution	BDMW distribution
q_0	1	0.96
q_1	0.78	0.92
q_2	0.95	0.95
θ	4.98	0.34
c	—-	2.6
-L	293.08	67.17
AIC	594.16	144.33
AICc	596.07	147.33
BIC	599.19	141.41
HQC	595.61	135.48

5.1. *Testing of additional parameter in BDMW*

Let $\{(x_{11},x_{21}),(x_{12},x_{22}),...,(x_{1n},x_{2n})\}$ be independent observations from a bivariate discrete modified Weibull distribution with parameters q_0,q_1,q_2,θ and c.

To test the significance of the additional parameter c ofBDMW, we use GLRT discussed in Section 2.4. Here we test the null hypothesis $H_0 : c = 1$ against the alternative $H_1 : c > 1$.

The generalized likelihood ratio test procedure is based on the statistic

$$\Lambda = \frac{\sup_{c=1} L(\underline{x}; q_0, q_1, q_2, \theta, c)}{\sup_{c\in\Theta} L(\underline{x}; q_0, q_1, q_2, \theta, c)}, \tag{30}$$

where Θ is the parametric space and $L(\underline{x}; q_0, q_1, q_2, \theta, c)$ is the likelihood function of the distribution.

Then, the test statistic is

$$-2\log\Lambda = \log L(\underline{x}; q_0, q_1, q_2, \theta, \hat{c}) - \log L(\underline{x}; q_0, q_1, q_2, \theta, 1), \tag{31}$$

where $\hat{c}$ is the maximum likelihood estimator of c without restrictions. The test statistic -2 $\log\Lambda$ is asymptotically distributed as χ^2 with 1 degrees of freedom.

Here, $-2\log\Lambda = 360.25$ and, for a significance level α equal to 0.001, the theoretical value is 10.83. Therefore, we reject the null hypothesis.

6. Mixture of bivariate discrete modified Weibull distributions

Suppose $f_i(x_1, x_2)$, $i = 1, 2, \ldots, g$ be bivariate discrete modified Weibull distributions.

Let

$$H(x_1, x_2) = \sum_{i=1}^{g} \alpha_i f_i(x_1, x_2), \tag{32}$$

where α_i are nonnegative quantities with $0 \leq \alpha_i \leq 1$, $i = 1,2,\dots,g$ and $\sum_{i=1}^{g} \alpha_i = 1$, which are the weights or mixing proportions. $f_i(x_1,x_2)$ are component densities. Thus, $H(x_1,x_2)$ is a g-component finite mixture density, where

$$H(x_1,x_2) = \begin{cases} H_1(x_1,x_2) & \text{if } x_1 < x_2 \\ H_2(x_1,x_2) & \text{if } x_2 < x_1 \\ H_0(x) & \text{if } x_1 = x_2 = x, \end{cases} \tag{33}$$

$$H_1(x_1,x_2) = \sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_1^{\theta_i} c_i^{x_1}} - q_{1i}^{(x_1+1)^{\theta_i} c_i^{x_1+1}} \right) \left((q_{0i}q_{2i})^{x_2^{\theta_i} c_i^{x_2}} - (q_{0i}q_{2i})^{(x_2+1)^{\theta_i} c_i^{x_2+1}} \right), \tag{34}$$

$$H_2(x_1,x_2) = \sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{1i})^{x_1^{\theta_i} c_i^{x_1}} - (q_{0i}q_{1i})^{(x_1+1)^{\theta_i} c_i^{x_1+1}} \right) \left(q_{2i}^{x_2^{\theta_i} c_i^{x_2}} - q_{2i}^{(x_2+1)^{\theta_i} c_i^{x_2+1}} \right) \tag{35}$$

and

$$H_0(x) = \sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x^{\theta_i} c_i^{x}} ((q_{0i}q_{2i})^{x^{\theta_i} c_i^{x}} - (q_{0i}q_{2i})^{(x+1)^{\theta_i} c_i^{x+1}}) \right)$$
$$- \sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{1i})^{(x+1)^{\theta_i} c_i^{x+1}} (q_{2i}^{x^{\theta_i} c_i^{x}} - q_{2i}^{(x+1)^{\theta_i} c_i^{x+1}}) \right). \tag{36}$$

6.1. The joint survival function

The joint survival function of the mixture distribution is given by

$$\begin{aligned} S(x_1,x_2) &= \sum_{i=1}^{g} \alpha_i P(X_1 \geq x_1, X_2 \geq x_2) \\ &= \sum_{i=1}^{g} \alpha_i q_{1i}^{x_1^{\theta_i} c_i^{x_1}} q_{2i}^{x_2^{\theta_i} c_i^{x_2}} q_{0i}^{z^{\theta_i} c_i^{z}}, \end{aligned} \tag{37}$$

where $z = \max(x_1,x_2)$.

To find the marginal distribution we can consider the marginal survival function

$$\begin{aligned} S(x_1) &= \sum_{i=1}^{g} \alpha_i P(X_1 \geq x_1) \\ &= \sum_{i=1}^{g} \alpha_i P\big(\min(U_{0i},U_{1i}) > x_1\big) \\ &= \sum_{i=1}^{g} \alpha_i (q_{0i}q_{1i})^{x_1^{\theta_i} c_i^{x_1}}. \end{aligned} \tag{38}$$

Thus, we get the marginal distribution of the mixture distribution as

$$f(x_1) = \sum_{i=1}^{g} \alpha_i \left((q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} - (q_{0i} q_{1i})^{(x_1+1)^{\theta_i} c_i^{x_1+1}} \right), \tag{39}$$

which is a mixed discrete modified Weibull distribution.

Then, marginal distribution functions of X_1 and X_2 are given by

$$F(x_1) = 1 - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} \tag{40}$$

and

$$F(x_2) = 1 - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{2i})^{x_2^{\theta_i} c_i^{x_2}}. \tag{41}$$

6.2. *The joint distribution function*

The joint distribution function of mixture of bivariate discrete modified Weibull distribution can be obtained by

$$\begin{aligned} F(x_1, x_2) &= F(x_1) + F(x_2) + S(x_1, x_2) - 1 \\ &= \begin{cases} F_1(x_1, x_2) & \text{if } x_1 < x_2 \\ F_2(x_1, x_2) & \text{if } x_2 \le x_1. \end{cases} \end{aligned} \tag{42}$$

If $x_1 < x_2$, then

$$\begin{aligned} F_1(x_1, x_2) &= 1 - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{2i})^{x_2^{\theta_i} c_i^{x_2}} + \sum_{i=1}^{g} \alpha_i q_{1i}^{x_1^{\theta_i} c_i^{x_1}} (q_{0i} q_{2i})^{x_2^{\theta_i} c_i^{x_2}} \\ &= 1 - \sum_{i=1}^{g} \alpha_i \left((q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} + (q_{0i} q_{2i})^{x_2^{\theta_i} c_i^{x_2}} (1 - q_{1i}^{x_1^{\theta_i} c_i^{x_1}}) \right). \end{aligned} \tag{43}$$

If $x_2 \le x_1$, then

$$\begin{aligned} F_2(x_1, x_2) &= 1 - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} - \sum_{i=1}^{g} \alpha_i (q_{0i} q_{2i})^{x_2^{\theta_i} c_i^{x_2}} + \sum_{i=1}^{g} \alpha_i (q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} q_{2i}^{x_2^{\theta_i} c_i^{x_2}} \\ &= 1 - \sum_{i=1}^{g} \alpha_i \left((q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} + (q_{0i} q_{1i})^{x_1^{\theta_i} c_i^{x_1}} (1 - q_{2i}^{x_2^{\theta_i} c_i^{x_2}}) \right). \end{aligned} \tag{44}$$

6.3. Identifiability Condition

The identifiability condition of the bivariate discrete modified Weibull distribution with joint probability mass function

$$H(x_1,x_2)=\begin{cases} H_1(x_1,x_2) & \text{if } x_1<x_2 \\ H_2(x_1,x_2) & \text{if } x_2<x_1 \\ H_0(x) & \text{if } x_1=x_2=x, \end{cases} \tag{45}$$

where

$$H_1(x_1,x_2)=\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_1^{\theta_i}c_i^{x_1}}-q_{1i}^{(x_1+1)^{\theta_i}c_i^{x_1+1}}\right)\left((q_{0i}q_{2i})^{x_2^{\theta_i}c_i^{x_2}}-(q_{0i}q_{2i})^{(x_2+1)^{\theta_i}c_i^{x_2+1}}\right), \tag{46}$$

$$H_2(x_1,x_2)=\sum_{i=1}^{g}\alpha_i\left((q_{0i}q_{1i})^{x_1^{\theta_i}c_i^{x_1}}-(q_{0i}q_{1i})^{(x_1+1)^{\theta_i}c_i^{x_1+1}}\right)\left(q_{2i}^{x_2^{\theta_i}c_i^{x_2}}-q_{2i}^{(x_2+1)^{\theta_i}c_i^{x_2+1}}\right), \tag{47}$$

$$\begin{aligned} H_0(x)=&\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x^{\theta_i}c_i^{x}}((q_{0i}q_{2i})^{x^{\theta_i}c_i^{x}}-(q_{0i}q_{2i})^{(x+1)^{\theta_i}c_i^{x+1}})\right)\\ &-\sum_{i=1}^{g}\alpha_i\left((q_{0i}q_{1i})^{(x+1)^{\theta_i}c_i^{x+1}}(q_{2i}^{x^{\theta_i}c_i^{x}}-q_{2i}^{(x+1)^{\theta_i}c_i^{x+1}})\right) \end{aligned} \tag{48}$$

and $q_{0i}\neq q_{0j}$, $q_{1i}\neq q_{1j}$, $q_{2i}\neq q_{2j}$, $\theta_i\neq\theta_j$, $c_i\neq c_j$, for $i,j\in\{1,2,\dots,g\}$ such that $i\neq j$.

PROOF. See in the Appendix. □

6.4. Maximum Likelihood Estimation

To estimate unknown parameters q_0, q_1, q_2, θ, c of bivariate discrete Weibull distribution using maximum likelihood estimation, consider a sample of size n, $\{(x_{11},x_{21}),(x_{12},x_{22}),...,$
$(x_{1n},x_{2n})\}$ from bivariate discrete modified Weibull distribution. $I_1=\{i;x_{1i}<x_{2i}\}$, $I_2=\{i;x_{1i}>x_{2i}\}$ and $I_0=\{i;x_{1i}=x_{2i}=x_i\}$ and n_j is the number of elements in the set $I_j, j=0,1,2.$ù

The likelihood function $L(\underline{x};q_0,q_1,q_2,\theta,c)$ is given by

$$L(\underline{x};q_0,q_1,q_2,\theta,c)=\prod_{j=1}^{n_1}f(x_{1j},x_{2j})\prod_{j=1}^{n_2}f(x_{1j},x_{2j})\prod_{j=1}^{n_0}f(x_{1j},x_{2j}). \tag{49}$$

Thus, the log likelihood function l is

$$\begin{aligned} l = &\sum_{j=1}^{n_1} \log\left(\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)\right) \\ &+ \sum_{j=1}^{n_2} \log\left(\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)\right) \\ &+ \sum_{j=1}^{n_0} \log\left(\sum_{i=1}^{g} \alpha_i \Big(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) \right. \\ &\left. - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i)\Big)\right). \end{aligned} \tag{50}$$

Moreover, we have

$$\begin{aligned} \frac{\partial l}{\partial \theta_i} = &\sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i)\big(g_1(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i) - g_1(x_{2j}+1, q_{0i}q_{2i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\ &+ \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)\big(g_1(x_{1j}, q_{1i}, \theta_i, c_i) - g_1(x_{1j}+1, q_{1i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\ &+ \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i)\big(g_1(x_{2j}, q_{2i}, \theta_i, c_i) - g_1(x_{2j}+1, q_{2i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)} \\ &+ \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)\big(g_1(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) - g_1(x_{1j}+1, q_{0i}q_{1i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)} \\ &+ \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left(g_1(x_j, q_{0i}q_{1i}q_{2i}, \theta_i, c_i) + g_1(x_j+1, q_{0i}q_{1i}q_{2i}, \theta_i, c_i)\right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i)\right)} \\ &- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} g_1(x_j+1, q_{0i}q_{2i}, \theta_i, c_i) + (q_{0i}q_{2i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_1(x_j+1, q_{1i}, \theta_i, c_i)\right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i)\right)} \\ &- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_1(x_j, q_{2i}, \theta_i, c_i)\right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i)\right)} \\ &- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{2i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_1(x_j+1, q_{1i}, \theta_i, c_i)\right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i)\right)}, \end{aligned} \tag{51}$$

$$
\begin{aligned}
\frac{\partial l}{\partial c_i} =& \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i)\big(g_3(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i) - g_3(x_{2j}+1, q_{0i}q_{2i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\
&+ \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)\big(g_3(x_{1j}, q_{1i}, \theta_i, c_i) - g_3(x_{1j}+1, q_{1i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i)\big(g_3(x_{2j}, q_{2i}, \theta_i, c_i) - g_3(x_{2j}+1, q_{2i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)\big(g_3(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) - g_3(x_{1j}+1, q_{0i}q_{1i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)} \\
&+ \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \big(g_3(x_j, q_{0i}q_{1i}q_{2i}, \theta_i, c_i) + g_3(x_j+1, q_{0i}q_{1i}q_{2i}, \theta_i, c_i)\big)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i) \right)} \\
&- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} g_3(x_j+1, q_{0i}q_{2i}, \theta_i, c_i) + (q_{0i}q_{2i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_3(x_j+1, q_{1i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i) \right)} \\
&- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_3(x_j, q_{2i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i) \right)} \\
&- \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \alpha_i \left((q_{0i}q_{2i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} g_3(x_j+1, q_{1i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i \left(q_{1i}^{x_j^{\theta_i} c_i^{x_j}} f_{\mathrm{DMW}}(x_j, q_{0i}q_{2i}, \theta_i, c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i} c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j, q_{2i}, \theta_i, c_i) \right)},
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\frac{\partial l}{\partial q_{0i}} =& \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i)\left(q_{2i}^{x_{2j}^{\theta_i} c_i^{x_{2j}}} g_6(x_{2j}, q_{0i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\
&- \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i)\left(q_{2i}^{(x_{2j}+1)^{\theta_i} c_i^{x_{2j}+1}} g_6(x_{2j}+1, q_{0i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{1j}, q_{1i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{2j}, q_{0i}q_{2i}, \theta_i, c_i)} \\
&+ \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i)\left(q_{1i}^{x_{1j}^{\theta_i} c_i^{x_{1j}}} g_6(x_{1j}, q_{0i}, \theta_i, c_i) \right)}{\sum_{i=1}^{g} \alpha_i f_{\mathrm{DMW}}(x_{2j}, q_{2i}, \theta_i, c_i) f_{\mathrm{DMW}}(x_{1j}, q_{0i}q_{1i}, \theta_i, c_i)}
\end{aligned}
$$

$$-\sum_{j=1}^{n_2}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)\left(q_{1i}^{(x_{1j}+1)^{\theta_i}c_i^{x_{1j}+1}}g_6(x_{1j}+1,q_{0i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)}$$
$$+\sum_{j=1}^{n_0}\frac{\sum_{i=1}^{g}\alpha_i\left((q_{1i}q_{2i})^{x_j^{\theta_i}c_i^{x_j}}g_6(x_j,q_{0i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}}f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)-(q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}}f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)}$$
$$-\sum_{j=1}^{n_0}\frac{\sum_{i=1}^{g}\alpha_i\left((q_{1i}^{x_j^{\theta_i}c_i^{x_j}}q_{2i}^{(x_j+1)^{\theta_i}c_i^{x_j+1}})g_6(x_j+1,q_{0i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}}f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)-(q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}}f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)}$$
$$-\sum_{j=1}^{n_0}\frac{\sum_{i=1}^{g}\alpha_i q_{1i}^{(x_j+1)^{\theta_i}c_i^{x_j+1}}g_6(x_j,q_{0i},\theta_i,c_i)f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)}{\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}}f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)-(q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}}f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)},\tag{53}$$

$$\frac{\partial l}{\partial q_{1i}}=\sum_{j=1}^{n_1}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)\big(g_6(x_{1j},q_{1i},\theta_i,c_i)-g_6(x_{1j}+1,q_{1i},\theta_i,c_i)\big)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i)f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)}$$
$$+\sum_{j=1}^{n_2}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)\left(q_{0i}^{x_{1j}^{\theta_i}c_i^{x_{1j}}}g_6(x_{2j},q_{1i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)}$$
$$-\sum_{j=1}^{n_2}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)\left(q_{0i}^{(x_{1j}+1)^{\theta_i}c_i^{x_{1j}+1}}g_6(x_{1j}+1,q_{1i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i)f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)}$$
$$+\sum_{j=1}^{n_0}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)g_6(x_j,q_{1i},\theta_i,c_i)}{\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}}f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)-(q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}}f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)}$$
$$-\sum_{j=1}^{n_0}\frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)q_{0i}^{(x_j+1)^{\theta_i}c_i^{x_j+1}}g_6(x_j+1,q_{1i},\theta_i,c_i)}{\sum_{i=1}^{g}\alpha_i\left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}}f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i)-(q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}}f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)},\tag{54}$$

$$\begin{aligned}
\frac{\partial l}{\partial q_{2i}} = & \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i)\left(q_{0i}^{x_{2j}^{\theta_i}c_i^{x_{2j}}}\, g_6(x_{2j},q_{2i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)} \\
& - \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i)\left(q_{0i}^{(x_{2j}+1)^{\theta_i}c_i^{x_{2j}+1}}\, g_6(x_{2j}+1,q_{2i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)} \\
& + \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)\big(g_6(x_{2j},q_{2i},\theta_i,c_i) - g_6(x_{2j}+1,q_{2i},\theta_i,c_i)\big)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)} \\
& + \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g}\alpha_i \left((q_{0i}q_{1i})^{x_j^{\theta_i}c_i^{x_j}}\, g_6(x_j,q_{2i},\theta_i,c_i) - q_{0i}^{(x_j+1)^{\theta_i}c_i^{x_j+1}}\, q_{1i}^{x_j^{\theta_i}c_i^{x_j}}\, g_6(x_j+1,q_{2i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i \left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}} f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)} \\
& - \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g}\alpha_i (q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}} \big(g_6(x_j,q_{2i},\theta_i,c_i) - g_6(x_j+1,q_{2i},\theta_i,c_i)\big)}{\sum_{i=1}^{g}\alpha_i \left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}} f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)}
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
\frac{\partial l}{\partial \alpha_i} = & \sum_{j=1}^{n_1} \frac{\sum_{i=1}^{g} f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{1j},q_{1i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{2j},q_{0i}q_{2i},\theta_i,c_i)} \\
& + \sum_{j=1}^{n_2} \frac{\sum_{i=1}^{g} f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)}{\sum_{i=1}^{g}\alpha_i f_{\mathrm{DMW}}(x_{2j},q_{2i},\theta_i,c_i) f_{\mathrm{DMW}}(x_{1j},q_{0i}q_{1i},\theta_i,c_i)} \\
& + \sum_{j=1}^{n_0} \frac{\sum_{i=1}^{g} \left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}} f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)}{\sum_{i=1}^{g}\alpha_i \left(q_{1i}^{x_j^{\theta_i}c_i^{x_j}} f_{\mathrm{DMW}}(x_j,q_{0i}q_{2i},\theta_i,c_i) - (q_{0i}q_{1i})^{(x_j+1)^{\theta_i}c_i^{x_j+1}} f_{\mathrm{DMW}}(x_j,q_{2i},\theta_i,c_i)\right)},
\end{aligned} \tag{56}$$

where $g_1(x,q,\theta,c) = x^{\theta}c^{x}q^{x^{\theta}c^{x}}\log(x+q)$, $g_3(x,q,\theta,c) = x^{\theta+1}c^{x-1}q^{x^{\theta}c^{x}}\log(q)$ and $g_6(x,q,\theta,c) = x^{\theta}c^{x}q^{x^{\theta}c^{x}-1}$.

The maximum likelihood estimators of the parameters q_{0i}, q_{1i}, q_{2i}, θ_i and c_i, $i = 1,2,\ldots,g$, can be obtained by solving these $6g$ non linear equations. We need a numerical technique to get the maximum likelihood estimators, since the solution of these equations is not easy.

7. Conclusion

In this paper we developed a bivariate discrete modified Weibull (BDMW) distribution and some of its important properties, such as joint survival function, joint probability mass function, joint cumulative distribution, marginal survival function, conditional probability mass function and joint hazard rate function. The maximum likelihood estimates of the parameters of BDMW are obtained. The suitability of BDMW is illustrated through a real life data set.

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Appendix

Proofs

Derivation of Survival Function

We have

$$\begin{aligned}
S_{\mathrm{BDMW}}(x_1,x_2) &= P(X_1 \geq x_1, X_2 \geq x_2)\\
&= P\big(\min(U_1,U_0) \geq x_1, \min(U_2,U_0) \geq x_2\big)\\
&= P\Big(U_1 \geq x_1, U_2 \geq x_2, U_0 \geq \max(x_1,x_2)\Big)\\
&= q_1^{x_1^\theta c^{x_1}} q_2^{x_2^\theta c^{x_2}} q_0^{z^\theta c^z}, \quad \text{where } z = \max(x_1,x_2).
\end{aligned}$$

If $x_1 < x_2$, then

$$\begin{aligned}
S_{\mathrm{BDMW}}(x_1,x_2) &= q_1^{x_1^\theta c^{x_1}} (q_0 q_2)^{x_2^\theta c^{x_2}}\\
&= S_{\mathrm{DMW}}(x_1; q_1, \theta, c) S_{\mathrm{DMW}}(x_2; q_0 q_2, \theta, c).
\end{aligned}$$

If $x_2 < x_1$, then

$$\begin{aligned}
S_{\mathrm{BDMW}}(x_1,x_2) &= (q_0 q_1)^{x_1^\theta c^{x_1}} q_2^{x_2^\theta c^{x_2}}\\
&= S_{\mathrm{DMW}}(x_1; q_0 q_1, \theta, c) S_{\mathrm{DMW}}(x_2; q_2, \theta, c).
\end{aligned}$$

If $x_1 = x_2 = x$, then

$$\begin{aligned}
S_{\mathrm{BDMW}}(x_1,x_2) &= (q_0 q_1 q_2)^{x^\theta c^x}\\
&= S_{\mathrm{DMW}}(x; q_0 q_1 q_2, \theta, c).
\end{aligned}$$

Thus, we have

$$S_{\mathrm{BDMW}}(x_1,x_2)=\begin{cases} S_{\mathrm{DMW}}(x_1;q_1,\theta,c)S_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c) & \text{if } x_1<x_2\\ S_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)S_{\mathrm{DMW}}(x_2;q_2,\theta,c) & \text{if } x_2<x_1\\ S_{\mathrm{DMW}}(x;q_0q_1q_2,\theta,c) & \text{if } x_1=x_2=x. \end{cases}$$

Derivation of Joint Probability Mass Function

We have

$$f_{X_1,X_2}(x_1,x_2)=S_{X_1,X_2}(x_1,x_2)-S_{X_1,X_2}(x_1,x_2+1)-S_{X_1,X_2}(x_1+1,x_2)+S_{X_1,X_2}(x_1+1,x_2+1).$$

If $x_1<x_2$, then

$$\begin{aligned} f&=q_1^{x_1^\theta c^{x_1}}\left((q_0q_2)^{x_2^\theta c^{x_2}}-(q_0q_2)^{(x_2+1)^\theta c^{x_2+1}}\right)\\ &\quad-q_1^{(x_1+1)^\theta c^{x_1+1}}\left((q_0q_2)^{x_2^\theta c^{x_2}}-(q_0q_2)^{(x_2+1)^\theta c^{x_2+1}}\right)\\ &=(q_1^{x_1^\theta c^{x_1}}-q_1^{(x_1+1)^\theta c^{x_1+1}})\left((q_0q_2)^{x_2^\theta c^{x_2}}-(q_0q_2)^{(x_2+1)^\theta c^{x_2+1}}\right)\\ &=f_{\mathrm{DMW}}(x_1;q_1,\theta,c)f_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c). \end{aligned}$$

If $x_2<x_1$, then

$$\begin{aligned} f&=(q_0q_1)^{x_1^\theta c^{x_1}}(q_2^{x_2^\theta c_2^x}-q_2^{(x_2+1)^\theta c^{x_2+1}})-(q_0q_1)^{(x_1+1)^\theta c^{x_1+1}}(q_2^{x_2^\theta c_2^x}-q_2^{(x_2+1)^\theta c^{x_2+1}})\\ &=\left((q_0q_1)^{x_1^\theta c^{x_1}}-(q_0q_1)^{(x_1+1)^\theta c^{x_1+1}}\right)(q_2^{x_2^\theta c^{x_2}}-q_2^{(x_2+1)^\theta c^{x_2+1}})\\ &=f_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)f_{\mathrm{DMW}}(x_2;q_2,\theta,c). \end{aligned}$$

If $x_1=x_2=x$, then

$$\begin{aligned} f&=(q_0q_1q_2)^{x^\theta c^x}-q_1^{x^\theta c^x}(q_0q_2)^{(x+1)^\theta c^{x+1}}-(q_0q_1)^{(x+1)^\theta c^{x+1}}q_2^{x^\theta c^x}+(q_0q_1q_2)^{(x+1)^\theta c^{x+1}}\\ &=q_1^{x^\theta c^x}\left((q_0q_2)^{x^\theta c^x}-(q_0q_2)^{(x+1)^\theta c^{x+1}}\right)-(q_0q_1)^{(x+1)^\theta c^{x+1}}(q_2^{x^\theta c^x}-q_2^{(x+1)^\theta c^{x+1}})\\ &=q_1^{x^\theta c^x}f_{\mathrm{DMW}}(x;q_0q_2,\theta,c)-(q_0q_1)^{(x+1)^\theta c^{x+1}}f_{\mathrm{DMW}}(x;q_2,\theta,c). \end{aligned}$$

From the previous three equations, we obtain

$$f_{\mathrm{BDMW}}(x_1,x_2)=\begin{cases} f_1(x_1,x_2;q_0,q_1,q_2,\theta,c) & \text{if } x_1<x_2\\ f_2(x_1,x_2;q_0,q_1,q_2,\theta,c) & \text{if } x_2<x_1\\ f_0(x;q_0,q_1,q_2,\theta,c) & \text{if } x_1=x_2=x, \end{cases}$$

where

$$
\begin{aligned}
f_1(x_1,x_2;q_0,q_1,q_2,\theta,c) &= f_{\mathrm{DMW}}(x_1;q_1,\theta,c) f_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c) \\
f_2(x_1,x_2;q_0,q_1,q_2,\theta,c) &= f_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c) f_{\mathrm{DMW}}(x_2;q_2,\theta,c) \\
f_0(x;q_0,q_1,q_2,\theta,c) &= q_1^{x^{\theta}c^{x}} f_{\mathrm{DMW}}(x;q_0q_2,\theta,c) - (q_0q_1)^{(x+1)^{\theta}c^{x+1}} f_{\mathrm{DMW}}(x;q_2,\theta,c).
\end{aligned}
$$

Derivation of Joint Cumulative Distribution Function

The joint cumulative distribution function of (X_1, X_2) can be easily obtained from the following relation

$$F_{X_1,X_2}(x_1,x_2) = F_{X_1}(x_1) + F_{X_2}(x_2) + S_{X_1,X_2}(x_1,x_2) - 1.$$

Then,

$$
\begin{aligned}
F_{\mathrm{BDMW}}(x_1,x_2) &= F_{\mathrm{DMW}}(x_1) + F_{\mathrm{DMW}}(x_2) + S_{\mathrm{BDMW}}(x_1,x_2) - 1 \\
&= 1 - q_1^{x_1^{\theta}c^{x_1}} + 1 - q_2^{x_2^{\theta}c^{x_2}} + q_1^{x_1^{\theta}c^{x_1}} q_2^{x_2^{\theta}c^{x_2}} q_0^{z^{\theta}c^{z}} - 1 \\
&= 1 - q_1^{x_1^{\theta}c^{x_1}} - q_2^{x_2^{\theta}c^{x_2}} + q_1^{x_1^{\theta}c^{x_1}} q_2^{x_2^{\theta}c^{x_2}} q_0^{z^{\theta}c^{z}} \\
&= \begin{cases} 1 - q_1^{x_1^{\theta}c^{x_1}} - q_2^{x_2^{\theta}c^{x_2}} + q_1^{x_1^{\theta}c^{x_1}} (q_0q_2)^{x_2^{\theta}c^{x_2}} & \text{if } x_1 < x_2 \\ 1 - q_1^{x_1^{\theta}c^{x_1}} - q_2^{x_2^{\theta}c^{x_2}} + (q_0q_1)^{x_1^{\theta}c^{x_1}} q_2^{x_2^{\theta}c^{x_2}} & \text{if } x_2 < x_1 \\ 1 - q_1^{x^{\theta}c^{x}} - q_2^{x^{\theta}c^{x}} + (q_0q_1q_2)^{x^{\theta}c^{x}} & \text{if } x_1 = x_2 = x \end{cases} \\
&= \begin{cases} 1 - q_1^{x_1^{\theta}c^{x_1}} \left(1 - (q_0q_2)^{x_2^{\theta}c^{x_2}}\right) - q_2^{x_2^{\theta}c^{x_2}} & \text{if } x_1 < x_2 \\ 1 - q_1^{x_1^{\theta}c^{x_1}} - q_2^{x_2^{\theta}c^{x_2}} \left(1 - (q_0q_1)^{x_1^{\theta}c^{x_1}}\right) & \text{if } x_2 \le x_1 \end{cases} \\
&= \begin{cases} F_{\mathrm{DMW}}(x_2;q_2,\theta,c) - q_1^{x_1^{\theta}c^{x_1}} F_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c) & \text{if } x_1 < x_2 \\ F_{\mathrm{DMW}}(x_1;q_1,\theta,c) - q_2^{x_2^{\theta}c^{x_2}} F_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c) & \text{if } x_2 \le x_1. \end{cases}
\end{aligned}
$$

Derivation of Joint Hazard Rate Function

The joint hazard rate function can be obtained by using the following relations

$$h_{\mathrm{BDMW}}(x_1,x_2) = \frac{f_{\mathrm{BDMW}}(x_1,x_2)}{S_{\mathrm{BDMW}}(x_1,x_2)}.$$

If $x_1 < x_2$, then

$$h_{\mathrm{BDMW}}(x_1,x_2) = \frac{(q_1^{x_1^\theta c^{x_1}} - q_1^{(x_1+1)^\theta c^{x_1+1}})\left((q_0q_2)^{x_2^\theta c^{x_2}} - (q_0q_2)^{(x_2+1)^\theta c^{x_2+1}}\right)}{q_1^{x_1^\theta c^{x_1}}(q_0q_2)^{x_2^\theta c^{x_2}}}$$
$$= (1-q_1^{(x_1+1)^\theta c^{x_1+1}-x_1^\theta c^{x_1}})(1-(q_0q_2)^{(x_2+1)^\theta c^{x_2+1}-x_2^\theta c^{x_2}}).$$

If $x_2 < x_1$, then

$$h_{\mathrm{BDMW}}(x_1,x_2) = \frac{\left((q_0q_1)^{x_1^\theta c^{x_1}} - (q_0q_1)^{(x_1+1)^\theta c^{x_1+1}}\right)(q_2^{x_2^\theta c^{x_2}} - q_2^{(x_2+1)^\theta c^{x_2+1}})}{(q_0q_1)^{x_1^\theta c^{x_1}}q_2^{x_2^\theta c^{x_2}}}$$
$$= \left(1-(q_0q_1)^{(x_1+1)^\theta c^{x_1+1}-x_1^\theta c^{x_1}}\right)(1-q_2^{(x_2+1)^\theta c^{x_2+1}-x_2^\theta c^{x_2}}).$$

If $x_1 = x_2 = x$, then

$$h_{\mathrm{BDMW}}(x_1,x_2) = \frac{q_1^{x^\theta c^x}\left((q_0q_2)^{x^\theta c^x} - (q_0q_2)^{(x+1)^\theta c^{x+1}}\right)}{(q_0q_1q_2)^{x^\theta c^x}}$$
$$- \frac{(q_0q_1)^{(x+1)^\theta c^{x+1}}(q_2^{x^\theta c^x} - q_2^{(x+1)^\theta c^{x+1}})}{(q_0q_1q_2)^{x^\theta c^x}}$$
$$= \left(1-(q_0q_2)^{(x+1)^\theta c^{x+1}-x^\theta c^x}\right) - (q_0q_1)^{(x+1)^\theta c^{x+1}}(1-q_2^{(x+1)^\theta c^{x+1}-x^\theta c^x}).$$

Thus, we obtain

$$h_{\mathrm{BDMW}}(x_1,x_2) = \begin{cases} \left(1-\frac{S_{\mathrm{DMW}}(x_1+1;q_1,\theta,c)}{S_{\mathrm{DMW}}(x_1;q_1,\theta,c)}\right)\left(1-\frac{S_{\mathrm{DMW}}(x_2+1;q_0q_2,\theta,c)}{S_{\mathrm{DMW}}(x_2;q_0q_2,\theta,c)}\right) & \text{if } x_1 < x_2 \\ \left(1-\frac{S_{\mathrm{DMW}}(x_1+1;q_0q_1,\theta,c)}{S_{\mathrm{DMW}}(x_1;q_0q_1,\theta,c)}\right)\left(1-\frac{S_{\mathrm{DMW}}(x_2+1;q_2,\theta,c)}{S_{\mathrm{DMW}}(x_2;q_2,\theta,c)}\right) & \text{if } x_2 < x_1 \\ \left(1-\frac{S_{\mathrm{DMW}}(x+1;q_0q_2,\theta,c)}{S_{\mathrm{DMW}}(x;q_0q_2,\theta,c)}\right)- & \\ \frac{S_{\mathrm{DMW}}(x+1;q_0q_1,\theta,c)}{S_{\mathrm{DMW}}(x;q_0q_1,\theta,c)}\left(1-\frac{S_{\mathrm{DMW}}(x+1;q_2,\theta,c)}{S_{\mathrm{DMW}}(x;q_2,\theta,c)}\right) & \text{if } x_1 = x_1 = x. \end{cases}$$

Derivation of Joint Probability Generating Function

$$
\begin{aligned}
G_{\text{BDMW}}(x_1,x_2) &= E(z_1^{X_1} z_2^{X_2}) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P(X_1=i, X_2=j) z_1^i z_2^j \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i-1} P(X_1=i, X_2=j) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} P(X_1=i, X_2=j) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty} P(X_1=i, X_2=i) z_1^i z_2^i \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i-1} f_{\text{DMW}}(i; q_0q_1, \theta, c) f_{\text{DMW}}(i; q_2, \theta, c) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} f_{\text{DMW}}(i; q_1, \theta, c) f_{\text{DMW}}(i; q_0q_2, \theta, c) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty}\Big[q_1^{i^\theta c^i} f_{\text{DMW}}(i; q_0q_2, \theta, c) - (q_0q_1)^{(i+1)^\theta c^{i+1}} f_{\text{DMW}}(i; q_2, \theta, c)\Big] z_1^i z_2^i \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{i-1} \Big((q_0q_1)^{i^\theta c^i} - (q_0q_1)^{(i+1)^\theta c^{i+1}}\Big)(q_2^{i^\theta c^i} - q_2^{(i+1)^\theta c^{i+1}}) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty}\sum_{j=i+1}^{\infty} (q_1^{i^\theta c^i} - q_1^{(i+1)^\theta c^{i+1}})\Big((q_0q_2)^{i^\theta c^i} - (q_0q_2)^{(i+1)^\theta c^{i+1}}\Big) z_1^i z_2^j \\
&\quad + \sum_{i=0}^{\infty} q_1^{i^\theta c^i}\Big((q_0q_2)^{i^\theta c^i} - (q_0q_2)^{(i+1)^\theta c^{i+1}}\Big) z_1^i z_2^i \\
&\quad - \sum_{i=0}^{\infty} (q_0q_1)^{(i+1)^\theta c^{i+1}} (q_2^{i^\theta c^i} - q_2^{(i+1)^\theta c^{i+1}}) z_1^i z_2^i.
\end{aligned}
$$

Identifiability Condition

First, we consider $g=2$.

For the condition $x_1 < x_2$, let

$$a_1A_1(x_1,x_2) + a_2A_2(x_1,x_2) = 0,$$

where a_1 and a_2 are any two arbitrary real numbers, $A_1(x_1,x_2)$ and $A_2(x_1,x_2)$ are distribution function, in which $A_2(x_1,x_2)$ is obtained from $A_1(x_1,x_2)$ by replacing q_{0i} by p_{0i},

q_{1i} by p_{1i}, q_{2i} by p_{2i}, θ_i by δ_i and c_i by ρ_i. Assume that for each $i=1,2$, $q_{0i}\neq p_{0i}$, $q_{1i}\neq p_{1i}$, $q_{2i}\neq p_{2i}$, $\theta_i\neq\delta_{1i}$ and $c_i\neq\rho_i$. Then, we have

$$\begin{aligned}A_1(x_1,x_2)=\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[&\alpha(q_{11}^{i^{\theta_1}c_1^i}-q_{11}^{(i+1)^{\theta_1}c_1^{i+1}})\Big((q_{01}q_{21})^{j^{\theta_1}c_1^j}-(q_{01}q_{21})^{(j+1)^{\theta_1}c_1^{j+1}}\Big)\\&+(1-\alpha)(q_{12}^{i^{\theta_2}c_2^i}-q_{12}^{(i+1)^{\theta_2}c_2^{i+1}})\Big((q_{02}q_{22})^{j^{\theta_2}c_2^j}-(q_{02}q_{22})^{(j+1)^{\theta_2}c_2^{j+1}}\Big)\Big],\end{aligned}$$

$$\begin{aligned}A_2(x_1,x_2)=\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[&\alpha(p_{11}^{i^{\delta_1}\rho_1^i}-p_{11}^{(i+1)^{\delta_1}\rho_1^{i+1}})\Big((p_{01}p_{21})^{j^{\delta_1}\rho_1^j}-(p_{01}p_{21})^{(j+1)^{\delta_1}\rho_1^{j+1}}\Big)\\&+(1-\alpha)(p_{12}^{i^{\delta_2}\rho_2^i}-p_{12}^{(i+1)^{\delta_2}\rho_2^{i+1}})\Big((p_{02}p_{22})^{j^{\delta_2}\rho_2^j}-(p_{02}p_{22})^{(j+1)^{\delta_2}\rho_2^{j+1}}\Big)\Big],\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[&a_1(q_{11}^{i^{\theta_1}c_1^i}-q_{11}^{(i+1)^{\theta_1}c_1^{i+1}})\Big((q_{01}q_{21})^{j^{\theta_1}c_1^j}-(q_{01}q_{21})^{(j+1)^{\theta_1}c_1^{j+1}}\Big)\\&+a_2(p_{11}^{i^{\delta_1}\rho_1^i}-p_{11}^{(i+1)^{\delta_1}\rho_1^{i+1}})\Big((p_{01}p_{21})^{j^{\delta_1}\rho_1^j}-(p_{01}p_{21})^{(j+1)^{\delta_1}\rho_1^{j+1}}\Big)\Big]\quad=0\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[&a_1(q_{12}^{i^{\theta_2}c_2^i}-q_{12}^{(i+1)^{\theta_2}c_2^{i+1}})\Big((q_{02}q_{22})^{j^{\theta_2}c_2^j}-(q_{02}q_{22})^{(j+1)^{\theta_2}c_2^{j+1}}\Big)\\&+a_2(p_{12}^{i^{\delta_2}\rho_2^i}-p_{12}^{(i+1)^{\delta_2}\rho_2^{i+1}})\Big((p_{02}p_{22})^{j^{\delta_2}\rho_2^j}-(p_{02}p_{22})^{(j+1)^{\delta_2}\rho_2^{j+1}}\Big)\Big]\quad=0.\end{aligned}$$

For $k=1,2$, we have

$$\Delta_{k,i,j}=(q_{1k}^{i^{\theta_k}c_k^i}-q_{1k}^{(i+1)^{\theta_k}c_k^{i+1}})\Big((q_{0k}q_{2k})^{j^{\theta_k}c_k^j}-(q_{0k}q_{2k})^{(j+1)^{\theta_k}c_k^{j+1}}\Big),$$

$$\Lambda_{k,i,j}=(p_{1k}^{i^{\delta_k}\rho_k^i}-p_{1k}^{(i+1)^{\delta_k}\rho_k^{i+1}})\Big((p_{0k}p_{2k})^{j^{\delta_k}\rho_k^j}-(p_{0k}p_{2k})^{(j+1)^{\delta_k}\rho_k^{j+1}}\Big),$$

$$\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[a_1\Delta_{1,i,j}+a_2\Lambda_{1,i,j}\Big]=0,$$

$$\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[a_1\Delta_{2,i,j}+a_2\Lambda_{2,i,j}\Big]=0$$

and

$$a_1\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Delta_{1,i,j}\Lambda_{2,i,j}=a_1\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Delta_{2,i,j}\Lambda_{1,i,j}.$$

This implies that $a_1 = 0$. Similarly, we get $a_2 = 0$.

If $x_1 > x_2$, we consider

$$b_1 B_1(x_1, x_2) + b_2 B_2(x_1, x_2) = 0,$$

where b_1 and b_2 are any two arbitrary real numbers, $B_1(x_1, x_2)$ and $B_2(x_1, x_2)$ are distribution function, in which $B_2(x_1, x_2)$ is obtained from $B_1(x_1, x_2)$ by replacing q_{0i} by p_{0i}, q_{1i} by p_{1i}, q_{2i} by p_{2i}, θ_i by δ_i and c_i by ρ_i. Assume that, for each $i = 1, 2$, $q_{0i} \neq p_{0i}$, $q_{1i} \neq p_{1i}$, $q_{2i} \neq p_{2i}$, $\theta_i \neq \delta_{1i}$ and $c_i \neq \rho_i$. Then, we have

$$B_1(x_1, x_2) = \sum_{i=1}^{x_1} \sum_{j=1}^{x_2} \Big[\alpha \Big((q_{01} q_{11})^{i^{\theta_1} c_1^i} - (q_{01} q_{11})^{(i+1)^{\theta_1} c_1^{i+1}} \Big) \big(q_{21}^{j^{\theta_1} c_1^j} - q_{21}^{(j+1)^{\theta_1} c_1^{j+1}} \big) + (1-\alpha) \Big((q_{02} q_{12})^{i^{\theta_2} c_2^i} - (q_{02} q_{12})^{(i+1)^{\theta_2} c_2^{i+1}} \Big) \big(q_{22}^{j^{\theta_2} c_2^j} - q_{22}^{(j+1)^{\theta_2} c_2^{j+1}} \big) \Big],$$

$$B_2(x_1, x_2) = \sum_{i=1}^{x_1} \sum_{j=1}^{x_2} \Big[\alpha \Big((p_{01} p_{11})^{i^{\delta_1} \rho_1^i} - (p_{01} p_{11})^{(i+1)^{\delta_1} \rho_1^{i+1}} \Big) \big(p_{21}^{j^{\delta_1} \rho_1^j} - p_{21}^{(j+1)^{\delta_1} \rho_1^{j+1}} \big) + (1-\alpha) \Big((p_{02} p_{12})^{i^{\delta_2} \rho_2^i} - (p_{02} p_{12})^{(i+1)^{\delta_2} \rho_2^{i+1}} \Big) \big(p_{22}^{j^{\delta_2} \rho_2^j} - p_{22}^{(j+1)^{\delta_2} \rho_2^{j+1}} \big) \Big],$$

$$\sum_{i=1}^{x_1} \sum_{j=1}^{x_2} \Big[b_1 \Big((q_{01} q_{11})^{i^{\theta_1} c_1^i} - (q_{01} q_{11})^{(i+1)^{\theta_1} c_1^{i+1}} \Big) \big(q_{21}^{j^{\theta_1} c_1^j} - q_{21}^{(j+1)^{\theta_1} c_1^{j+1}} \big) + b_2 \Big((p_{01} p_{11})^{i^{\delta_1} \rho_1^i} - (p_{01} p_{11})^{(i+1)^{\delta_1} \rho_1^{i+1}} \Big) \big(p_{21}^{j^{\delta_1} \rho_1^j} - p_{21}^{(j+1)^{\delta_1} \rho_1^{j+1}} \big) \Big] = 0$$

and

$$\sum_{i=1}^{x_1} \sum_{j=1}^{x_2} \Big[b_1 \Big((q_{02} q_{12})^{i^{\theta_2} c_2^i} - (q_{02} q_{12})^{(i+1)^{\theta_2} c_2^{i+1}} \Big) \big(q_{22}^{j^{\theta_2} c_2^j} - q_{22}^{(j+1)^{\theta_2} c_2^{j+1}} \big) + b_2 \Big((p_{02} p_{12})^{i^{\delta_2} \rho_2^i} - (p_{02} p_{12})^{(i+1)^{\delta_2} \rho_2^{i+1}} \Big) \big(p_{22}^{j^{\delta_2} \rho_2^j} - p_{22}^{(j+1)^{\delta_2} \rho_2^{j+1}} \big) \Big] = 0.$$

For $k = 1, 2$, we have

$$\Psi_{k,i,j} = \Big((q_{0k} q_{1k})^{i^{\theta_k} c_2^i} - (q_{0k} q_{1k})^{(i+1)^{\theta_k} c_k^{i+1}} \Big) \big(q_{2k}^{j^{\theta_k} c_k^j} - q_{2k}^{(j+1)^{\theta_k} c_k^{j+1}} \big),$$

$$\Phi_{k,i,j} = \Big((p_{0k} p_{1k})^{i^{\delta_k} \rho_k^i} - (p_{0k} p_{1k})^{(i+1)^{\delta_k} \rho_k^{i+1}} \Big) \big(p_{2k}^{j^{\delta_k} \rho_k^j} - p_{2k}^{(j+1)^{\delta_k} \rho_k^{j+1}} \big),$$

$$\sum_{i=1}^{x_1} \sum_{j=1}^{x_2} \Big[b_1 \Psi_{1,i,j} + b_2 \Phi_{1,i,j} \Big] = 0,$$

$$\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Big[b_1\Psi_{2,i,j}+b_2\Phi_{2,i,j}\Big]=0$$

and

$$b_1\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Psi_{1,i,j}\Phi_{2,i,j}=b_1\sum_{i=1}^{x_1}\sum_{j=1}^{x_2}\Psi_{2,i,j}\Phi_{1,i,j}.$$

This implies that $b_1=0$. Similarly we obtain $b_2=0$.

For $x_1=x_2=x$, let

$$t_1T_1(x_1,x_2)+t_2T_2(x_1,x_2)=0,$$

where t_1 and t_2 are any two arbitrary real numbers, $T_1(x_1,x_2)$ and $T_2(x_1,x_2)$ are distribution function, in which $T_2(x_1,x_2)$ is obtained from $T_1(x_1,x_2)$ by replacing q_{0i} by p_{0i}, q_{1i} by p_{1i}, q_{2i} by p_{2i}, θ_i by δ_i and c_i by ρ_i. Assume that for each $i=1,2$, $q_{0i}\neq p_{0i}$, $q_{1i}\neq p_{1i}$, $q_{2i}\neq p_{2i}$, $\theta_i\neq\delta_{1i}$ and $c_i\neq\rho_i$. Then, we have

$$\begin{aligned}T_1(x_1,x_2)=\sum_{i=1}^{x}\Bigg[\alpha\Bigg(& q_{11}^{i^{\theta_1}c_1^i}\Big((q_{01}q_{21})^{i^{\theta_1}c_1^i}-(q_{01}q_{21})^{(i+1)^{\theta_1}c_1^{i+1}}\Big)\\&-(q_{01}q_{11})^{(i+1)^{\theta_1}c_1^{i+1}}(q_{21}^{i^{\theta_1}c_1^i}-q_{21}^{(i+1)^{\theta_1}c_1^{i+1}})\Bigg)\\+(1-\alpha)\Bigg(& q_{12}^{i^{\theta_2}c_2^i}\Big((q_{02}q_{22})^{i^{\theta_2}c_2^i}-(q_{02}q_{22})^{(i+1)^{\theta_2}c_2^{i+1}}\Big)\\&-(q_{02}q_{12})^{(i+1)^{\theta_2}c_2^{i+1}}(q_{22}^{i^{\theta_2}c_2^i}-q_{22}^{(i+1)^{\theta_2}c_2^{i+1}})\Bigg)\Bigg]\end{aligned}$$

and

$$\begin{aligned}T_2(x_1,x_2)=\sum_{i=1}^{x}\Bigg[\alpha\Bigg(& p_{11}^{i^{\delta_1}\rho_1^i}\Big((p_{01}p_{21})^{i^{\delta_1}\rho_1^i}-(p_{01}p_{21})^{(i+1)^{\delta_1}\rho_1^{i+1}}\Big)\\&-(p_{01}p_{11})^{(i+1)^{\delta_1}\rho_1^{i+1}}(p_{21}^{i^{\delta_1}\rho_1^i}-p_{21}^{(i+1)^{\delta_1}\rho_1^{i+1}})\Bigg)\\+(1-\alpha)\Bigg(& p_{12}^{i^{\delta_2}\rho_2^i}\Big((p_{02}p_{22})^{i^{\theta_2}\rho_2^i}-(p_{02}p_{22})^{(i+1)^{\delta_2}\rho_2^{i+1}}\Big)\\&-(p_{02}p_{12})^{(i+1)^{\delta_2}\rho_2^{i+1}}(p_{22}^{i^{\delta_2}\rho_2^i}-p_{22}^{(i+1)^{\delta_2}\rho_2^{i+1}})\Bigg)\Bigg].\end{aligned}$$

Using the same procedure we get $t_1=0$ and $t_2=0$.

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SUMMARY

In this paper we develop a bivariate discrete modified Weibull (BDMW) distribution and derived some of its important properties such as joint survival function, marginal survival function, conditional survival function, joint probability generating function and hazard rate function. The significance of additional parameter in BDMW is tested using generalized likelihood ratio test. Also we construct a finite mixture of BDMW and established its identifiability condition. Certain properties of the mixture model are derived. The parameters of BDMW are estimated through method of maximum likelihood.

Keywords: Weibull distribution; Joint survival function; Probability mass function; Maximum likelihood estimation