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A NOTE ON ESTIMATION OF STRESS-STRENGTH RELIABILITY UNDER GENERALIZED UNIFORM DISTRIBUTION WHEN STRENGTH STOCHASTICALLY DOMINATES STRESS

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1. Introduction

Every system or device is subjected to one or more stresses, which cause the system to breakdown when it is not strong enough to withstand the stresses. In the case of a single stress Y acting on a system with strength X, $P(X > Y)$ defines the stress-strength reliability of the system, where both X and Y are generally random variables. This probability finds usefulness in many other areas of research, like psychology, economics, medicine, biometry, environmental risk assessment, etc.

The problem of estimating stress-strength reliability plays an important role in reliability analysis, and it has been addressed by many authors. A few of the studies are as follows. Tong *et al.* (1974) obtained the minimum variance unbiased estimator of $P(X > Y)$ in a closed form when X and Y are independently exponentially distributed. McCool (1991) studied the problem of estimating the probability when X and Y are independent Weibull variables having the same, but unknown, shape parameter. Pal *et al.* (2005) investigated the minimum variance unbiased estimator of the probability when X and Y follow independent two parameter exponential distributions, and also discussed tests for stress-strength reliability. Ali *et al.* (2005) considered independent generalized uniform distributions for stress and strength and obtained the minimum variance unbiased estimator of the reliability. Problem of Bayesian and non-Bayesian estimation of stress-strength reliability when X and Y are independent random variables having Gompertz distributions have been considered by Saraçoglu *et al.* (2009). Saraçoğlu *et al.* (2012) dealt with the estimation of the stress-strength reliability, when X and Y are independent exponential random variables, and the data obtained from both distributions are progressively type-II censored. Valiollahi *et al.* (2013) estimated

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$P(Y < X)$ for X and Y following Weibull distributions, using progressive type - II censored data. Rezaei *et al.* (2015) derived the maximum likelihood estimator and Bayes estimator from progressively censored samples for the stress-strength reliability under independent Pareto distributions of the strength and stress. Jia *et al.* (2017) discussed the problem of Bayes estimation of $P(Y < X)$ for the Weibull distribution with arbitrary parameters. Iranmanesh *et al.* (2018) studied estimation of stress strength reliability when stress and strength have inverted gamma distributions, while Bashir *et al.* (2019) addressed the problem of estimating stress-strength reliability in single component models under different distributions. Jana *et al.* (2019) studied the problem of estimating the stress–strength reliability when the stress and strength follow two-parameter exponential distributions having different location parameters but a common scale parameter. Nadeb *et al.* (2019) considered estimation of stress-strength reliability under exponentiated Fréchet distribution based on Type-II censored data. The ten frequentist estimation methods have been discussed by Almarashi *et al.* (2020) for estimating $P(Y < X)$ when X and Y are independent Weibull distributions with the same shape parameter. Kundu *et al.* (2020) addressed the problem of estimating stress-strength reliability under two parameter exponential distributions with common location parameter. Inference about the stress-strength reliability for the two-parameter exponential distribution using generalized order statistics has been studied by Jafari and Bafekri (2021).

In order to have at least a considerable value of the stress-strength reliability it is desirable that the strength of the system be stochastically higher than the stress on it. In this note we assume strength X and stress Y to be independently distributed, each following a generalized uniform distribution, and X stochastically dominates Y. The aim of the note is to find suitable estimators of the reliability. It may be noted that the uniform distribution, and also its generalization, has applicability in practice. For design of anchorage regions in traditional set-up, the stress on the anchorage length is commonly assumed to be uniformly distributed. Further, force of water flow, stress on venting valve, etc. may have generalized uniform distributions.

The note is organized as follows. In Section 2, the stress-strength reliability ξ is deduced. In Section 3, uniformly minimum variance unbiased estimator (UMVUE) of the reliability is obtained, and the UMVUE of its variance is indicated in Section 4. In Section 5, a consistent estimator of ξ is proposed, which has lower mean-squared error (MSE) as compared to the UMVUE of ξ. A simulation study is carried out in Section 6, and real life data have been analyzed in Section 7. Finally, in Section 8, we give a short discussion on our findings.

2. Stress-strength reliability

The density function of a generalized uniform distribution with shape parameter α and threshold parameter θ is given by

$$f(x) = \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha}, \quad 0 < x < \theta, \quad \alpha > -1 \quad \text{and} \quad \theta > 0. \tag{1}$$

The distribution may be denoted by $\mathrm{GU}(\alpha,\theta)$.

This distribution was introduced by Tiwari *et al.* (1996), who obtained Bayes estimates of the parameters of the Pareto distribution using generalized uniform distribution. It reduces to a uniform distribution over $(0,\theta)$ when $\alpha=0$ and to a standard power function distribution for $\theta=1$.

Suppose the strength X of a system is distributed as $\mathrm{GU}(\alpha_1,\theta_1)$ and it is independent of the stress Y acting on it, where Y follows $\mathrm{GU}(\alpha_2,\theta_2)$ distribution. We assume that X stochastically dominates Y. This means that

$$\bar{F}_X(x)\geq \bar{F}_Y(x), \tag{2}$$

for all x, with strict inequality for some x and where $\bar{F}_X$ and $\bar{F}_X$ denote the reliability functions of X and Y, respectively. In the present case, Eq.(2) is equivalent to

$$\left(\frac{x}{\theta_1}\right)^{\alpha_1+1}\leq\left(\frac{x}{\theta_2}\right)^{\alpha_2+1}, \tag{3}$$

for all x and with strict inequality for some x.

For simplicity sake, let us assume that the distributions of X and Y have the same shape parameters, i.e., $\alpha_1=\alpha_2=\alpha$. Then, Eq.(3) is achieved by taking $\theta_1>\theta_2$ and the stress-strength reliability is obtained as

$$\xi=1-\frac{1}{2}\rho^{(\alpha+1)},\quad \rho=\frac{\theta_2}{\theta_1}. \tag{4}$$

Clearly, $\xi>\frac{1}{2}$. Our aim is to estimate ξ. We shall assume that α is known.

3. UMVUE OF ξ

Let $(X_1,X_2,\ldots,X_n)$ and $(Y_1,Y_2,\ldots,Y_n)$ be independent random samples of strength of the system and stress on it, respectively. Let, $X_{(n)}$ and $Y_{(n)}$ denote, respectively, the n-th order statistics in the samples on strength and stress. Then, $X_{(n)}\sim\mathrm{GU}(n(\alpha+1)-1,\theta_1)$ and $Y_{(n)}\sim\mathrm{GU}(n(\alpha+1)-1,\theta_2)$.

Let, $U=\max(X_{(n)},Y_{(n)})$ and $V=\min(X_{(n)},Y_{(n)})$. In order to find the UMVUE of ξ, we make use of the following proposition:

PROPOSITION 1. *(U,V) is complete sufficient for (θ_1,θ_2).*

PROOF. Keeping in mind that for $\theta_1>\theta_2$, we can write the joint density of $(X_1,X_2,\ldots,X_n)$ and $(Y_1,Y_2,\ldots,Y_n)$ as

$$f_{X,Y}(x,y)=\frac{(\alpha+1)^{2n}}{\theta_1^{n(\alpha+1)}\theta_2^{n(\alpha+1)}}\prod_{i=1}^{n}x_i^{\alpha}\prod_{i=1}^{n}y_i^{\alpha}I_{(u,\theta_1)}I_{(v,\theta_2)}, \tag{5}$$

where

$$I_{(u,\theta_1)} = \begin{cases} 1, & \text{if } u < \theta_1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

and

$$I_{(v,\theta_2)} = \begin{cases} 1, & \text{if } v < \theta_2 \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Then, by Fisher-Neyman Factorization Theorem, (U,V) is sufficient for (θ_1,θ_2). To show completeness, we note that the joint density of U and V is given by

$$f_{U,V}(u,v) = \begin{cases} 2n^2(\alpha+1)^2\left(\frac{uv}{\theta_1\theta_2}\right)^{n(\alpha+1)-1}, & \text{for } 0 < v < u < \theta_2 \\ n^2(\alpha+1)^2\left(\frac{uv}{\theta_1\theta_2}\right)^{n(\alpha+1)-1}, & \text{for } 0 < v < \theta_2 < u < \theta_1. \end{cases} \tag{8}$$

Writing $\underline{\theta} = (\theta_1,\theta_2)$, for any real-valued function $g(u,v)$ of u and v we have

$$E_{\underline{\theta}}[g(U,V)] = 0, \text{ for all } \theta_1,\theta_2 > 0 \text{ and } \theta_1 > \theta_2. \tag{9}$$

Therefore,

$$2\int_0^{\theta_2}\int_0^{u} g(u,v)(uv)^{n(\alpha+1)-1}\mathrm{d}v\mathrm{d}u + \int_{\theta_2}^{\theta_1}\int_0^{\theta_2} g(u,v)(uv)^{n(\alpha+1)-1}\mathrm{d}v\mathrm{d}u = 0, \tag{10}$$

$$\int_0^{\theta_2} g(\theta_1,v)v^{n(\alpha+1)-1}\mathrm{d}v = 0, \tag{11}$$

$$g(\theta_1,\theta_2) = 0, \tag{12}$$

for all $\theta_1,\theta_2 > 0$ and $\theta_1 > \theta_2$. Then,

$$g(u,v) = 0, \text{ for all } u,v > 0 \text{ and } u > v. \tag{13}$$

Hence, (U,V) is a complete statistic for $\underline{\theta}$. □

By Lehmann-Scheffé Theorem, any function of the complete sufficient statistic for $\underline{\theta}$, which is an unbiased estimator of $\xi \equiv \xi(\underline{\theta})$, will be the UMVUE of ξ.

We obtain UMVUE of ξ from the following Theorem:

THEOREM 2. *The UMVUE of ξ is given by*

$$\hat{\xi} = \begin{cases} 1 - C_1\left(\frac{V}{U}\right)^{\alpha+1}, & \text{if } U = X_{(n)} \\ 1 - C_2\left(\frac{V}{U}\right)^{\alpha+1}, & \text{if } U = Y_{(n)}, \end{cases} \tag{14}$$

where $C_1 = \frac{n^2-1}{2n^2}$ *and* $C_2 = \frac{(n+1)^2}{2n^2}$.

PROOF. It is sufficient to show that $\hat{\xi}$ is an unbiased estimator of ξ. We have

$$\begin{aligned} E(\hat{\xi}) &= 1-C_1\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\Bigg[\int_0^{\theta_2}\left(\int_0^x y^{(n+1)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-1)(\alpha+1)-1}\mathrm{d}x \\ &+ \int_{\theta_2}^{\theta_1}\left(\int_0^{\theta_2} y^{(n+1)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-1)(\alpha+1)-1}\mathrm{d}x\Bigg] \\ &- C_2\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\int_0^{\theta_2}\left(\int_0^y x^{(n+1)(\alpha+1)-1}\mathrm{d}x\right)y^{(n-1)(\alpha+1)-1}\mathrm{d}y \\ &= 1-C_1\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\Bigg[\frac{\theta_2^{(n+1)(\alpha+1)}\{\theta_1^{(n-1)(\alpha+1)}-\theta_2^{(n-1)(\alpha+1)}\}}{(n^2-1)(\alpha+1)^2} \\ &+ \frac{\theta_2^{2n(\alpha+1)}}{2n(n+1)(\alpha+1)^2}\Bigg]-C_2\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}.\frac{\theta_2^{2n(\alpha+1)}}{2n(n+1)(\alpha+1)^2} \\ &= 1-C_1\frac{n^2}{n^2-1}\rho^{\alpha+1}-C_1\left[\frac{n}{2(n+1)}-\frac{n^2}{n^2-1}\right]\rho^{n(\alpha+1)} \\ &- C_2\frac{n}{2(n+1)}\rho^{n(\alpha+1)}=1-\frac{1}{2}\rho^{\alpha+1}=\xi. \end{aligned} \tag{15}$$

□

4. UMVUE OF THE VARIANCE OF $\hat{\xi}$

In order to find the UMVUE of $\mathrm{Var}(\hat{\xi})$, we first find the UMVUE of ξ^2 for $n \geq 2$.

THEOREM 3. *The UMVUE of ξ^2 is given by*

$$\begin{aligned} V_2 &= 1-D_1\left(\frac{V}{U}\right)^{(\alpha+1)}+D_3\left(\frac{V}{U}\right)^{2(\alpha+1)}, \text{ if } U=X_{(n)} \\ &= 1-D_2\left(\frac{V}{U}\right)^{(\alpha+1)}+D_4\left(\frac{V}{U}\right)^{2(\alpha+1)}, \text{ if } U=Y_{(n)}, \end{aligned} \tag{16}$$

where $D_1=\frac{n^2-1}{n^2}, D_2=\frac{(n+1)^2}{n^2}, D_3=\frac{n^2-4}{4n^2}$ *and* $D_4=\frac{(n+2)^2}{4n^2}$.

PROOF. By Lehmann-Scheffé Theorem, it is sufficient to show that V_2 is an unbiased estimator of ξ^2.

We have,

$$
\begin{aligned}
E(V_2) &= 1 - D_1 \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \left[\int_0^{\theta_2} \left(\int_0^{x} y^{(n+1)(\alpha+1)-1} \mathrm{d}y \right) x^{(n-1)(\alpha+1)-1} \mathrm{d}x \right. \\
&+ \left. \int_{\theta_2}^{\theta_1} \left(\int_0^{\theta_2} y^{(n+1)(\alpha+1)-1} \mathrm{d}y \right) x^{(n-1)(\alpha+1)-1} \mathrm{d}x \right] \\
&- D_2 \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \int_0^{\theta_2} \left(\int_0^{y} x^{(n+1)(\alpha+1)-1} \mathrm{d}x \right) y^{(n-1)(\alpha+1)-1} \mathrm{d}y \\
&+ D_3 \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \left[\int_0^{\theta_2} \left(\int_0^{x} y^{(n+2)(\alpha+1)-1} \mathrm{d}y \right) x^{(n-2)(\alpha+1)-1} \mathrm{d}x \right. \\
&+ \left. \int_{\theta_2}^{\theta_1} \left(\int_0^{\theta_2} y^{(n+2)(\alpha+1)-1} \mathrm{d}y \right) x^{(n-2)(\alpha+1)-1} \right] \mathrm{d}x \\
&+ D_4 \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \int_0^{\theta_2} \left(\int_0^{y} x^{(n+2)(\alpha+1)-1} \mathrm{d}x \right) y^{(n-2)(\alpha+1)-1} \mathrm{d}y \\
&= 1 - D_1 \frac{n^2}{n^2-1} \rho^{\alpha+1} + D_3 \frac{n^2}{n^2-4} \rho^{2(\alpha+1)} - D_1 \left[\frac{n}{2(n+1)} - \frac{n^2}{n^2-1} \right] \rho^{n(\alpha+1)} \\
&- D_2 \frac{n}{2(n+1)} \rho^{n(\alpha+1)} + D_3 \left[\frac{n}{2(n+2)} - \frac{n^2}{n^2-4} \right] \rho^{n(\alpha+1)} + D_4 \frac{n}{2(n+2)} \rho^{n(\alpha+1)} \\
&= 1 - \rho^{\alpha+1} + \frac{1}{4} \rho^{2(\alpha+1)} = \xi^2. \qquad (17)
\end{aligned}
$$

The UMVUE of $\mathrm{Var}(\hat{\xi})$ is, therefore, given by $\hat{\xi}^2 - V_2$. □

5. Alternative estimator of ξ

It is noted that ξ is greater than $\frac{1}{2}$, as $\theta_1 > \theta_2$. When $U = X_{(n)}$, $\hat{\xi} > \frac{1}{2}$, as $C_1 = \frac{n^2-1}{2n^2} < \frac{1}{2}$ and $\left(\frac{V}{U}\right)^{\alpha+1} < 1$.

However, when $U = Y_{(n)}$, $\hat{\xi} \geq \frac{1}{2}$ for $C_2\left(\frac{V}{U}\right)^{\alpha+1} \leq \frac{1}{2}$, i.e. $X_{(n)} \leq \left(\frac{n}{n+1}\right)^{\frac{2}{\alpha+1}} Y_{(n)}$, while $\hat{\xi} < \frac{1}{2}$ for $C_2\left(\frac{V}{U}\right)^{\alpha+1} > \frac{1}{2}$, i.e.$X_{(n)} > \left(\frac{n}{n+1}\right)^{\frac{2}{\alpha+1}} Y_{(n)}$. This is clear from the expression of $\hat{\xi}$. As such, the estimator may be modified as

$$\begin{aligned}\tilde{\xi} &= 1 - C_1\left(\frac{V}{U}\right)^{\alpha+1}, \text{ if } U = X_{(n)} \\ &= 1 - C_2\left(\frac{V}{U}\right)^{\alpha+1}, \text{ if } U = Y_{(n)} \text{ and } X_{(n)} \le (2C_2)^{-\frac{1}{\alpha+1}} Y_{(n)} \\ &= \frac{1}{2}, \text{ if } U = Y_{(n)} \text{ and } X_{(n)} > (2C_2)^{-\frac{1}{\alpha+1}} Y_{(n)}, \end{aligned} \tag{18}$$

where $C_1 = \frac{n^2-1}{2n^2}$ and $C_2 = \frac{(n+1)^2}{2n^2}$.

But $\tilde{\xi}$ is a biased estimator of ξ. However, we have the following propositions:

PROPOSITION 4. *$\tilde{\xi}$ is a consistent estimator of ξ.*

PROOF. Let, $h = (2C_2)^{-\frac{1}{\alpha+1}} = \left(\frac{n}{n+1}\right)^{\frac{2}{\alpha+1}} < 1$. Then

$$\begin{aligned} E(\tilde{\xi}) &= \xi + C_2 \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \int_0^{\theta_2} \left(\int_{hy}^{y} x^{(n+1)(\alpha+1)-1} \mathrm{d}x \right) y^{(n-1)(\alpha+1)-1} \mathrm{d}y \\ &- \frac{1}{2} \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \int_0^{\theta_2} \left(\int_{hy}^{y} x^{(n+1)(\alpha+1)-1} \mathrm{d}x \right) y^{(n-1)(\alpha+1)-1} \mathrm{d}y \\ &= \xi + \left(1 - h^{(n+1)(\alpha+1)}\right) \left[C_2 \frac{n}{2(n+1)} - \frac{n}{4(n+1)} \right] \rho^{n(\alpha+1)} \\ &= \xi + \left[1 - \left(\frac{n}{n+1}\right)^{2(n+1)} \right] \left(\frac{2n+1}{4n(n+1)} \right) \rho^{n(\alpha+1)} \\ &\to \xi \text{ as } n \to \infty. \end{aligned} \tag{19}$$

Now, $\mathrm{Var}(\tilde{\xi}) = E(\tilde{\xi}^2) - E^2(\tilde{\xi})$. We note that

$$\begin{aligned} \frac{E^2(\tilde{\xi})}{n} &= \frac{1}{n} \left[\xi^2 + \left[1 - \left(\frac{n}{n+1}\right)^{2(n+1)} \right] \left(\frac{2n+1}{2n(n+1)} \right) \rho^{n(\alpha+1)} \xi \right. \\ &+ \left. \left[1 - \left(\frac{n}{n+1}\right)^{2(n+1)} \right]^2 \left(\frac{2n+1}{4n(n+1)} \right)^2 \rho^{2n(\alpha+1)} \right], \end{aligned} \tag{20}$$

which tends to 0 as $n \to \infty$, since $\rho, \xi < 1$. Again,

$$
\begin{aligned}
\frac{E(\tilde{\xi}^2)}{n} &= \frac{1}{n}\Bigg[1-2C_1\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\Big[\int_0^{\theta_2}\left(\int_0^{x} y^{(n+1)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-1)(\alpha+1)-1}\mathrm{d}x \\
&+ \int_{\theta_2}^{\theta_1}\left(\int_0^{\theta_2} y^{(n+1)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-1)(\alpha+1)-1}\mathrm{d}x\Big] \\
&- 2\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\Bigg\{C_2\int_0^{\theta_2}\left(\int_0^{hy} x^{(n+1)(\alpha+1)-1}\mathrm{d}x\right)y^{(n-1)(\alpha+1)-1}\mathrm{d}y \\
&+ \frac{1}{2}\int_0^{\theta_2}\left(\int_{hy}^{y} x^{(n+1)(\alpha+1)-1}\mathrm{d}x\right)y^{(n-1)(\alpha+1)-1}\mathrm{d}y\Bigg\} + C_1^2\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \\
&\times \Bigg\{\int_0^{\theta_2}\left(\int_0^{x} y^{(n+2)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-2)(\alpha+1)-1}\mathrm{d}x \\
&+ \int_{\theta_2}^{\theta_1}\left(\int_0^{\theta_2} y^{(n+2)(\alpha+1)-1}\mathrm{d}y\right)x^{(n-2)(\alpha+1)-1}\mathrm{d}x\Bigg\} \\
&+ \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}}\Bigg\{C_2^2\int_0^{\theta_2}\left(\int_0^{hy} x^{(n+2)(\alpha+1)-1}\mathrm{d}x\right)y^{(n-2)(\alpha+1)-1}\mathrm{d}y \\
&+ \frac{1}{4}\int_0^{\theta_2}\left(\int_{hy}^{y} x^{(n+2)(\alpha+1)-1}\mathrm{d}x\right)y^{(n-2)(\alpha+1)-1}\mathrm{d}y\Bigg\}\Bigg] \\
&= \frac{1}{n}\Bigg[1-2C_1\frac{n^2}{n^2-1}\rho^{\alpha+1}+{C_1}^2\frac{n^2}{n^2-4}\rho^{2(\alpha+1)} \\
&+ \left(\frac{n}{n+1}\right)^{2(n+1)}\left(\frac{2n+1}{2n(n+1)}\right)\rho^{n(\alpha+1)} \\
&+ {C_2}^2\frac{n}{2(n+2)}\rho^{n(\alpha+1)}\times h^{(n+2)(\alpha+1)} + \frac{1-h^{(n+2)(\alpha+1)}}{4}\times\frac{n}{2(n+2)}\rho^{n(\alpha+1)}\Bigg] \\
&= \frac{1}{n}\Bigg[1-\rho^{\alpha+1}+\frac{(n^2-1)^2}{4n^2(n^2-4)}\rho^{2(\alpha+1)}+\left(\frac{n}{n+1}\right)^{2(n+1)}\left(\frac{2n+1}{2n(n+1)}\right)\rho^{n(\alpha+1)} \\
&+ \frac{(n+1)^4}{8n^3(n+2)}\rho^{n(\alpha+1)}\left(\frac{n}{n+1}\right)^{2(n+2)} \\
&+ \frac{n}{8(n+2)}\times\left\{1-\left(\frac{n}{n+1}\right)^{2(n+2)}\right\}\rho^{n(\alpha+1)}\Bigg],
\end{aligned} \tag{21}
$$

which tends to 0 as $n\to\infty$, since $\rho<1$. Hence, $\frac{\mathrm{Var}(\tilde{\xi})}{n}\to 0$ as $n\to\infty$.

Thus, $\tilde{\xi}$ is a consistent estimator of ξ. □

PROPOSITION 5. *$MSE(\tilde{\xi}) < Var(\hat{\xi})$, whatever be $\theta_1 > \theta_2$.*

PROOF.

$$\begin{aligned}
\text{MSE}(\tilde{\xi}) &= \text{E}(\tilde{\xi}-\xi)^2 \\
&= \text{E}\left(\tilde{\xi}-\hat{\xi}\right)^2 + \text{Var}(\hat{\xi}) + 2\text{E}\left[(\tilde{\xi}-\hat{\xi})(\hat{\xi}-\xi)\right] \\
&= \text{Var}(\hat{\xi}) + \frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \\
&\times \int_0^{\theta_2}\left(\int_{hy}^{y}\left\{C_2\left(\frac{x}{y}\right)^{\alpha+1}-\frac{1}{2}\right\}^2 x^{n(\alpha+1)-1}\text{d}x\right)y^{n(\alpha+1)-1}\text{d}y \\
&- 2\frac{n^2(\alpha+1)^2}{(\theta_1\theta_2)^{n(\alpha+1)}} \\
&\times \int_0^{\theta_2}\left(\int_{hy}^{y}\left\{C_2\left(\frac{x}{y}\right)^{\alpha+1}-\frac{1}{2}\right\}\right. \\
&\times \left.\left\{C_2\left(\frac{x}{y}\right)^{\alpha+1}-\frac{1}{2}\rho^{\alpha+1}\right\}x^{n(\alpha+1)-1}\text{d}x\right)y^{n(\alpha+1)-1}\text{d}y \\
&< \text{Var}(\hat{\xi}),
\end{aligned} \tag{22}$$

since for $hy_{(n)} < x_{(n)} < y_{(n)}$,$0 < y_{(n)} < \theta_2$, we get $C_2\left(\frac{x_{(n)}}{y_{(n)}}\right)^{\alpha+1} > \frac{1}{2}$, so that

$$\left\{C_2\left(\frac{x_{(n)}}{y_{(n)}}\right)^{\alpha+1}-\frac{1}{2}\right\}\left\{C_2\left(\frac{x_{(n)}}{y_{(n)}}\right)^{\alpha+1}-\frac{1}{2}\rho^{\alpha+1}\right\} > \left\{C_2\left(\frac{x_{(n)}}{y_{(n)}}\right)^{\alpha+1}-\frac{1}{2}\right\}^2, \tag{23}$$

whatever be $\theta_2 < \theta_1$, as $\rho < 1$.

Thus, $\tilde{\xi}$ is a consistent estimator of ξ, which is better than the UMVUE $\hat{\xi}$ in terms of mean squared error. □

6. SIMULATION STUDY

Consider $X \sim \text{GU}(2,4)$ and $Y \sim \text{GU}(2,3.98)$. To obtain the UMVUE of ξ, random samples of size $n = 20$ are taken on X and Y, respectively. The sample observations are in Table 1.

The stress-strength reliability for the given distributions is $\xi = 0.9975$. From the data we have, $X_{(n)} = 3.9612 < Y_{(n)} = 3.9705$, so that $U = Y_{(n)} = 3.9705$. Therefore, the UMVUE of ξ is $\hat{\xi} = 0.4526 < 0.5$, and its estimated variance is 0.2346. The alternative consistent estimator is $\tilde{\xi} = 0.5$, with estimated MSE $= 0.1167$.

TABLE 1
Sample observations - Case 1.

Variable	Sample observations
X	1.0125 3.4886 3.6466 3.2555 2.5390 3.5855 2.9751 3.9612 1.5243 2.2444 3.7036 3.8322 1.5527 3.4464 2.2802 3.4935 3.8710 3.0522 3.7397 3.0033
Y	3.8059 2.8348 1.4655 3.3005 3.15775 3.5812 2.1519 3.3644 2.6642 2.5573 3.2125 2.5011 2.5125 2.5130 3.9705 3.7078 2.9588 1.9375 2.3406 3.3837

Now suppose $Y \sim \mathrm{GU}(2,3)$ and the random observations on Y are those reported in Table 2. Let us consider the distribution of X to be same as before, and the random sample on X be also that used earlier.

TABLE 2
Sample observations - Case 2.

Variable	Sample observations
Y	1.9384 2.7673 2.8330 2.8620 0.8226 2.0712 0.9885 1.4046 1.4960 2.5452 2.2953 2.8093 2.0824 2.0459 1.2570 2.0826 1.8152 1.7058 1.9652 2.8359

In this case, the true stress-strength reliability is $\xi = 0.9373$. The data give $X_{(n)} = 3.9612 > Y_{(n)} = 2.862012$, so that $U = X_{(n)} = 3.9612$. Therefore, the UMVUE of ξ is $\hat{\xi} = 0.811889$, with estimated variance 0.00078.

7. Data analysis

In this section, we analyse real life data sets for the purpose of illustration. We consider the breaking strengths of jute fiber at two different gauge lengths, namely, 10 mm and 20 mm. The data have been taken from Xia *et al.* (2009) and are presented in Table 3 and Table 4, where Z denotes the breaking strength of jute fiber of gauge length 10 mm and W is the breaking strength of gauge length 20 mm.

TABLE 3
Breaking strength of jute fiber of gauge length 10 mm (Z).

Variable	Breaking strength
Z	693.73 704.66 323.83 778.17 123.06 637.66 383.43 151.48 108.94 50.16 671.49 183.16 257.44 727.23 291.27 101.15 376.42 163.40 141.38 700.74 262.90 353.24 422.11 43.93 590.48 212.13 303.90 506.60 530.55 177.25

TABLE 4
Breaking strength of jute fiber of gauge length 20 mm (W).

Variable	Breaking strength
W	71.46 419.02 284.64 585.57 456.60 113.85 187.85 688.16 662.66 45.58 578.62 756.70 594.29 166.49 99.72 707.36 765.14 187.13 145.96 350.70 547.44 116.99 375.81 581.60 119.86 48.01 200.16 36.75 244.53 83.55

As the gauge length has a negative effect on the breaking strength, one would be interested to estimate $\xi = P(Z > W)$. It may be noted that, because of the negative effect, one may assume that $\bar{F}_Z(x) \geq \bar{F}_W(x)$, $\forall x$, that is, Z stochastically dominates W. Further, the breaking strength of fibre for any gauge length will be bounded below.

Consider one-to-one transformations as follows:

$$\begin{aligned} X &= \exp(-W/100) \\ Y &= \exp(-Z/100). \end{aligned} \tag{24}$$

Then, $\xi = P(Z > W) = P(X > Y)$. Further,

$$\bar{F}_Z(x) \geq \bar{F}_W(x),\ \forall x \iff \bar{F}_X(w) \geq \bar{F}_Y(w),\ \forall w, \tag{25}$$

which implies that X stochastically dominates Y. Further, since W and Z are bounded below, X and Y will be bounded above.

We take $\alpha = -0.74$. So, assuming $X \sim \mathrm{GU}(-0.74, \theta_1)$ and $Y \sim \mathrm{GU}(-0.74, \theta_2)$, we obtain the MLEs of θ_1 and θ_2 as $\hat{\theta}_1 = X_{(n)} = 0.6924$, $\hat{\theta}_2 = Y_{(n)} = 0.6445$. The P-P plots are in Figure 1.

From Kolmogorov-Smirnov test (test statistic $= 0.11716$ for (a), test statistic $= 0.11489$ for (b) in Figure 1, and in each case p-value $= 0.240$), we have reasons to believe that the transformed data sets fit the corresponding theoretical distributions.

Since, $U = \max\{X_{(n)}, Y_{(n)}\} = X_{(n)} = 0.6924$, we obtain the UMVUE of ξ as $\hat{\xi} = 0.5098$, with its estimated variance 0.00053.

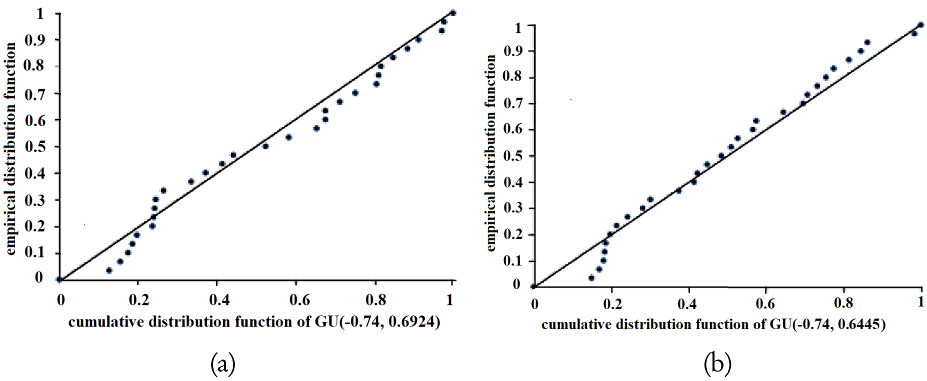

Figure 1 – P-P plots of the transformed data sets.

8. Discussion

This note investigates the problem of estimating the stress-strength reliability of a system when strength and stress have independent generalized uniform distributions with same shape parameter, but unequal threshold parameters. Further, the strength of the system is known to stochastically dominate the stress on it, which indicates a lower bound, namely 0.5, to the stress-strength reliability. UMVU estimator of the reliability has been worked out, but it does not satisfy the lower bound for certain values of the highest order statistics of the random samples taken from the strength and stress distributions. A modification of the estimator to satisfy the lower bound leads to a consistent estimator of the reliability, which is found to have lower mean squared error as compared to the UMVU estimator. The case of both shape and threshold parameters unknown, but with stochastic dominance of strength distribution, will be considered in a future study. It is expected to be rather complex.

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SUMMARY

In this note we find the UMVUE and a consistent estimator of the stress strength reliability of a system, whose strength stochastically dominates the stress. Strength of the system and the stress on it are assumed to be independently distributed, each having a generalized uniform distribution. Simulation study has been carried out, and an application to real life data has also been cited.

Keywords: Stress-strength reliability; Generalized uniform distribution; Stochastic dominance; Estimation.