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EXTENDED ODD LOMAX FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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1. Introduction

Probability distributions are very useful in describing real world phenomena. Due to this, it is a fundamental practice to obtain distributions which best describe a given data. Though there is a rich collection of distributions to which data can be modeled with, these distributions sometimes fail to capture all the variations in the data. Because of this, there is the need for the development of new distributions, the extension and generalization of already existing distributions. These usually result in more flexible distributions which best describe real world scenarios. The Lomax distribution has had extensive applications in many fields, and hence has had several extensions and generalizations. Some of these include the transmuted Weibull Lomax distribution by Afify *et al.* (2015), beta exponentiated Lomax distribution by Mead (2016), Gompertz Lomax distribution by Oguntunde *et al.* (2017) and recently a generator of odd Lomax distributions by Cordeiro *et al.* (2019).

In the development of new distributions, several techniques have been developed and employed by different researchers. In this study, the method proposed by Alzaatreh *et al.* (2013) is used to develop a new family of distributions using the Lomax distribution. The new distribution adds three extra parameters to a baseline distribution.

Given a random variable X with a probability density function (PDF) $f(x)$ and a cumulative distribution function (CDF) $F(x)$, and also let T be a continuous random variable with PDF $r(t)$ defined on the support $[a, b]$. Alzaatreh *et al.* (2013) defined a

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new family of distributions as

$$F(x) = \int_a^{W(F(x))} r(t)dt, \tag{1}$$

where $W(F(x)) \in [a, b]$ is differentiable and monotonically non-decreasing. Furthermore, $W(F(x)) \to a$ as $x \to -\infty$ and $W(F(x)) \to b$ as $x \to +\infty$. This method of generating new distributions has been used recently by Haq and Elgarhy (2018) to develop the odd Fréchet-G family of distributions and by Nasiru (2018) to develop the extended odd Fréchet-G family of distributions, among others. Cordeiro *et al.* (2019) introduced the odd Lomax families of distributions by considering the random variable T to follow the Lomax distribution. In this study, the odds of a baseline distribution with an extra shape parameter is considered with T following the Lomax distribution.

The rest of the paper is organized as follows: In Section 2, the CDF, PDF, quantile function and hazard rate function of the new family of distributions are presented. Section 3 presents the mixture representation of the density function of the family of distributions. Section 4 presents four special distributions of the family of distributions. In Section 5, some of the properties of the family of distributions are given. These include the moments, moment generating function, mean residual, mean waiting time and the order statistics of the family of distributions. Section 6 presents the parameter estimation and Section 7 presents the assessment of the parameters of the family of distributions using Monte Carlo simulations. Section 8 presents applications of the family of distributions to two datasets to illustrate its flexibility and usefulness. The conclusion of the study is presented in Section 9.

2. The new family of distributions

In this section, the odd Lomax family of distributions is developed in the framework of Eq. (1). Consider a random variable T following the Lomax distribution with PDF and CDF defined by $r(t) = \alpha\theta^{\alpha}(\theta + t)^{-(\alpha+1)}$ and $R(t) = 1 - \theta^{\alpha}(\theta + t)^{-\alpha}$, $t > 0, \alpha > 0, \theta > 0$, respectively. Given a baseline distribution $G(x; \boldsymbol{\omega})^{\beta}$, where $\boldsymbol{\omega}$ is a vector of parameters, then the function $W(F(x))$ in Eq. (1) can be defined as the odds ratio of the distribution as $\frac{G(x;\boldsymbol{\omega})^{\beta}}{1-G(x;\boldsymbol{\omega})^{\beta}}$. Substituting the odds ratio and the density $r(t)$ into Eq. (1) gives

$$F(x) = \alpha\theta^{\alpha} \int_0^{\frac{G(x;\boldsymbol{\omega})^{\beta}}{(1-G(x;\boldsymbol{\omega})^{\beta})}} (\theta + t)^{-(\alpha+1)} dt. \tag{2}$$

Solving Eq. (2) gives the CDF of the new family of distributions as

$$F(x) = 1 - \left(\frac{\theta(1 - G(x; \boldsymbol{\omega})^{\beta})}{\theta + (1 - \theta)G(x; \boldsymbol{\omega})^{\beta}} \right)^{\alpha}, \tag{3}$$

for $x \in \mathbb{R}$, where $\alpha > 0$ and $\beta > 0$ are extra shape parameters and $\theta > 0$ is an extra scale parameter. Differentiating Eq. (3) gives the PDF of the new family of distributions for $x \in \mathbb{R}, \alpha > 0, \beta > 0$ and $\theta > 0$ as

$$f(x) = \frac{\alpha\beta\theta^{\alpha} g(x;\omega)G(x;\omega)^{\beta-1}}{(1-G(x;\omega)^{\beta})^{1-\alpha}}(\theta+(1-\theta)G(x;\omega)^{\beta})^{-(\alpha+1)}. \tag{4}$$

The new family of distributions will hence forth be referred to as the extended odd Lomax-G (EOLxG) distribution. A random variable X with a PDF defined by Eq. (4) is therefore denoted by $X \sim EOLxG(x;\omega)$.

Adopting the interpretation of the CDF of the odd Weibull and extended odd Fréchet distributions by Cooray (2006) and Nasiru (2018), respectively, the interpretation of the CDF of the EOLxG family of distributions is given as follows. Let the random variable Z follow a continuous distribution with CDF $G(x;\omega)^{\beta}$, the odds that a failure time will not exceed x is given by $\frac{G(x;\omega)^{\beta}}{1-G(x;\omega)^{\beta}}$. If the randomness or variability of the odds of the random variable Z is defined by the Lomax distribution, then the distribution of Z is as given as

$$F_Z(x) = \mathbb{P}(Z \le x) = \mathbb{P}\left(X \le \frac{G(x;\omega)^{\beta}}{1-G(x;\omega)^{\beta}}\right) = F_X\left(\frac{G(x;\omega)^{\beta}}{1-G(x;\omega)^{\beta}}\right).$$

To generate random numbers from the EOLxG family of distributions, the quantile function of EOLxG is necessary. The quantile function of a random variable from the EOLxG family of distributions is obtained as the inverse function of the CDF given in Eq. (3). Thus, the quantile function is given by

$$Q(u) = G^{-1}\left\{\left[\left(\theta\left[(1-u)^{-\frac{1}{\alpha}}-1\right]\right)^{-1}+1\right]^{-\frac{1}{\beta}}\right\}, \qquad u \in [0,1], \tag{5}$$

where $G^{-1}(\cdot)$ is the inverse function of the CDF of the baseline distribution.

The quantile function is also useful in obtaining the skewness and kurtosis measures, especially when the moments of the distribution do not exist. These measures can be computed using Galton's skewness and Moor's kurtosis measures, given respectively as

$$\mathscr{B} = \frac{Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)-2Q\left(\frac{4}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)} \text{ and } \mathscr{M} = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}.$$

The hazard rate function of the family of distributions is given by

$$h(x) = \frac{\alpha\beta g(x;\omega)G(x;\omega)^{\beta-1}}{(1-G(x;\omega)^{\beta})(\theta+(1-\theta)G(x;\omega)^{\beta})}, \qquad x \in \mathbb{R}. \tag{6}$$

REMARK 1. *When $\beta = 1$, we obtain the odd Lomax family of distributions by* Cordeiro et al. *(2019).*

3. Mixture representation

The mixture representation of the EOLxG density is given in this section. The density function in Eq. (4) can be re-written as

$$f(x) = \frac{\alpha\beta\theta^{-1} g(x;\omega)G(x;\omega)^{\beta-1}(1-G(x;\omega)^{\beta})^{\alpha-1}}{\left(1+(\theta^{-1}-1)G(x;\omega)^{\beta}\right)^{(\alpha+1)}}. \tag{7}$$

Using the generalized binomial expansions given as

$$(1+z)^{-n} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}(-1)^i z^i \quad \text{and} \quad (1-z)^n = \sum_{i=0}^{\infty}\binom{n}{i}(-1)^i z^i$$

for $|z| < 1$, then we have

$$(1-G(x;\omega)^{\beta})^{\alpha-1} = \sum_{i=0}^{\infty}\binom{\alpha-1}{i}(-1)^i G(x;\omega)^{\beta i} \tag{8}$$

and

$$(1+(\theta^{-1}-1)G(x;\omega)^{\beta})^{-(\alpha+1)} = \sum_{j=0}^{\infty}\binom{\alpha+j}{j}(-1)^j(\theta^{-1}-1)^j G(x;\omega)^{\beta j}. \tag{9}$$

Substituting Eq. (8) and Eq. (9) into Eq. (7) gives the mixture representation as

$$f(x) = \frac{\alpha\beta}{\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{ij}\, g(x;\omega)G(x;\omega)^{\beta(i+j+1)-1}, \tag{10}$$

where $\varphi_{ij} = \binom{\alpha-1}{i}\binom{\alpha+j}{j}(-1)^{i+j}(\theta^{-1}-1)^j$. Alternatively, the density function can be represented as

$$f(x) = \frac{\alpha\beta}{\theta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi^{*}_{ij}\,\varkappa_{i+j+1}(x), \tag{11}$$

where $\varphi^{*}_{ij} = \frac{\varphi_{ij}}{\beta(i+j+1)}$ and $\varkappa_{i+j+1} = \beta(i+j+1)g(x;\omega)G(x;\omega)^{\beta(i+j+1)}$. $\varkappa_{i+j+1}$ is the exponentiated-G (Exp-G) class density function with power parameter $\beta(i+j+1)$. The Exp-G density can be used to obtain some of the properties of the new distribution.

4. Some special distributions

4.1. *Extended odd Lomax-Weibull (EOLxW) distribution*

Suppose that the baseline distribution is defined as the Weibull distribution with CDF and PDF for $x > 0$ and $\lambda > 0, \sigma > 0$ defined by $G(x) = 1 - e^{-\lambda x^{\sigma}}$ and $g(x) = \lambda\sigma x^{\sigma-1}e^{-\lambda x^{\sigma}}$, respectively. The PDF and hazard functions of EOLxW distribution are obtained by substituting the CDF and PDF of the Weibull distribution into Eq. (4) and Eq. (6), respectively. This gives the following PDF and CDF for $x > 0$ and $\alpha > 0, \beta > 0, \theta > 0, \lambda > 0, \sigma > 0$, respectively as

$$f(x) = \frac{\alpha\beta\theta^{\alpha}\lambda\sigma x^{\sigma-1}e^{-\lambda x^{\sigma}}\left(1-e^{-\lambda x^{\sigma}}\right)^{\beta-1}}{\left(\theta+(1-\theta)(1-e^{-\lambda x^{\sigma}})^{\beta}\right)^{(\alpha+1)}}\left(1-\left(1-e^{-\lambda x^{\sigma}}\right)^{\beta}\right)^{\alpha-1}$$

and

$$F(x) = 1-\left(\frac{\theta\left(1-\left[1-e^{-\lambda x^{\sigma}}\right]^{\beta}\right)}{\theta+(1-\theta)\left[1-e^{-\lambda x^{\sigma}}\right]^{\beta}}\right)^{\alpha}.$$

Figure 1(a) and Figure 1(b) show the plots for some parameter values indicating possible shapes of the PDF and hazard rate function of the EOLxW distribution. It can be observed that the density function can assume different shapes including decreasing, right skewed and approximately symmetric shapes. Also, it can be observed that the hazard rate function exhibit decreasing, increasing and up-side down bathtub shapes among others.

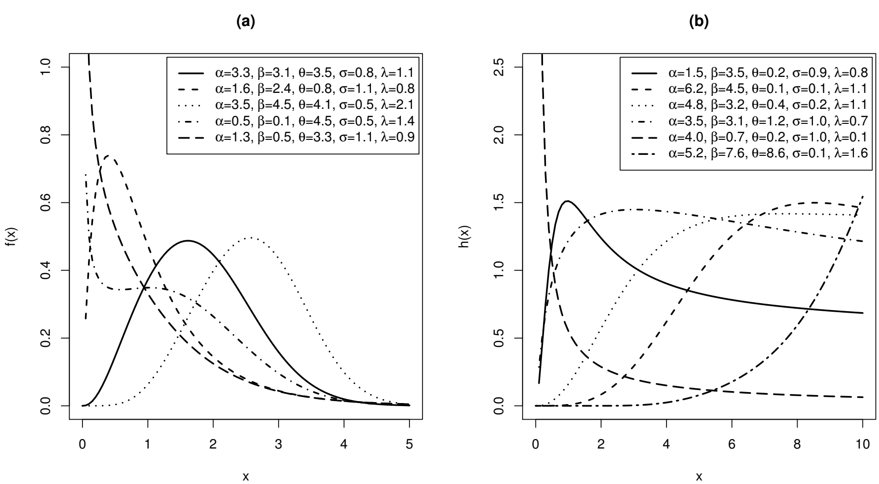

Figure 1 – Plots of the density and hazard functions of EOLxW for some possible parameter values.

4.2. Extended odd Lomax-Lomax (EOLxLx) distribution

If we consider the baseline distribution to follow the Lomax distribution with PDF and CDF given as $g(x) = \delta\sigma^{\delta}(\sigma+x)^{-(\delta+1)}$ and $G(x) = 1-\sigma^{\delta}(\sigma+x)^{-\delta}$ for $x > 0$ and $\sigma > 0, \delta > 0$ respectively, we obtain EOLxLx distribution, with its PDF and CDF functions given respectively as

$$f(x)=\frac{\alpha\beta\theta^{\alpha}\delta\sigma^{\delta}(\sigma+x)^{-(\delta+1)}\left(1-\sigma^{\delta}(\sigma+x)^{-\delta}\right)^{\beta-1}}{\left(1-\left(1-\sigma^{\delta}(\sigma+x)^{-\delta}\right)^{\beta}\right)^{1-\alpha}\left(\theta+(1-\theta)\left(1-\sigma^{\delta}(\sigma+x)^{-\delta}\right)^{\beta}\right)^{\alpha+1}}$$

and

$$F(x)=1-\left(\frac{\theta\left(1-\left[1-\sigma^{\delta}(\sigma+x)^{-\delta}\right]^{\beta}\right)}{\theta+(1-\theta)\left[1-\sigma^{\delta}(\sigma+x)^{-\delta}\right]^{\beta}}\right)^{\alpha}.$$

Figure 2(a) and Figure 2(b) show different possible shapes for different parameter values of the density and hazard rate functions plots of the EOLxLx distribution, respectively. The density and hazard rate functions exhibit different shapes indicating the ability of EOLxLx to model a range of datasets.

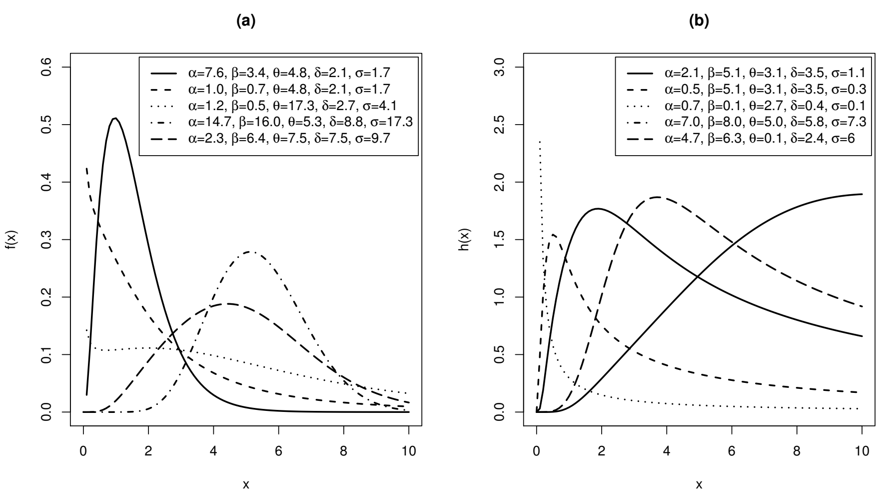

Figure 2 – Plot of the density and hazard functions of EOLxLx for some possible parameter values.

4.3. *Extended odd Lomax-Burr XII (EOLxB) distribution*

Again, if the we consider the Burr XII distribution as the baseline distribution with its PDF and CDF defined for $x > 0$ and $\tau > 0, \gamma > 0, \xi > 0$ as $g(x) = \tau\gamma\xi^{-\gamma}x^{\gamma-1}(1+(x/\xi)^{\gamma})^{-(\tau+1)}$ and $G(x) = 1-(1+(x/\xi)^{\gamma})^{-\tau}$, respectively, we obtain the EOLxB distribution. The PDF and CDF of the EOLxB distribution are given respectively as

$$f(x) = \frac{\alpha\beta\theta^{\alpha}\tau\gamma\xi^{-\gamma}x^{\gamma-1}\Big(1+(\frac{x}{\xi})^{\gamma}\Big)^{-(\tau+1)}\Big(1-\Big(1-\Big(1+(\frac{x}{\xi})^{\gamma}\Big)^{-\tau}\Big)^{\beta}\Big)^{\alpha-1}}{\Big(1-\Big(1+(\frac{x}{\xi})^{\gamma}\Big)^{-\tau}\Big)^{1-\beta}\Big(\theta+(1-\theta)\Big(1-\Big(1+(\frac{x}{\xi})^{\gamma}\Big)^{-\tau}\Big)^{\beta}\Big)^{\alpha+1}}$$

and

$$F(x) = 1-\left(\frac{\theta\Big(1-\Big[1-(1+(\frac{x}{\xi})^{\gamma})^{-\tau}\Big]^{\beta}\Big)}{\theta+(1-\theta)\Big[1-(1+(\frac{x}{\xi})^{\gamma})^{-\tau}\Big]^{\beta}}\right)^{\alpha}.$$

The density and hazard rate functions plots of the EOLxB distribution for some parameter values are shown in Figure 3(a) and Figure 3(b) respectively. The density function exhibit right skewed, left skewed, decreasing and approximately symmetric shapes. Also, the hazard rate function exhibit different shapes for selected parameter values.

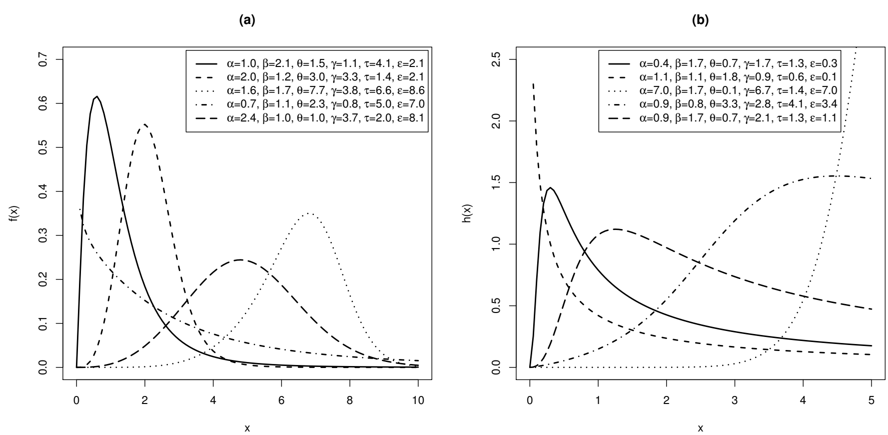

Figure 3 – Plot of the density and hazard functions of EOLxB for some possible parameter values.

4.4. *Extended odd Lomax-Paralogistic (EOLxPL) distribution*

Finally, suppose the Paralogistic distribution is considered as the baseline distribution with PDF and CDF given, for $x > 0$ and $\tau > 0, \sigma > 0$, as $g(x) = \tau^2\sigma^{-\tau}x^{-(\tau+1)}(1 + (x/\sigma)^\tau)^{-(\tau+1)}$ and $G(x) = 1-(1+(x/\sigma)^\tau)^{-\tau}$, respectively, we obtain the EOLxPL distribution with PDF and CDF given respectively as

$$f(x) = \frac{\alpha\beta\theta^\alpha\tau^2\sigma^{-\tau}x^{-(\tau+1)}\left(1+(\frac{x}{\sigma})^\tau\right)^{-(\tau+1)}\left(1-\left(1+(\frac{x}{\sigma})^\tau\right)^{-\tau}\right)^{\beta-1}}{\left(1-\left(1-\left(1+(\frac{x}{\sigma})^\tau\right)^{-\tau}\right)^\beta\right)^{1-\alpha}\left(\theta+(1-\theta)\left(1-\left(1+(\frac{x}{\sigma})^\tau\right)^{-\tau}\right)^\beta\right)^{\alpha+1}}$$

and

$$F(x) = 1-\left(\frac{\theta\left(1-\left[1-(1+(\frac{x}{\sigma})^\tau)^{-\tau}\right]^\beta\right)}{\theta+(1-\theta)\left[1-(1+(\frac{x}{\sigma})^\tau)^{-\tau}\right]^\beta}\right)^\alpha.$$

Figure 4(a) and Figure 4(b) show the plots for various parameter values for the PDF and hazard rate function, respectively, of the EOLxPL distribution. The density function can assume right skewed, left skewed and decreasing shapes among other shapes. The hazard rate function exhibit different non-monotonic failure rate shapes.

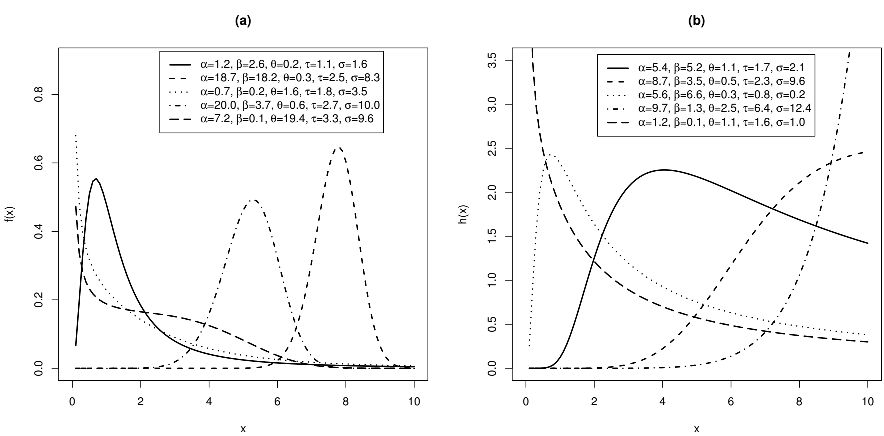

Figure 4 – Plot of the density and hazard functions of EOLxPL for some possible parameter values.

5. Statistical properties

The statistical properties of EOLxG family of distributions are obtained in this section. These include the moments, moment generating function, the mean residual life, mean waiting time and order statistics.

5.1. Moments

The moments of a distribution are useful in estimating some key measures of a distribution including the central tendencies and dispersions.

The rth non-central moment of a distribution with PDF $f(x)$, by definition, is given by

$$E[X^r] = \int_{-\infty}^{\infty} x^r f(x) dx. \tag{12}$$

Substituting Eq. (7) into Eq. (12) gives the non-central moment of the EOLxG family of distributions as

$$E[X^r] = \frac{\alpha\beta}{\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{ij} \varpi_{i+j+1}(x), \tag{13}$$

where $\varpi_{i+j+}(x) = \int_{-\infty}^{\infty} x^r g(x;\omega) G(x;\omega)^{\beta(i+j+1)-1} dx$. When $r = 1$ in Eq. (13), we obtain $E[X] = \mu$.

Also, the incomplete moment of a random variable X is defined as

$$M_r(t) = \int_{-\infty}^{t} x^r f(x) dx. \tag{14}$$

Substituting Eq. (7) into Eq. (14) gives the incomplete moment of the EOLxG family of distributions as

$$M_r(t) = \frac{\alpha\beta}{\theta} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \varphi_{ij} \varpi^*_{i+j+1}(t), \tag{15}$$

where $\varpi^*_{i+j+}(t) = \int_{-\infty}^{t} x^r g(x;\omega) G(x;\omega)^{\beta(i+j+1)-1} dx$.

5.2. Moment generating function (MGF)

The moment generating function (MGF) of a distribution, by definition, is given as

$$M_X(t) = E[e^{tX}]. \tag{16}$$

Using the Tailor expansion of e^{tX}, the MGF can be written as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E[X^r]. \tag{17}$$

Substituting Eq. (7) into (17) gives the MGF of the EOLxG family of distributions as

$$M_X(t)=\frac{\alpha\beta}{\theta}\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{t^r}{r!}\varphi_{ij}\varpi_{i+j+1}(x). \tag{18}$$

5.3. *Mean residual life and mean waiting time*

The mean residual life of a random variable X, by definition, is given by

$$M^*(x)=E[X-x|X>x]=\frac{1}{S(X)}\left[E(X)-\int_{-\infty}^{x}tf(t)dt\right]-x, \tag{19}$$

where $S(X)=1-F(X)$, $E(X)$ is the first non-central moment of the random variable X and $\int_{-\infty}^{x}tf(t)dt$ is the first incomplete moment. Thus, the mean residual life for the EOLxG distribution is obtained by substituting Eq. (13) and Eq. (15), with $r=1$ in both cases, into Eq. (19). This gives

$$M^*(x)=\frac{\alpha\beta}{\theta[1-F(X)]}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{ij}\varpi_{i+j+1}(x)-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{ij}\varpi^*_{i+j+1}(x)\right]-x. \tag{20}$$

Also, mean waiting time is defined as

$$\overline{M}(x)=x-\frac{1}{F(x)}\int_{-\infty}^{x}tf(t)dt.$$

Substituting the first incomplete moment of the EOLxG family of distributions into the definition gives the mean waiting time of the distributions as

$$\overline{M}(x)=x-\frac{\alpha\beta}{\theta F(x)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\varphi_{ij}\varpi^*_{i+j+1}(x).$$

5.4. *Order statistics*

Let $X_1,X_2,\dots,X_n$ be a sample of size n from the EOLxG distribution and $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ denote the ordered statistics of the sample. The PDF of the ith order statistic, denoted by $f_{i:n}(x)$, is defined as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x). \tag{21}$$

But $[1-F(x)]^{n-i}=\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}[F(x)]^k$. Therefore, Eq. (21) becomes

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}[F(x)]^{k+i-1}. \tag{22}$$

Substituting the CDF and PDF given by Eq. (3) and Eq. (4), respectively, into Eq. (22) gives the PDF of the ith order statistic as

$$f_{i:n}(x)=\frac{\alpha\beta\theta^{\alpha}g(x;\omega)G(x;\omega)^{\beta-1}(1-G(x;\omega)^{\beta})^{\alpha-1}n!}{\left(\theta+(1-\theta)G(x;\omega)^{\beta}\right)^{\alpha+1}(i-1)!(n-i)!}$$
$$\times\sum_{j=0}^{n-i}(-1)^{j}\binom{n-i}{k}\left[1-\left(\frac{\theta\left(1-G(x;\omega)^{\beta}\right)}{\theta+(1-\theta)G(x;\omega)^{\beta}}\right)^{\alpha}\right]^{i+j-1}. \quad (23)$$

The PDF of first order statistic is defined as

$$f_{1:n}(x)=n[1-F(x)]^{n-1}f(x). \quad (24)$$

Again, substituting Eq. (3) and Eq. (4) into (24) gives the PDF as

$$f_{1:n}(x)=\alpha\beta\theta^{\alpha n}g(x;\omega)G(x;\omega)^{\beta-1}n\frac{\left(1-G(x;\omega)^{\beta}\right)^{\alpha n-1}}{\left(\theta+(1-\theta)G(x;\omega)^{\beta}\right)^{\alpha n+1}}. \quad (25)$$

Furthermore, the PDF of the nth order statistic is defined by

$$f_{n:n}(x)=n[F(x)]^{n-1}f(x). \quad (26)$$

Using the PDF and CDF of the EOLxG family of distributions, the PDF of the nth order statistic is defined as

$$f_{n:n}(x)=\frac{\alpha\beta\theta^{\alpha}g(x;\omega)G(x;\omega)^{\beta-1}n}{\left(\theta+(1-\theta)G(x;\omega)^{\beta}\right)^{\alpha+1}}\left(1-G(x;\omega)^{\beta}\right)^{\alpha-1}$$
$$\times\left[1-\left(\frac{\theta\left(1-G(x;\omega)^{\beta}\right)}{\theta+(1-\theta)G(x;\omega)^{\beta}}\right)^{\alpha}\right]^{n-1}. \quad (27)$$

6. Parameter estimation

Maximum likelihood method is used for the parameter estimation of the EOLxG family of distributions. Suppose $X_1,X_2,\ldots,X_n$ are n random samples from the EOLxG distribution, then the total log-likelihood function, given the $p\times 1$ parameter vector $\psi=(\alpha,\beta,\theta,\omega)^T$, is given by

$$\ell(x)=n\log[\alpha\beta\theta^{\alpha}]+\sum_{i=1}^{n}\log[g(x_i;\omega)]-(\alpha+1)\sum_{i=1}^{n}\log\left[\theta+(1-\theta)G(x_i;\omega)^{\beta}\right]$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left[1-G(x_i;\omega)^{\beta}\right]+(\beta-1)\sum_{i=1}^{n}\log[G(x_i;\omega)]. \quad (28)$$

The log-likelihood function given in Eq. (28) is differentiated with respect to each parameter to obtain the score function, $U(\psi)=\left(\frac{\partial \ell(x)}{\partial \alpha}, \frac{\partial \ell(x)}{\partial \beta}, \frac{\partial \ell(x)}{\partial \theta}, \frac{\partial \ell(x)}{\partial \omega}\right)^T$ as follows

$$\frac{\partial l(x)}{\partial \alpha}=\frac{n}{\alpha}+n\log\theta+\sum_{i=1}^{n}\log\left[1-G(x_i;\boldsymbol{\omega})^{\beta}\right]-\sum_{i=1}^{n}\log\left[\theta+(1-\theta)G(x_i;\boldsymbol{\omega})^{\beta}\right],$$

$$\frac{\partial l(x)}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log[G(x_i;\boldsymbol{\omega})]+\beta(\alpha-1)\sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\omega})G(x_i;\boldsymbol{\omega})^{\beta-1}}{\left(1-G(x_i;\boldsymbol{\omega})^{\beta}\right)}$$
$$-\beta(\alpha+1)(1-\theta)\sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\omega})G(x_i;\boldsymbol{\omega})^{\beta-1}}{\left(\theta+(1-\theta)G(x_i;\boldsymbol{\omega})^{\beta}\right)},$$

$$\frac{\partial l(x)}{\partial \theta}=\frac{n\alpha}{\theta}+(\alpha+1)\sum_{i=1}^{n}\frac{\left(1-G(x_i;\boldsymbol{\omega})^{\beta}\right)}{\left(\theta+(1-\theta)G(x_i;\boldsymbol{\omega})^{\beta}\right)},$$

$$\frac{\partial l(x)}{\partial \boldsymbol{\omega}}=\sum_{i=1}^{n}\frac{g'(x_i;\boldsymbol{\omega})}{g(x_i;\boldsymbol{\omega})}+(\beta-1)\sum_{i=1}^{n}\frac{G'(x_i;\boldsymbol{\omega})}{G(x_i;\boldsymbol{\omega})}-\beta(\alpha-1)\sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\omega})G(x_i;\boldsymbol{\omega})^{\beta-1}}{\left(1-G(x_i;\boldsymbol{\omega})^{\beta}\right)}$$
$$-\beta(\alpha+1)(1-\theta)\sum_{i=1}^{n}\frac{g(x_i;\boldsymbol{\omega})G(x_i;\boldsymbol{\omega})^{\beta-1}}{\left(\theta+(1-\theta)G(x_i;\boldsymbol{\omega})^{\beta}\right)},$$

where $G'(x;\boldsymbol{\omega})$ and $g'(x;\boldsymbol{\omega})$ means the derivative of the functions $G(x;\boldsymbol{\omega})$ and $g(x;\boldsymbol{\omega})$ with respect to $\boldsymbol{\omega}$ respectively.

The maximum likelihood estimates of the parameters are obtained by equating the score function to zero and numerically solving the system of equations. To achieve this, it is convenient to use non-linear methods of optimization, such as the quasi-Newton algorithm. In this study, `R` (the `mle2` function) is used to achieve this. The $p\times p$ observed information matrix $J(\psi)$ can be used for the interval estimation of the parameters of the distribution. When the standard likelihood conditions are satisfied for the EOLxG family of distributions, with $n\to\infty$, the distribution of $\widehat{\psi}$ can be approximated by the multivariate normal $N_4(\widehat{\psi},J^{-1}(\widehat{\psi}))$, where $J(\widehat{\psi})$ is the expected information matrix. $N_4(\widehat{\psi},J^{-1}(\widehat{\psi}))$ can be used to construct an approximate confidence interval for the parameters of the EOLxG family of distributions.

7. SIMULATION

In this section, Monte Carlo simulations are performed to assess the accuracy and consistency of the maximum likelihood estimators of the new distribution. For illustrative purposes, the EOLxW distribution is considered. The following procedure is used:

- The quantile function in Eq. (5) is used to generate random samples from the distribution. This is done by substituting $u \sim U(0,1)$ into the quantile function for sample sizes $n = 50, 100, 250, 400$ and 600.
- This is repeated for $N = 3000$ times for each sample size and for two (2) sets of parameters: $\alpha = 0.2, \beta = 0.2, \theta = 0.9, \lambda = 2.6$ and $\sigma = 1.8$; $\alpha = 0.2, \beta = 0.3, \theta = 0.6, \lambda = 1.5$ and $\sigma = 1.5$.
- For each experiment, the average estimate (AE), the average bias (AB) and root mean square error (RMSE) for the estimators are computed. For the parameter estimates, $\widehat{\psi} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda}, \widehat{\sigma})'$, the following formulas were used to compute the measures above:

$$\mathrm{AV} = \frac{1}{N}\sum_{i=1}^{N}\widehat{\psi},\ \mathrm{AB} = \frac{1}{N}\sum_{i=1}^{N}(\psi - \psi)\ \text{ and }\ \mathrm{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widehat{\psi} - \psi)^2},$$

  where $\psi = (\alpha, \beta, \lambda, \sigma)'$.

Table 1 shows the results for the first set of parameters. It can be observed that as the sample size increase, the average estimates tend to be closer to the true parameters, the RMSE and bias also tend to reduce with increasing sample size for each parameter. A similar trend can be observed for the second set of parameters in Table 2. This shows the accuracy and consistency of the estimators of the EOLxW distribution.

The asymptotic distribution of the parameters of the EOLxW distribution is investigated via Monte Carlo simulations. For parameter values $(\alpha, \beta, \theta, \lambda, \sigma) = (0.9, \beta = 0.9, \theta = 1.5, \lambda = 2.1, \sigma = 2.5)$, random samples of size 100 are simulated and the parameter estimates obtained. This is repeated 1000 times. The histograms of the parameter estimates are shown in Figure 5. It can be observed that the distributions of the parameter estimates can be approximated to the normal distribution with sufficiently large number of simulations.

TABLE 1
Simulation results for $(\alpha,\beta,\theta,\lambda,\sigma)=(0.2,0.2,0.9,2.6,1.8)$.

Parameter	Sample size (n)	AV	AB	RMSE
$\alpha=0.2$	50	0.242	0.042	0.175
	100	0.228	0.028	0.112
	250	0.210	0.010	0.054
	400	0.207	0.007	0.040
	600	0.204	0.004	0.017
$\beta=0.2$	50	0.253	0.053	0.463
	100	0.226	0.026	0.082
	250	0.216	0.016	0.068
	400	0.214	0.013	0.060
	600	0.209	0.009	0.052
$\theta=0.9$	50	1.036	0.136	2.137
	100	0.917	0.017	0.598
	250	0.900	0.000	0.197
	400	0.908	0.008	0.126
	600	0.905	0.005	0.072
$\lambda=2.6$	50	2.386	-0.213	0.957
	100	2.448	-0.152	0.627
	250	2.568	-0.032	0.246
	400	2.593	-0.007	0.146
	600	2.598	-0.002	0.102
$\sigma=1.5$	50	2.044	0.244	0.612
	100	1.879	0.079	0.295
	250	1.805	0.005	0.124
	400	1.791	-0.009	0.089
	600	1.791	-0.009	0.078

TABLE 2
Simulation results for $(\alpha, \beta, \theta, \lambda, \sigma) = (0.2, 0.3, 0.6, 1.5, 1.5)$.

Parameter	Sample size (n)	AV	AB	RMSE
	50	0.279	0.079	0.258
	100	0.278	0.078	0.219
$\alpha = 0.2$	250	0.234	0.034	0.128
	400	0.215	0.015	0.074
	600	0.210	0.010	0.054
	50	0.406	0.106	2.193
	100	0.337	0.037	0.153
$\beta = 0.2$	250	0.320	0.020	0.097
	400	0.313	0.013	0.074
	600	0.311	0.011	0.065
	50	0.712	0.112	1.937
	100	0.653	0.053	0.454
$\theta = 0.9$	250	0.624	0.024	0.217
	400	0.613	0.013	0.122
	600	0.611	0.011	0.086
	50	1.329	-0.171	0.749
	100	1.349	-0.151	0.527
$\lambda = 2.6$	250	1.435	-0.065	0.298
	400	1.477	-0.023	0.176
	600	1.491	-0.009	0.126
	50	1.826	0.326	0.686
	100	1.659	0.159	0.393
$\sigma = 1.5$	250	1.534	0.034	0.178
	400	1.509	0.009	0.120
	600	1.497	-0.003	0.077

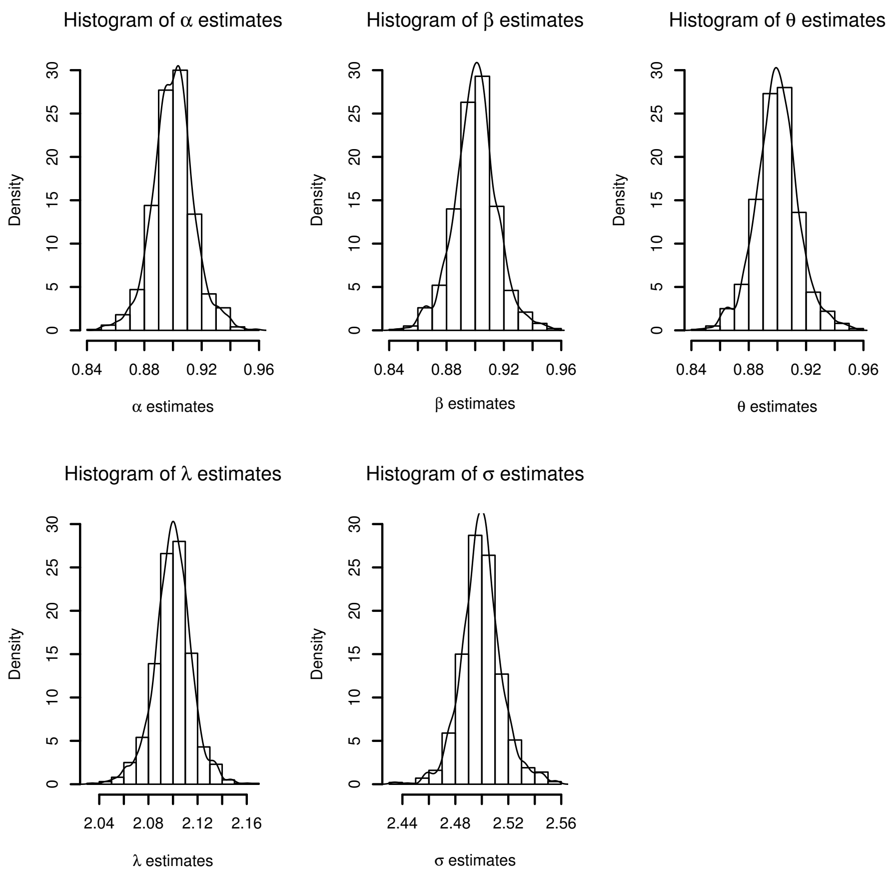

Figure 5 – Histograms of estimated parameters.

8. APPLICATIONS

In this section, the flexibility and usefulness of the EOLxG family of distributions are illustrated by using two different datasets. The performance of the EOLxLx and EOLxPL distributions are compared with other distributions. Generally, the larger the number of parameters of a distribution, the better fit it is likely to provide to a given data set. Hence, the maximum likelihood estimates of the distributions are computed with goodness-of-fit measures, some of which penalize the model for large number of parameters. The goodness-of-fit measures computed for each distribution include Akaike information criterion (AIC), Bayesian information criterion (BIC), Kolmogrove-Smirnov (K-S) and Anderson-Darling (A-D) statistics. The distribution with the least of these measures is considered the best.

8.1. Application 1: Time-to-failure of turbocharger data

The first data set considered is the time-to-failure ($10^3 h$) rate of turbocharger of a type of engine. The data set consists of 40 observations and can be found in Xu *et al.* (2003). The fit of EOLxPL is compared with Kumaraswamy transmuted log-logistic (KwTLL) (Afify *et al.*, 2016), odd Lomax-log-logistic (OLxLL) (Cordeiro *et al.*, 2019) and Burr X Lomax (BXLx) (Yousof *et al.*, 2017). The density functions of KwTLL, OLxLL and BXLx distributions are given respectively as follows:

$$\begin{aligned} f(x) = {} & a b \beta \alpha^{-\beta} x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{\alpha-1} \\ & \times\left\{1-\lambda\left[1-2\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right]\right\}\left\{1+\lambda\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{\alpha-1} \\ & \times\left\{1-\left[1+\lambda\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right]\right\}^{\alpha}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{b-1}, \\ & x>0, |\lambda| \leq 1, \alpha>0, \beta>0, a>0, b>0, \end{aligned}$$

$$\begin{aligned} f(x) = {} & \frac{\alpha\theta^{\alpha} b a^{-b} x^{b-1}\left[1+\left(\frac{x}{a}\right)^{b}\right]^{-2}}{\left[1-\left(1-\left[1+\left(\frac{x}{a}\right)^{b}\right]^{-1}\right)\right]^{2}}\left[\theta+\frac{1-\left[1+\left(\frac{x}{a}\right)^{b}\right]^{-1}}{1-\left(1-\left[1+\left(\frac{x}{a}\right)^{b}\right]^{-1}\right)}\right]^{-(\alpha+1)}, \\ & x>0, \alpha>0, \theta>0, \lambda>0, \sigma>0 \end{aligned}$$

and

$$\begin{aligned} f(x) = {} & \frac{2\theta\alpha}{\beta}\left[1+\frac{x}{\beta}\right]^{\alpha-1}\left(\left[1+\frac{x}{\beta}\right]^{\alpha}-1\right)\exp\left\{-\left(\left[1+\frac{x}{\beta}\right]^{\alpha}-1\right)^{2}\right\} \\ & \times\left\{\exp\left[-\left(\left[1+\frac{x}{\beta}\right]^{\alpha}-1\right)^{2}\right]\right\}^{\theta-1}, \quad x>0, \beta>0, \theta>0. \end{aligned}$$

Table 3 shows the maximum likelihood estimates of EOLxPL, KwTLL, OLxLL and BXLx distributions with their corresponding standard errors in brackets. It also shows the model selection criteria. It can be observed that the EOLxPL distribution has the least values for the model selection criteria. This suggests that EOLxPL outperforms KwTLL, OLxLL and BXLx distributions.

TABLE 3
Model comparison for turbochargers failure ($10^3 h$).

Model	Parameter estimates	-2ℓ	AIC	BIC	K-S	A-D
EOLxPL	$\alpha = 0.483$ (0.436) $\beta = 0.151$ (0.073) $\theta = 0.474$ (0.777) $\tau = 17.419$ (2.815) $\sigma = 9.487$ (0.755)	155.998	165.998	174.442	0.052	0.098
KwTLL	$\lambda = 3.362$ (0.294) $\alpha = 18.734$ (0.022) $\beta = 32.956$ (0.002) $a = 264.060$ (0.001) $b = 32.908$ (0.030)	166.217	176.217	184.661	0.115	0.783
OLxLL	$\alpha = 19.867$ (0.002) $\theta = 0.007$ (0.007) $\lambda = 59.777$ (0.001) $\sigma = 3.701$ (0.509)	165.861	173.861	180.616	0.109	0.692
BXLx	$\alpha = 0.008$ (0.043) $\beta = 0.001$ (0.048) $\theta = 1.556$ (0.032)	164.906	170.905	175.973	0.110	0.110

Figure 6 shows the histogram of the turbocharger failures with the densities of the fitted distributions. The plot shows that the EOLxPL distribution tend to describe the shape of the data better than the KwTLL, OLxLL and BXLx distributions.

Figure 7 also shows the P-P plots for the fitted distributions. The plots show that the EOLxPL distribution best describes the data.

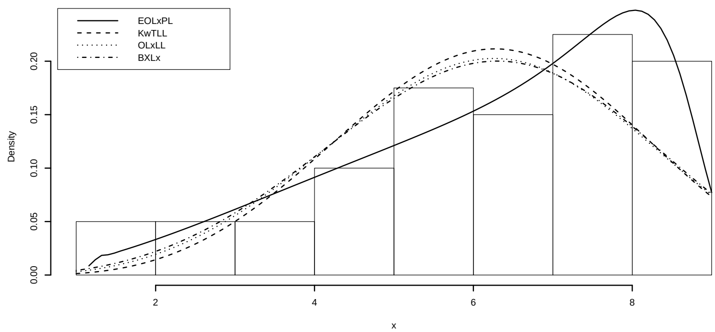

Figure 6 – The estimated densities of EOLxPL, KwTLL, OLxLL and OP distributions.

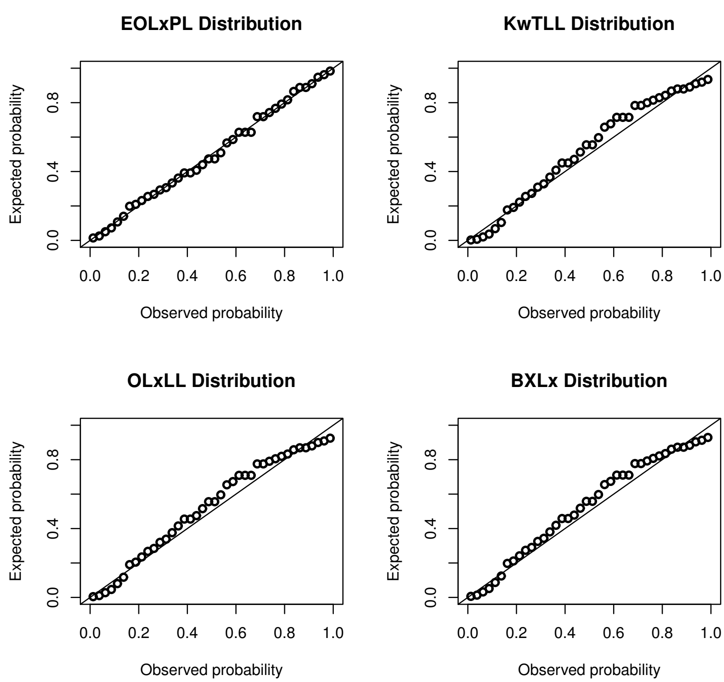

Figure 7 – P-P plots of EOLxPL, KwTLL, OLxLL and OP distributions.

8.2. Application 2: Fatigue time of 101 6061-T6 aluminum coupons

The second data set consists of the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second. The data set has been analyzed my many authors and can be found in Birnbaum and Saunders (1969). The performance of EOLxLx is compared with beta exponentiated Lomax (BELx) (Mead, 2016), extended odd Fréchet-Nadarajah-Haghighi (EOFNH) (Nasiru, 2018) and the odd Lomax-Weibull (OLxW) (Cordeiro *et al.*, 2019) distributions. The BELx, EOFNH and OLxW density functions are given respectively as follows:

$$f(x)=\frac{\beta\theta\lambda}{B(a,b)}(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{a\beta-1}\left\{1-\left[1-(1+\lambda x)^{-\theta}\right]^{\beta}\right\}^{b-1},$$
$$x>0,a>0,b>0,\lambda>0,\theta>0,\beta>0,$$

$$f(x)=\frac{\alpha\beta\lambda\theta(1+\lambda x)^{\beta-1}e^{\left(1-(1+\lambda x)^{\beta}\right)}e^{-\left[\left(1-e^{\left(1-(1+\lambda x)^{\beta}\right)}\right)^{-\alpha}-1\right]^{\theta}}}{\left(1-e^{\left(1-(1+\lambda x)^{\beta}\right)}\right)^{\alpha\theta+1}\left[1-\left(1-e^{\left(1-(1+\lambda x)^{\beta}\right)}\right)^{\alpha}\right]^{1-\theta}},$$
$$x>0,\alpha>0,\beta>0,\lambda>0,\theta>0,$$

and

$$f(x)=\frac{\alpha\theta^{\alpha}\sigma\lambda^{\sigma}x^{\sigma-1}e^{-\lambda x^{\sigma}}}{[1-(1-e^{-\lambda x^{\sigma}})]^{2}}\left[\theta+\frac{1-e^{-\lambda x^{\sigma}}}{1-(1-e^{-\lambda x^{\sigma}})}\right]^{-(\alpha+1)}.$$
$$x>0,\alpha>0,\theta>0,\lambda>0,\sigma>0.$$

The maximum likelihood estimates of EOLxLx, BELx, EOFNH and OLxW distributions with their corresponding standard errors in brackets and model selection criteria are shown Table 4. It can be observed that the EOLxLx distribution has the least of the measures. This indicates that the EOLxLx distribution outperforms the BELx, EOFNH and OLxW distributions in modeling the data.

The histogram of the data with the densities of the EOLxLx, BELx, EOFNH and OLxW distributions are shown in Figure 8. It can be observed that the EOLxLx distribution fits the data better than the other distributions.

Again, the performance of the distributions is illustrated with P-P plots of the fitted distributions shown in Figure 9. It can be observed that EOLxLx describes the data better than the other competing distributions.

TABLE 4
Model comparison for fatigue life of 6061-T6 aluminum coupons data.

Model	Parameter estimates	-2ℓ	AIC	BIC	K-S	A-D
EOLxLx	$\alpha = 3.524$ (2.529) $\beta = 46.407$ (0.150) $\theta = 0.034$ (0.061) $\delta = 2.501$ (0.227) $\sigma = 88.349$ (0.036)	927.948	937.948	951.073	0.065	0.295
BELx	$a = 4.111$ (0.282) $b = 15.075$ (0.081) $\lambda = 0.008$ (0.002) $\theta = 2.846$ (0.489) $\beta = 11.346$ (0.100)	930.776	940.776	953.901	0.082	0.658
EOFNH	$\alpha = 1.500$ (0.316) $\beta = 1.100$ (0.286) $\lambda = 0.007$ (0.003) $\theta = 2.600$ (0.372)	956.506	964.506	975.005	0.139	3.303
OLxW	$\alpha = 0.896$ (0.233) $\theta = 84.688$ (0.001) $\lambda = 0.004$ (0.003) $\sigma = 1.464$ (0.142)	954.073.01	962.073	972.573	0.152	4.505

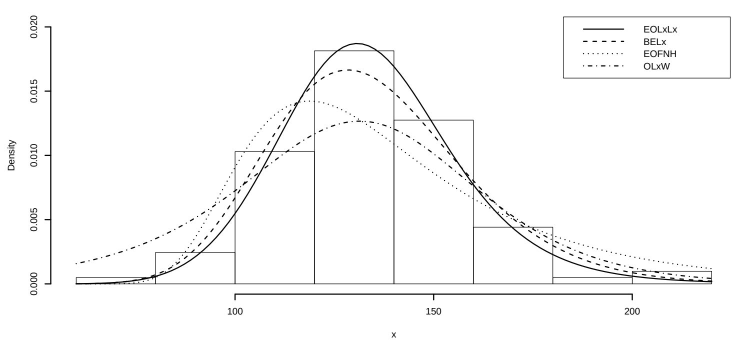

Figure 8 – The estimated densities of EOLxPL, KwTLL, OLxLL and OP distributions.

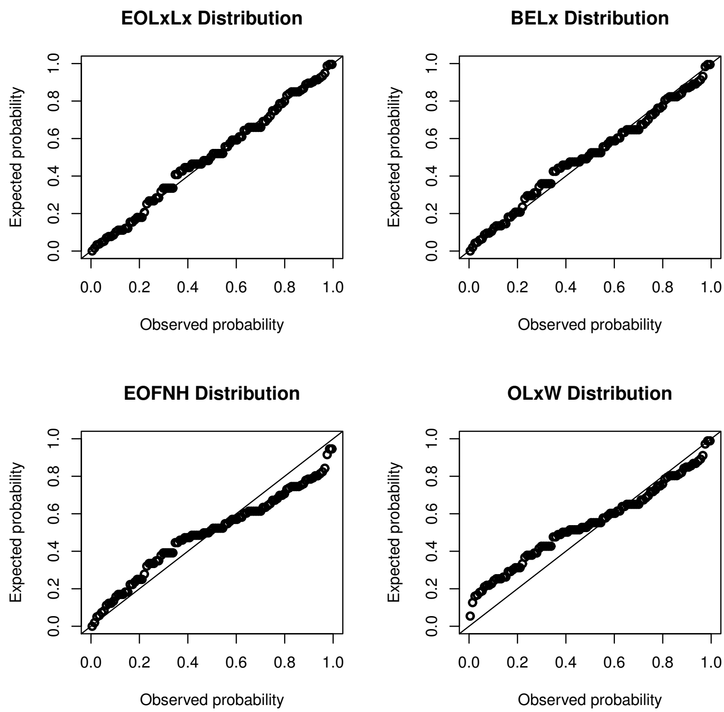

Figure 9 – P-P plots of EOLxPL, KwTLL, OLxLL and OP distributions.

9. Conclusion

A new family of distributions called the extended odd Lomax family of distributions is proposed in this study with two extra shape parameters and one scale parameter. For a baseline distribution $G(x, \omega)$, several special distributions are derived and the shapes of their density and hazard rate functions obtained to illustrate the flexibility of the new family of distributions. Various statistical properties of the family of distributions are also derived. The parameters of the family of distributions are obtained using maximum likelihood method of estimation. The consistency of the maximum likelihood estimators are investigated via Monte Carlo simulations. The results show that the estimators exhibit desirable properties of estimators with increasing sample size. Two special distributions of the family of distributions are applied to two real datasets to illustrate the flexibility and usefulness of the new family of distributions. The results reveal that the distributions perform better and adequately describe the datasets.

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Summary

The Lomax distribution has a wide range of applications. Due to this, it has had many extensions to render it more flexible and useful to model real world data. In this study, a new family of distributions called the extended odd Lomax family of distributions is introduced by adding two extra shape parameters and one scale parameter. We derived several statistical properties of the new family of distributions. The parameters of the family of distributions are estimated by the use of maximum likelihood method and the consistency of the estimators investigated via Monte Carlo simulations. The usefulness and flexibility of the new family of distributions are illustrated by the use of two real datasets. The results show that the distributions adequately describe the datasets.

Keywords: Odd Lomax distribution; Family of distributions; Quantile function.