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UNIVARIATE DISCRETE NADARAJAH AND HAGHIGHI DISTRIBUTION: PROPERTIES AND DIFFERENT METHODS OF ESTIMATION

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1. INTRODUCTION

Researchers in many fields regularly encounter variables that are discrete in nature. Thus, continuous models may not be appropriate or suitable when data are discrete. For example, the number of rounds fired from a weapon till the first failure, the number of deaths at a given place over a given time period, the number of cycles prior to the first failure when devices work in cycles, the marks obtained by students in an examination, the number of forest fires during a given period of time, the number of breakdowns of a computer, the length of stay (usually measured as number of days) in an observation ward of leukemia patients, etc. In all these cases, the lifetimes are not measured on continuous scale but are simply counted and hence are discrete random variables. During the last few decades, several continuous lifetime distributions have been proposed, and some of them have been extensively studied and modified. However, studies on discrete distributions are comparatively less than continuous distributions and therefore more studies are required in this area. The discretization of a continuous lifetime distribution retains the same functional form of the survival function, therefore, many reliability characteristics and properties shall remain unchanged. Thus, discretization of a continuous lifetime model is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one.

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During the last two decades, many continuous distributions have been discretized. For example, Nakagawa and Osaki (1975) proposed the discrete Weibull distribution, Nekhoukhou *et al.* (2013) proposed a new discrete version of the generalized exponential distribution (Gupta and Kundu, 1999) and compared it to Nekoukhou *et al.* (2011), Bakouch *et al.* (2014) and Nekoukhou *et al.* (2015) suggested a discrete version of the beta-exponential distribution (Nadarajah and Kotz, 2006), Para and Jan (2017) proposed discrete analogue of generalized Weibull distribution. Alamatsaz *et al.* (2016) proposed discrete generalized Rayleigh distribution and the references cited therein.

This paper introduces a new discrete probability model from the extended exponential distribution introduced by Nadarajah and Haghighi (2011) and provides a detailed study of its properties. The survival function of the Nadarajah and Haghighi (NH) distribution is:

$$S(t)=\exp(1-\lambda(1+t)^{\alpha}), \tag{1}$$

where $\alpha, \lambda > 0$ are the shape and the scale parameters of the NH distribution. The resulting cumulative distribution function (CDF), probability distribution function (PDF) and hazard rate function (hrf) are given as follows:

$$F(t)=1-\exp(1-(1+\lambda t)^{\alpha}), \tag{2}$$

$$f(t)=\alpha\lambda(1+\lambda t)^{\alpha-1}\exp(1-(1+\lambda t)^{\alpha}), \tag{3}$$

$$h(t)=\alpha\lambda(1+\lambda t)^{\alpha-1}. \tag{4}$$

The hrf of NH distribution has a closed form (4) as in the case of Weibull distribution and the generalized exponential distribution. Eq. (3) reduces to the exponential distribution for $\alpha = 1$, and has an interesting property of having zero mode and allowing increasing, decreasing and constant hrfs.

Discretization of continuous distribution can be done using different methodologies. For example, Nekhoukhou *et al.* (2013) used a technique to convert a continuous distribution into discrete analogue. To this end, for any continuous distribution on $\Re^{+}=[0,+\infty)$ with f(t), one can construct a discrete counterpart supported on the set of integers $N_0 = 0, 1, 2, \cdots$ whose probability mass function (pmf) is of the form

$$P_t = P(T=t) = s(t)-s(t+1), \qquad t=0,1,2,3,\cdots, \tag{5}$$

where $s(t)$ is the survival function of $f(t)$. Substituting (1) in (5), the pmf of the resulting discrete NH distribution is given by

$$P(T=t)=\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}}-\theta^{(\frac{1}{\lambda}+t+1)^{\alpha}}), \qquad t=0,1,2,3,\cdots, \tag{6}$$

where $\theta=\exp(-\lambda)^{\alpha}$ and $0<\theta<1$. The new distribution is named as the discrete NH (DNH) distribution denoted by DNH(α,λ,θ) with parameters $\alpha,\lambda>0$ and $0<\theta<1$.

The objective of this article is twofold: First, we introduce a new three-parameter discrete distribution, with the aim that the new distribution will produce a better fit compared to the Poisson, negative binomial, and zero inflated Poisson distributions in certain practical situations. Additionally, we will provide a comprehensive details of the mathematical properties of the proposed new distribution. Second, we will estimate the parameters of the model using eight different methods of estimation and to provide a guideline for choosing the best estimation method for the DNH distribution, which we think would be of deep interest to applied statisticians. Also, to the best of our knowledge, no attempt have been made to compare all these estimators for the DNH distribution along with mathematical and statistical properties. Additionally, all results are new, unless otherwise indicated. Other attractive features of the DNH distribution include heavy tail and closed forms for its cumulative distribution function (CDF) and hazard rate function.

The rest of the paper is organized as follows: In Section 2, we investigate some of its mathematical properties such as the quantile function, hazard rate behavior, Lorenz and Bonferroni curves, order statistics, stochastic ordering, mean, and variance. Estimation of the parameters of the DNH distribution is discussed in Section 3 using eight different estimation methods. A simulation study is carried out to assess the performance of different estimation methods in Section 4. Two applications of the DNH distribution and comparison with the Poisson, negative binomial and zero inflated Poisson distributions are presented in Section 5. Finally, some concluding remarks are given in Section 6.

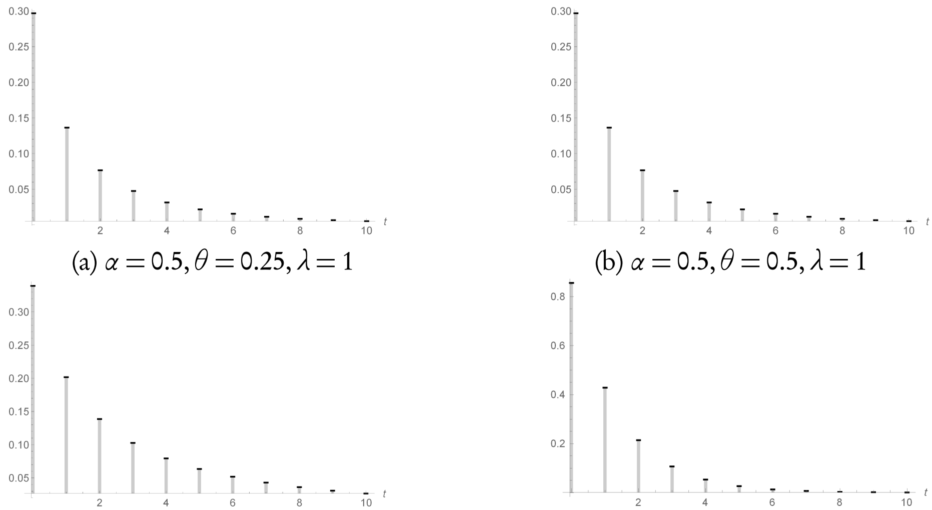
2. Properties of the DNH distribution

This section discusses some of the mathematical quantities of the DNH distribution, including the shape of the distribution, mean and variance, quantile function, Bonferroni and Lorenz curves, order statistics and stochastic ordering.

2.1. Shape of the DNH distribution

In Figure 1, PMF (6) of the $DNH(\alpha, \lambda, \theta)$ for different values of α, λ and θ has been depicted.

In Figure 1(a) and Figure 1(b), the shape parameters α are assumed fixed and $\theta = 0.25$ and 0.5 are used respectively. It is observed that the frequencies remain same regardless of the choice of θ, but, when we increase the value of α and decrease the values of λ and θ, the frequencies also increased. Contrary to this, the frequencies decreases by decreasing the value of α and increasing the values of λ and θ as shown in Figure 1(c) and Figure 1(d) .



(a) $\alpha = 0.5, \theta = 0.25, \lambda = 1$ (b) $\alpha = 0.5, \theta = 0.5, \lambda = 1$

(c) $\alpha = 5, \theta = 0.01, \lambda = 0.05$ (d) $\alpha = 1, \theta = 0.5, \lambda = 0.5$

Figure 1 – Comparison of the shape of DNH distribution for different values of α, θ and λ.

2.2. *The survival and hazard rate functions*

The survival and hazard rate functions of the DNH(α, λ, θ) distribution are given as

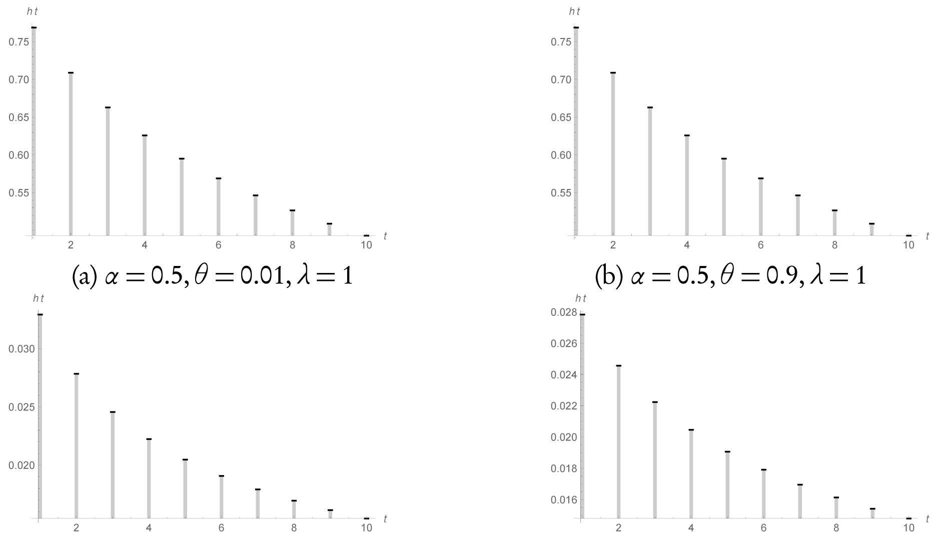
$$s(t) = \exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}}), \qquad t > 0, \tag{7}$$

and

$$h(t) = \frac{\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}}}{\theta^{(\frac{1}{\lambda}+t)^{\alpha}}}, \qquad t > 0, \tag{8}$$

respectively. Figure 2 depicts the hazard rate function of the DNH(α, λ, θ) distribution for different values of α, λ and θ.

For different choices of the parameters of the DNH(α, λ, θ) distribution, the hazard function has different frequencies. From Figure 2(a) and Figure 2(b), clearly, one can see that for different values of the parameter θ, frequencies remain the same. However, Figure 2(c) and Figure 2(d) show that the hazard function changes with the values of α and λ, i.e., the hazard rate decreases by increasing the values of α and λ.



(a) $\alpha = 0.5, \theta = 0.01, \lambda = 1$ (b) $\alpha = 0.5, \theta = 0.9, \lambda = 1$

(c) $\alpha = 0.5, \theta = 0.01, \lambda = 0.5$ (d) $\alpha = 5, \theta = 0.9, \lambda = 0.5$

Figure 2 – Comparison of the hazard function of DNH distribution assuming different values of α, λ and θ.

2.3. *Mean and variance of the distribution*

The mean and the variance of $DNH(\alpha, \lambda, \theta)$ distribution are given by

$$E(T) = \sum_{t=0}^{\infty} t \exp(1)(\theta^{(\frac{1}{\lambda}+t)^\alpha} - \theta^{(\frac{1}{\lambda}+t+1)^\alpha}), \tag{9}$$

and

$$\mathrm{Var}(T) = \sum_{t=0}^{\infty} t^2 \exp(1)(\theta^{(\frac{1}{\lambda}+t)^\alpha} - \theta^{(\frac{1}{\lambda}+t+1)^\alpha}) - (\sum_{t=0}^{\infty} t \exp(1)(\theta^{(\frac{1}{\lambda}+t)^\alpha} - \theta^{(\frac{1}{\lambda}+t+1)^\alpha}))^2, \tag{10}$$

respectively. If $\alpha > 0$ is an integer value, ∞ in sum expressions can be replaced by α, i.e., in (9), (10) the notation $\sum_{t=0}^{\infty}$ should be replaced by $\sum_{t=0}^{\alpha-1}$ (Nekhoukhou *et al.*, 2013). Since Eqs. (9) and (10) cannot be further simplified, the mean and the variance of $DNH(\alpha, \lambda, \theta)$ distribution are calculated numerically in Table 1.

It is observed that the mean and variance decrease by increasing the value of parameter α, whereas increase for values of λ. Further, the mean and variance increase by increasing the value of θ, for instance when $\lambda = 1$, $\alpha = 0.5$ and $\theta = 0.1$ the mean and variance are 0.2274 and 0.0517, respectively. However, if $\alpha = 0.5$ and $\theta = 0.5$, the mean and variance are 8.6068 and 74.07, respectively.

TABLE 1
Mean and variance of the DNH distribution for different values of α, λ and θ.

α/θ	0.1	0.2	0.5	0.7
		$\lambda=0.5$		
0.5	0.123 (0.542)	0.579 (4.031)	7.598 (131.760)	25.413 (415.461)
1	0.003(0.004)	0.027(0.040)	0.679 (1.577)	3.108 (7.952)
1.5	$1.0\times10^{-4}(1.3\times10^{-4})$	0.001 (0.001)	0.869 (0.105)	0.652 (0.869)
2	2.7×10^{-9} (2.7×10^{-9})	1.3×10^{-6} (1.3×10^{-6})	0.005 (0.005)	0.119 (0.124)
		$\lambda=1$		
0.5	0.227 (0.052)	0.859 (0.737)	8.607 (74.070)	26.851 (720.900)
1	0.030 (0.001)	0.136 (0.018)	1.359 (1.847)	4.439 (19.712)
1.5	0.004 (1.0×10^{-5})	0.029 (0.001)	0.469 (0.219)	1.643 (2.701)
2	3.0×10^{-4} (7.4×10^{-8})	0.004 (2.0×10^{-5})	0.175 (0.031)	0.772 (0.596)
		$\lambda=1.5$		
0.5	0.288 (1.005)	0.996 (5.444)	9.007 (129.318)	27.380 (364.448)
1	0.065 (0.075)	0.232 (0.295)	1.712 (2.205)	5.000 (3.332)
1.5	0.019 (0.019)	0.088 (0.085)	0.768 (0.545)	2.164 (0.227)
2	0.005 (0.004)	0.031 (0.030)	0.416 (0.283)	1.248 (0.218)

2.4. *Quantile function*

The quantile function plays an important role in probability distribution theory and it is also known as the percent-point function or the inverse cumulative distribution function. It is used to generate random number from a given distribution. The quantile function of the DNH distribution is given by

$$Q(p)=F^{-1}(p)=-\frac{1}{\lambda}+\left(\frac{\log(1-p)}{\log\theta}-\frac{1}{\log\theta}\right)^{\frac{1}{\alpha}}, \tag{11}$$

where $0<p<1$.

2.5. *Bonferroni and Lorenz curves*

The Bonferroni curve (BC) and Lorenz curve (LC) are also known as the income inequalities and are commonly used in economics to describe the wealth of a nation. For a random variable T, with PMF and CDF $p(t)$ and $F(t)$ respectively and quantile function $F^{-1}(p)$, the BC and the LC are given as

$$B_F(p)=\frac{1}{p\mu}\sum_{0}^{p}F^{-1}(t), \tag{12}$$

and

$$L_F(p)=\frac{1}{\mu}\sum_{0}^{p}F^{-1}(t), \tag{13}$$

respectively, where μ is the mean of DNH. The BC and LC of DNH distribution are given below:

$$B_F(p) = \frac{\sum_{t=0}^{p}(-\frac{1}{\lambda} + (\frac{\log(1-p)}{\log\theta} - \frac{1}{\log\theta})^{\frac{1}{\alpha}})}{p\sum_{t=0}^{\infty} t\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}})}, \tag{14}$$

and

$$L_F(p) = \frac{\sum_{t=0}^{p}(-\frac{1}{\lambda} + (\frac{\log(1-p)}{\log\theta} - \frac{1}{\log\theta})^{\frac{1}{\alpha}})}{\sum_{t=0}^{\infty} t\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}})}, \tag{15}$$

respectively. Eqs. (14) and (15) can be solved numerically.

2.6. *Order statistics*

Order statistics are required in many fields, such as climatology, engineering, industry, etc. (Arnold *et al.*, 1992). Joined with rank statistics, order statistics are among the most important tools in nonparametric statistics which is used to develop robust estimators. In statistics, the jth ordered value is equal to the jth order statistic of the sample. The largest order statistic is the maximum for a sample of size n, the $n-th$ order statistic is, $T_{(n)} = \max(T_1, T_2, \cdots, T_n)$. Moreover, the sample range, $Range(T_1, T_2, \cdots, T_n) = T_{(n)} - T_{(1)}$ is the difference between the maximum and the minimum values of a given data set which is also a function of order statistics. If $T_{(1)}, T_{(2)}, \cdots, T_{(n)}$ denote the order statistics of a random sample $T_1, T_2, \cdots, T_n$ from a population with cdf $F(t)$ and the pmf $p(t)$, then the pmf of $T_{(j)}$ is given by

$$F_{T_{(j)}}(t) = \frac{n!}{(j-1)!(n-j)!} p(t)[F(t)]^{j-1}[1-F(t)]^{n-j}, \tag{16}$$

for $j = 1, 2, \cdots, n$. The pmf of the j-th order statistic of the DNH distribution is given by

$$\begin{aligned} F_{T_{(j)}}(t) = & \frac{n!}{(j-1)!(n-j)!}[\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}})][(1-\exp(1)\theta^{(\frac{1}{\lambda}+t)^{\alpha}})]^{j-1} \\ & \times [\exp(1)\theta^{(\frac{1}{\lambda}+t)^{\alpha}}]^{n-j}. \end{aligned} \tag{17}$$

Thus, the PMF of the first order statistic is obtained by $j = 1$ in (17). The resulting cdf is given as:

$$F_{T_{(1)}}(t) = \frac{n!}{(n-1)!}[\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}})][\exp(1)\theta^{(\frac{1}{\lambda}+t)^{\alpha}}]^{n-1}. \tag{18}$$

Similarly, the PMF of the $n-th$ order statistic is obtained by $j = n$, which is given as:

$$F_{T_{(n)}}(t) = \frac{n!}{(n-1)!}[\exp(1)(\theta^{(\frac{1}{\lambda}+t)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha}})][(1-\exp(1)\theta^{(\frac{1}{\lambda}+t)^{\alpha}})]^{n-1}. \tag{19}$$

2.7. Stochastic ordering

Stochastic ordering is an interesting and important area of statistics and it is basically the concept concerning of one random variable being bigger than another random variable. Stochastic ordering, especially hazard rate ordering, mean residual life ordering and likelihood ratio ordering have the following relationship:

1. Hazard rate order ($T_1 \leq_{hr} T_2$) if $h_{T_1}(t) \leq h_{T_2}(x)$ for all t.
2. Mean residual life order ($T_1 \leq_{mlr} T_2$) if $m_{T_1} \leq m_{T_2}$ for all t.
3. Likelihood ratio order ($T_1 \leq_{lr} T_2$) if $\frac{f_{T_1}(t)}{f_{T_2}(t)}$ is a decreasing function t.

THEOREM 1. *Let $T_1 \sim DNH(\alpha_1, \lambda, \theta)$ and $T_2 \sim DNH(\alpha_2, \lambda, \theta)$ be two independent random variables. If $\alpha_1 \leq \alpha_2$ then $(T_1 \leq_{lr} T_2)$ for all t.*

PROOF. To prove the theorem, it is enough to show that $\frac{f_{T_1}(t)}{f_{T_2}(t)}$ is decreasing in t.

$$\frac{f_{T_1}(t)}{f_{T_2}(t)} = \frac{\exp(1)\left[\theta^{(\frac{1}{\lambda}+t)^{\alpha_1}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha_1}}\right]}{\exp(1)\left[\theta^{(\frac{1}{\lambda}+t)^{\alpha_2}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha_2}}\right]} \tag{20}$$

Now,

$$\frac{d}{dt}\log\frac{f_{T_1}(t)}{f_{T_2}(t)} = \frac{d}{dt}\log\frac{\left[\theta^{(\frac{1}{\lambda}+t)^{\alpha_1}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha_1}}\right]}{\left[\theta^{(\frac{1}{\lambda}+t)^{\alpha_2}} - \theta^{(\frac{1}{\lambda}+t+1)^{\alpha_2}}\right]} < 0 \tag{21}$$

if $\alpha_1 < \alpha_2, \forall\ t > 0$. Hence, for $\alpha_1 < \alpha_2$, $T_1 \leq_{lr} T_2$, $\forall$ t. □

3. PARAMETER ESTIMATION

This section describes eight methods of parameter estimation for the DNH distribution. To this end, assume that $\mathbf{t} = (t_1, \cdots, t_n)$ is a random sample of size n from the DNH distribution and we are interested to estimate the α, λ and θ using the information of the observed sample.

3.1. Maximum likelihood estimation

Method of the maximum likelihood estimation is widely used estimation method in statistics (Casella and Berger, 2002). This method of estimation has many attractive properties like consistency, asymptotic efficiency, etc. The likelihood function of the probability mass function (6) is given by

$$L(t;\alpha,\theta,\lambda) = \prod_{i=1}^{n} p(t_i) = \exp(1)(\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}} - \theta^{(\frac{1}{\lambda}+t_i+1)^{\alpha}}), \tag{22}$$

$$\log L(t;\alpha,\theta,\lambda)=\sum_{i=0}^{n}\log\exp(1)(\theta^{(\frac{1}{\lambda}+t_i)^\alpha}-\theta^{(\frac{1}{\lambda}+t_i+1)^\alpha}). \tag{23}$$

Taking partial derivatives of $\log L(t;\alpha,\theta,\lambda)$ with respect to α,λ and θ, and equating the resulting equations to zero, one can obtain the normal equations to find the ML estimators.

$$\frac{\partial \log L(t;\alpha,\theta)}{\partial \alpha}=0,$$

$$\sum_{i=1}^{n}\log(\theta^{(1+t)^\alpha}\log\theta\log(1+t)(1+t)^\alpha-\theta^{(2+t)^\alpha}\log\theta\log(2+t)(2+t)^\alpha=0, \tag{24}$$

and

$$\frac{\partial \log L(t;\alpha,\theta)}{\partial \theta}=0,$$

$$\sum_{i=1}^{n}\log(\theta^{(-1+(1+t)^\alpha)}(1+t)^\alpha-\theta^{(-1+(2+t)^\alpha)}(2+t)^\alpha=0. \tag{25}$$

If $\lambda\neq 1$ then

$$\frac{\partial \log L(t;\alpha,\lambda\theta)}{\partial \lambda}=0,$$

$$\sum_{i=1}^{n}\frac{-\alpha\theta^{(\frac{1}{\lambda}+t_i)^\alpha}\log\theta(\frac{1}{\lambda}+t_i)^{\alpha-1}+\alpha\theta^{(1+\frac{1}{\lambda}+t_i)^\alpha}\log\theta(1+\frac{1}{\lambda}+t_i)^{\alpha-1}}{\theta^{(\frac{1}{\lambda}+t_i)^\alpha}-\theta^{(1+\frac{1}{\lambda}+t_i)^\alpha}}=0. \tag{26}$$

The solution of above normal Eqs. (24), (25) and (26) cannot be obtained in closed form, hence require an iterative method like Newton Raphson to compute MLEs of (α,λ,θ).

3.2. Method of least squares

To estimate parameters by minimizing the sum of square of residuals, a standard approach is the least squares estimators (Swain *et al.*, 1988). For the estimation of parameters α,λ and θ of DNH distribution, the least squares estimators can be obtained by minimizing:

$$\sum_{i=1}^{n}\left[F(t_{(i)})-\frac{i}{n+1}\right]^2, \tag{27}$$

with respect to unknown parameters α,λ and θ. In other words, $\hat{\alpha}_{LSE}$, $\hat{\lambda}_{LSE}$ and $\hat{\theta}_{LSE}$ are obtained by minimizing

$$\sum_{i=1}^{n}\left[1-\exp(1)(\theta^{(\frac{1}{\lambda}+t_i)^\alpha})-\frac{i}{n+1}\right]^2 \tag{28}$$

with respect to α, λ and θ.

Taking partial derivative with respect to α, λ and θ, the following are obtained

$$\sum_{i=1}^{n} -2\exp(1)\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}}(1-\frac{i}{n+1}-\exp(1)\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}})\log[\theta]\log[\frac{1}{\lambda}+t_i](\frac{1}{\lambda}+t_i), \tag{29}$$

$$\sum_{i=1}^{n} \frac{2\exp(1)\alpha\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}}(1-\frac{i}{n+1}-\exp(1)\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}})\log[\theta](\frac{1}{\lambda}+t_i)^{\alpha-1}}{\lambda^2}, \tag{30}$$

$$\sum_{i=1}^{n} -2\exp(1)\alpha\theta^{-1+(\frac{1}{\lambda}+t_i)^{\alpha}}(1-\frac{i}{n+1}-\exp(1)\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}})(\frac{1}{\lambda}+t_i)^{\alpha}, \tag{31}$$

respectively. The weighted least squares estimators of the parameters of DNH distribution are obtained by minimizing

$$\sum_{i=1}^{n} w_i\left[F(t_{(i)})-\frac{i}{n+1}\right]^2 \tag{32}$$

with respect to parameters, where $w_i = \frac{1}{V(T_i)} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ are the weights. Thus, $\hat{\alpha}_{WLSE}$, $\hat{\lambda}_{WLSE}$ and $\hat{\theta}_{WLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{n-i+1}\left[1-\exp(1)(\theta^{(\frac{1}{\lambda}+t_i)^{\alpha}})-\frac{i}{n+1}\right]^2 \tag{33}$$

with respect to α, λ and θ.

3.3. *Method of percentile estimation*

The method of percentile estimation was originally introduced by Kao (1959) and it has been used for the parameter estimation of several distributions. In this study, we use this method to estimate the parameters of the DNH distribution. Since $F(t) = 1-\exp(1)\theta^{(\frac{1}{\lambda}+t)^{\alpha}}$ and $Q(p) = -\frac{1}{\lambda}+\left(\frac{\log(1-p)}{\log\theta}-\frac{1}{\log\theta}\right)^{\frac{1}{\alpha}}$, let T_i be the i-th order statistic, i.e., $T_{(1)} < T_{(2)} < \cdots < T_{(n)}$. If p_i denote an estimate of $F(t_{(i)})$, then the estimate of α, λ and θ can be obtained by minimizing

$$\sum_{i=1}^{n}\left(t_{(i)}-[-\frac{1}{\lambda}+(\frac{\log(1-p_i)-1}{\log\theta})^{\frac{1}{\alpha}}]\right)^2 \tag{34}$$

with respect to α, λ and θ. The estimators of α, λ and θ are obtained by solving the following non-linear equations

$$\sum_{i=1}^{n} \frac{2\log[\frac{-1+\log(1-p_i)}{\log\theta}](\frac{-1+\log(1-p_i)}{\log\theta})^{\frac{1}{\alpha}}(\frac{1}{\lambda}-(\frac{-1+\log(1-p_i)}{\log\theta})^{\frac{1}{\alpha}}+t_i)}{\alpha^2} = 0, \tag{35}$$

$$\sum_{i=1}^{n}\frac{-2[\frac{1}{\lambda}-(\frac{-1+\log[1-p_i]}{\log[\theta]})^{\frac{1}{\alpha}}+t_i]}{\lambda^2}=0, \tag{36}$$

$$\sum_{i=1}^{n}\frac{2(-1+\log(1-p_i))(\frac{-1+\log(1-p_i)}{\log\theta})^{-1+\frac{1}{\alpha}}(\frac{1}{\lambda}-(\frac{-1+\log(1-p_i)}{\log\theta}))^{\frac{1}{\alpha}+t_i}}{\alpha\theta(\log\theta)^2}=0. \tag{37}$$

The resulting estimators are denoted as the percentile estimators. Many estimators of p_i are proposed in the literature (Mann *et al.*, 1974). In this work we consider $p_i=\frac{i}{n+1}$.

3.4. *Method of maximum product spacings*

An alternative to MLE, Cheng and Amin (1983) suggested the method of maximum product of spacing (MPS) for parameter estimation. Independently, it was introduced by Ranneby (1984) as an approximation for the Kullback-Leibler information measure. For a random sample with uniform spacings from the DNH distribution are given by:

$$D_i=F(t_i|\alpha,\lambda,\theta)-F(t_{i-1}|\alpha,\lambda,\theta), \tag{38}$$

where $i=1,2,\cdots,n$. Cheng and Amin (1983) suggested $\sum_{i=1}^{n+1}D_i(\alpha,\lambda,\theta)=1$. The estimators under the MPS are $\hat{\alpha}_{MPS}$, $\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$ obtained by maximizing the following geometric mean of the spacings with respect to parameters,

$$G(\alpha,\lambda,\theta)=\left[\prod_{i=1}^{n+1}D_i(\alpha,\lambda,\theta)\right]^{\frac{1}{n+1}}, \tag{39}$$

or alternatively, by maximizing the function

$$H(\alpha,\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i(\alpha,\lambda,\theta), \tag{40}$$

$\hat{\alpha}_{MPS}$, $\hat{\lambda}_{MPS}$ and $\hat{\theta}_{MPS}$ can be obtained by solving the non-linear equations

$$\frac{\partial}{\partial\alpha}H(\alpha,\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda,\theta)}[\delta_1(t_i|\alpha,\lambda,\theta)]=0, \tag{41}$$

$$\frac{\partial}{\partial\lambda}H(\alpha,\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda,\theta)}[\delta_3(t_i|\alpha,\lambda,\theta)]=0, \tag{42}$$

$$\frac{\partial}{\partial\theta}H(\alpha,\lambda,\theta)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{D_i(\alpha,\lambda,\theta)}[\delta_2(t_i|\alpha,\lambda,\theta)]=0, \tag{43}$$

where

$$\delta_1(t_i|\alpha,\lambda,\theta)=\theta^{(\frac{1}{\lambda}+t_{i-1})^\alpha}\log[\theta]\log[\frac{1}{\lambda}+t_{i-1}](\frac{1}{\lambda}+t_{i-1})^\alpha+\theta^{(\frac{1}{\lambda}+t_i)^\alpha}\times \log[\theta]\log[\frac{1}{\lambda}+t_i](\frac{1}{\lambda}+t_i)^\alpha, \tag{44}$$

$$\delta_2(t_i|\alpha,\lambda,\theta)=-\frac{1}{\lambda^2}[\alpha\theta^{(\frac{1}{\lambda}+t_{i-1})^\alpha}\log[\theta](\frac{1}{\lambda}+t_{i-1})^{\alpha-1}+\alpha\theta^{(\frac{1}{\lambda}+t_i)^\alpha}\times \log[\theta](\frac{1}{\lambda}+t_i)^{\alpha-1}], \tag{45}$$

$$\delta_3(t_i|\alpha,\lambda,\theta)=\theta^{-1+(\frac{1}{\lambda}+t_i)^\alpha}(\frac{1}{\lambda}+t_{i-1})^\alpha+\theta^{-1+(\frac{1}{\lambda}+t_i)^\alpha}(\frac{1}{\lambda}+t_i)^\alpha, \tag{46}$$

Cheng and Amin (1983) proved that maximizing the product of spacings is as efficient and consistent as the MLE estimators under different conditions.

3.5. Cramèr-von-Mises method of estimation

This is another parameter estimation method. Macdonald (1971) proved that the Cramèr-von-Mises has a smaller bias as compare to other minimum distance type estimators. Assuming $\hat{\alpha}_{CME}$, $\hat{\lambda}_{CME}$ and $\hat{\theta}_{CME}$, the Cramèr-von-Mises estimators are obtained by minimizing

$$C(\alpha,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}[F(t_i)-\frac{2_i-1}{2n}]^2. \tag{47}$$

To obtain the estimators of DNH distribution, the following non-linear equations need to be solved numerically.

$$\sum_{i=1}^{n}[F(t_i)-\frac{2_i-1}{2n}]\delta_1(t_i|\alpha,\lambda,\theta)=0, \tag{48}$$

$$\sum_{i=1}^{n}[F(t_i)-\frac{2_i-1}{2n}]\delta_2(t_i|\alpha,\lambda,\theta)=0, \tag{49}$$

$$\sum_{i=1}^{n}[F(t_i)-\frac{2_i-1}{2n}]\delta_3(t_i|\alpha,\lambda,\theta)=0, \tag{50}$$

where $\delta_1(.|\alpha,\lambda,\theta)$, $\delta_2(.|\alpha,\lambda,\theta)$ and $\delta_3(.|\alpha,\lambda,\theta)$ are given by Eqs. (44), (45) and (46), respectively.

3.6. Method of Anderson-Darling and right-tail Anderson-Darling

The Anderson-Darling (AD) test was proposed by Anderson and Darling (1952) as an alternative to other statistical tests for detecting sample distribution departure from the normality. A nice feature of the AD test is that it converge towards the asymptote very quickly (Anderson and Darling, 1954; Pettitt, 1976). Let $\hat{\alpha}_{ADE}$, $\hat{\lambda}_{ADE}$ and $\hat{\theta}_{ADE}$ are the AD estimators obtained by minimizing, with respect to α, λ and θ, as given below:

$$A(\alpha,\theta) = -n - \frac{1}{n}\sum_{i=1}^{n}(2_i - 1)[\log F(t_{i:n}|\alpha,\theta) + \log \bar{F}(t_{n+1-i:n}|\alpha,\theta)]. \tag{51}$$

These estimators can also be obtained by solving the non-linear equations:

$$\sum_{i=1}^{n}(2i-1)[\frac{\delta_1(t_{i:n}|\alpha,\theta)}{F(t_{i:n}|\alpha,\theta)} - \frac{\delta_1(t_{n+1-i:n}|\alpha,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\theta)}] = 0, \tag{52}$$

$$\sum_{i=1}^{n}(2i-1)[\frac{\delta_2(t_{i:n}|\alpha,\theta)}{F(t_{i:n}|\alpha,\theta)} - \frac{\delta_2(t_{n+1-i:n}|\alpha,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\theta)}] = 0, \tag{53}$$

$$\sum_{i=1}^{n}(2i-1)[\frac{\delta_3(t_{i:n}|\alpha,\theta)}{F(t_{i:n}|\alpha,\theta)} - \frac{\delta_3(t_{n+1-i:n}|\alpha,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\theta)}] = 0, \tag{54}$$

where $\delta_1(.|\alpha,\lambda,\theta)$, $\delta_2(.|\alpha,\lambda,\theta)$ and $\delta_3(.|\alpha,\lambda,\theta)$ are given by Eqs. (44), (45) and (46), respectively.

The estimates of the Right-tail Anderson-Darling $\hat{\alpha}_{RTADE}$, $\hat{\lambda}_{RTADE}$ and $\hat{\theta}_{RTADE}$ of the parameters α, λ and θ respectively, are obtained by minimizing

$$R(\alpha,\theta) = \frac{n}{2} - 2\sum_{i=1}^{n}F(t_{i:n}|\alpha,\lambda,\theta) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log \bar{F}(t_{n+1-i:n}|\alpha,\lambda,\theta). \tag{55}$$

Solving the following non-linear equations:

$$-2\sum_{i=1}^{n}\frac{\delta_1(t_{i:n}|\alpha,\lambda,\theta)}{F(t_{i:n}|\alpha,\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_1(t_{n+1-i:n}|\alpha,\lambda,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\lambda,\theta)} = 0, \tag{56}$$

$$-2\sum_{i=1}^{n}\frac{\delta_2(t_{i:n}|\alpha,\lambda,\theta)}{F(t_{i:n}|\alpha,\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_2(t_{n+1-i:n}|\alpha,\lambda,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\lambda,\theta)} = 0, \tag{57}$$

$$-2\sum_{i=1}^{n}\frac{\delta_3(t_{i:n}|\alpha,\lambda,\theta)}{F(t_{i:n}|\alpha,\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_3(t_{n+1-i:n}|\alpha,\lambda,\theta)}{\bar{F}(t_{n+1-i:n}|\alpha,\lambda,\theta)} = 0, \tag{58}$$

where $\delta_1(.|\alpha,\theta)$, $\delta_2(.|\alpha,\theta)$ and $\delta_3(.|\alpha,\theta)$ are given by Eqs. (44), (45) and (46), respectively, will result the RAD estimators.

4. Monte Carlo simulation study

In this section, a Monte Carlo simulation study is conducted to compare the performance of the different methods of estimation. To be more precise, the performance of different estimators is evaluated on the basis of bias, root mean squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and the empirical distribution functions. Methods are compared for sample sizes $n = 10, 20, 50, 75$, and 100. To this end, first we generate five thousand independent samples of size n from the DNH distribution with different parameter configurations. In particular, we consider $\alpha = 1, 5$; $\lambda = 1, 1.5$, and $\theta = 0.1, 0.3, 0.7$. It is observed that 5,000 repetitions are sufficiently large enough to have stable results. Next, the parameters are estimated using the method of maximum likelihood. Then for all other methods, we have used the maximum likelihood estimates as the initial values. In addition, the same randomly generated samples are used to compute and compare the other estimation methods. The results of the simulation studies are tabulated in Tables 2 to 6.

To generate the pseudo-random numbers from the DNH distribution, the following quantile function is used:

$$Q(p) = -\frac{1}{\lambda} + \left(\frac{\log(1-p)}{\log\theta} - \frac{1}{\log\theta}\right)^{\frac{1}{\alpha}},$$

where $p \sim \mathrm{UNIF}(0,1)$. The formulae to calculate the bias, root mean-squared error, and the average absolute difference between the theoretical and the empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions are given as follows:

$$\mathrm{Bias}(\hat{\alpha}) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\alpha}_i - \alpha), \qquad \mathrm{Bias}(\hat{\lambda}) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\lambda}_i - \lambda), \tag{59}$$

$$\mathrm{Bias}(\hat{\theta}) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\theta}_i - \theta), \qquad \mathrm{RMSE}(\hat{\alpha}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\alpha} - \alpha)^2}, \tag{60}$$

$$\mathrm{RMSE}(\hat{\lambda}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\lambda} - \lambda)^2}, \tag{61}$$

$$\mathrm{RMSE}(\hat{\theta}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\theta} - \theta)^2}, \tag{62}$$

$$D_{abs}(\hat{\alpha}) = \frac{1}{(nR)}\sum_{i=1}^{R}\sum_{j=1}^{n}|F(t_{ij}|\alpha,\lambda,\theta) - F(t_{ij}|\hat{\alpha},\hat{\lambda},\hat{\theta})|, \tag{63}$$

$$D_{max}(\hat{\alpha}) = \frac{1}{R}\sum_{i=1}^{R} \max_{j} |F(t_{ij}|\alpha,\lambda,\theta) - F(t_{ij}|\hat{\alpha},\hat{\lambda},\hat{\theta})|. \tag{64}$$

Using the above measures, we conducted a simulation study and the results are tabulated in Table 2 to Table 9. Partial sum of the ranks shown in row with label $\sum Ranks$. A superscript shows the rank of every estimations among the all considered eight estimation methods. For instance, Table 2 presents the bias of MLE, Bias($\hat{\alpha}$) as 6.312^8 for sample size $n = 10$. This shows that Bias($\hat{\alpha}$) calculated using MLE method is 6.312 and it is ranked 8^{th} among all consider estimators which are used in this study. The following conclusion are drawn from Table 2 to Table 9.

1. All the discussed methods of estimation have consistency property, i.e., the bias and RMSE decrease with the sample size n.

2. It is observed that for a given θ, Bias($\hat{\alpha}$) generally increases as the value of α increases. This pattern is observed among all eight estimation methods.

3. In case of RMSE, all estimators produce minimal RMSE for parameter $\hat{\alpha}$ as compared to $\hat{\theta}$ and $\hat{\lambda}$.

4. Comparing the D_{abs} and D_{max} for eight considered estimation methods, D_{abs} is smaller as compared to D_{max}. Further, it gets smaller as sample size n becomes large.

5. As the performance of different methods of estimation is concerned, it is observed that the MPS method performs better on the basis of least biases and RMSE as compared to other considered methods. Cramèr-von-Mises (CVM) is the next best method of estimation, followed by percentile estimators (PCE). The ML estimation method ranked 8th while weighted least squared (WLS) estimation ranked 5th, right-tail Anderson-Darling ranked 6th among the eight methods of estimation.

6. For all sample sizes and α, MPS performed uniformly the best while percentile and weighted least square methods also performed better for $\alpha \leq 1$. However, it is also observed that their performance is degraded for large value of α, e.g., $\alpha = 5$.

7. All assumed estimation methods perform better when $\lambda \neq 1$ as compared to $\lambda = 1$.

8. In most methods, parameters converge quickly when $\lambda \neq 1$ as compared to $\lambda = 1$.

TABLE 2
Simulation results for $\alpha = 0.5$, $\theta = 0.5$ and $\lambda = 1$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RAD
10	Bias($\hat{\alpha}$)	6.312^{8}	-0.059^{5}	-0.047^{4}	-0.031^{3}	-0.500^{7}	-0.005^{1}	0.299^{6}	-0.011^{1}
	RMSE($\hat{\alpha}$)	6.312^{8}	0.135^{3}	0.133^{2}	0.185^{5}	0.500^{6}	0.139^{7}	1.382^{7}	0.126^{1}
	Bias($\hat{\theta}$)	0.497^{8}	-0.056^{6}	-0.050^{5}	-0.036^{4}	-0.263^{7}	-0.028^{3}	0.024^{1}	-0.027^{2}
	RMSE($\hat{\theta}$)	0.497^{8}	0.080^{4}	0.072^{2}	0.144^{5}	0.263^{7}	0.067^{1}	0.153^{6}	0.073^{3}
	D_{abs}	0.999^{8}	0.110^{4}	0.103^{3}	0.143^{5}	0.293^{7}	0.103^{2}	0.179^{6}	0.101^{1}
	D_{max}	0.999^{8}	0.166^{4}	0.154^{3}	0.253^{5}	0.555^{7}	0.151^{1}	0.291^{6}	0.154^{2}
	$\sum$Ranks	48^{8}	26^{4}	19^{3}	27^{5}	41^{7}	12^{2}	32^{6}	11^{1}
20	Bias($\hat{\alpha}$)	-0.186^{7}	-0.056^{6}	-0.041^{3}	-0.046^{4}	0.244^{8}	-0.029^{2}	0.047^{5}	-0.026^{1}
	RMSE($\hat{\alpha}$)	0.691^{8}	0.096^{4}	0.088^{2}	0.146^{5}	0.248^{6}	0.089^{3}	0.651^{7}	0.083^{1}
	Bias($\hat{\theta}$)	531.905^{8}	-0.053^{6}	-0.044^{4}	-0.050^{5}	-0.497^{7}	-0.038^{3}	-0.020^{1}	-0.035^{2}
	RMSE($\hat{\theta}$)	934.627^{8}	0.064^{4}	0.052^{1}	0.133^{6}	0.497^{7}	0.053^{2}	0.096^{5}	0.057^{3}
	D_{abs}	0.999^{8}	0.084^{4}	0.075^{1}	0.122^{6}	0.545^{7}	0.080^{3}	0.103^{5}	0.078^{2}
	D_{max}	0.999^{8}	0.133^{4}	0.116^{1}	0.225^{6}	0.998^{7}	0.121^{3}	0.167^{5}	0.121^{2}
	$\sum$Ranks	47^{8}	$28^{4.5}$	12^{2}	32^{6}	42^{7}	16^{3}	$28^{4.5}$	11^{1}
50	Bias($\hat{\alpha}$)	-0.313^{8}	-0.054^{6}	-0.034^{1}	-0.050^{5}	0.268^{7}	-0.044^{4}	-0.042^{4}	-0.035^{2}
	RMSE($\hat{\alpha}$)	0.593^{8}	0.071^{4}	0.057^{1}	0.115^{6}	0.269^{7}	0.065^{3}	0.085^{5}	0.058^{2}
	Bias($\hat{\theta}$)	508.684^{8}	-0.050^{5}	-0.037^{1}	-0.055^{6}	-0.500^{7}	-0.044^{3}	-0.045^{4}	-0.039^{2}
	RMSE($\hat{\theta}$)	655.365^{8}	0.055^{4}	0.039^{1}	0.116^{6}	0.500^{7}	0.050^{3}	0.055^{5}	0.048^{2}
	D_{abs}	0.999^{8}	0.064^{4}	0.054^{1}	0.099^{6}	0.544^{7}	0.062^{3}	0.065^{5}	0.059^{2}
	D_{max}	0.999^{8}	0.110^{4}	0.088^{1}	0.188^{6}	0.998^{7}	0.103^{3}	0.110^{5}	0.098^{2}
	$\sum$Ranks	48^{8}	$27^{4.5}$	6^{1}	35^{6}	42^{7}	19^{3}	$27^{4.5}$	12^{2}
75	Bias($\hat{\alpha}$)	-0.362^{8}	-0.052^{6}	-0.029^{1}	-0.047^{5}	0.275^{7}	-0.045^{3}	-0.046^{4}	-0.036^{2}
	RMSE($\hat{\alpha}$	0.591^{8}	0.065^{5}	0.048^{1}	0.104^{6}	0.276^{7}	0.060^{3}	0.061^{4}	0.052^{2}
	Bias($\hat{\theta}$)	726.656^{8}	-0.049^{5}	-0.034^{1}	-0.054^{6}	-0.500^{7}	-0.045^{3}	-0.048^{4}	-0.040^{2}
	RMSE($\hat{\theta}$)	1146.157^{8}	0.052^{5}	0.036^{1}	0.108^{6}	0.500^{7}	0.049^{3}	0.051^{4}	0.046^{2}
	D_{abs}	0.999^{8}	0.059^{4}	0.049^{1}	0.091^{6}	0.545^{7}	0.058^{3}	0.059^{4}	0.055^{2}
	D_{max}	0.999^{8}	0.106^{5}	0.080^{1}	0.173^{6}	0.999^{7}	0.100^{3}	0.105^{4}	0.094^{2}
	$\sum$Ranks	48^{8}	30^{5}	6^{1}	35^{6}	42^{7}	18^{3}	25^{4}	12^{2}
100	Bias($\hat{\alpha}$)	-0.444^{8}	-0.052^{6}	-0.027^{1}	-0.046^{3}	0.278^{7}	-0.047^{4}	-0.048^{5}	-0.037^{2}
	RMSE($\hat{\alpha}$)	0.579^{8}	0.062^{5}	0.042^{1}	0.096^{6}	0.279^{7}	0.058^{4}	0.057^{3}	0.049^{2}
	Bias($\hat{\theta}$)	2379.126^{8}	-0.049^{5}	-0.032^{1}	-0.053^{6}	-0.500^{7}	-0.046^{3}	-0.048^{4}	-0.040^{2}
	RMSE($\hat{\theta}$)	3125.464^{8}	0.051^{5}	0.034^{1}	0.103^{6}	0.500^{7}	0.048^{3}	0.050^{4}	0.044^{2}
	D_{abs}	0.999^{8}	0.056^{4}	0.045^{1}	0.086^{6}	0.545^{7}	0.055^{3}	0.057^{5}	0.052^{2}
	D_{max}	0.999^{8}	0.103^{4}	0.076^{1}	0.164^{6}	0.999^{7}	0.099^{3}	0.104^{5}	0.091^{2}
	$\sum$Ranks	48^{8}	29^{5}	6^{1}	33^{6}	42^{7}	20^{3}	26^{4}	12^{2}

TABLE 3
Simulation results for $\alpha = 1$, $\theta = 0.7$ and $\lambda = 1$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RAD
10	Bias($\hat{\alpha}$)	-1.122^{7}	-0.196^{5}	-0.172^{4}	-0.566^{6}	-0.049^{1}	-0.101^{3}	1.838^{8}	-0.079^{2}
	RMSE($\hat{\alpha}$)	1.257^{7}	0.294^{4}	0.300^{5}	0.568^{6}	0.058^{1}	0.269^{3}	3.804^{8}	0.245^{2}
	Bias($\hat{\theta}$)	619.539^{8}	-0.118^{4}	-0.110^{3}	-0.403^{6}	-0.697^{7}	-0.080^{2}	0.129^{5}	-0.070^{1}
	RMSE($\hat{\theta}$)	777.302^{8}	0.145^{4}	0.143^{3}	0.410^{6}	0.697^{7}	0.119^{2}	0.191^{5}	0.110^{1}
	D_{abs}	0.999^{8}	0.146^{4}	0.142^{3}	0.330^{5}	0.599^{7}	0.138^{2}	0.472^{6}	0.135^{1}
	$D_{\max}$	0.999^{8}	0.263^{4}	0.261^{3}	0.655^{5}	0.998^{7}	0.235^{2}	0.731^{6}	0.220^{1}
	$\sum$Ranks	46^{8}	25^{4}	21^{3}	34^{6}	30^{5}	14^{2}	38^{7}	8^{1}
20	Bias($\hat{\alpha}$)	5.620^{8}	-0.200^{4}	-0.200^{5}	-0.533^{6}	-0.049^{1}	-0.154^{3}	1.002^{7}	-0.107^{2}
	RMSE($\hat{\alpha}$)	5.632^{8}	0.245^{4}	0.258^{5}	0.536^{6}	0.057^{1}	0.217^{3}	2.396^{7}	0.183^{2}
	Bias($\hat{\theta}$)	3.900^{8}	-0.116^{4}	-0.118^{5}	-0.453^{6}	-0.699^{7}	-0.097^{2}	0.109^{3}	-0.078^{1}
	RMSE($\hat{\theta}$)	113.998^{8}	0.129^{3}	0.135^{4}	0.461^{6}	0.699^{7}	0.113^{2}	0.173^{5}	0.097^{1}
	D_{abs}	0.999^{8}	0.126^{4}	0.126^{3}	0.411^{6}	0.601^{7}	0.121^{2}	0.367^{5}	0.117^{1}
	$D_{\max}$	0.999^{8}	0.258^{3}	0.270^{4}	0.852^{6}	0.999^{7}	0.234^{2}	0.567^{5}	0.209^{1}
	$\sum$Ranks	48^{8}	22^{3}	26^{4}	36^{7}	30^{5}	14^{2}	32^{6}	8^{1}
50	Bias($\hat{\alpha}$)	7.103^{8}	-0.200^{4}	-0.241^{5}	-0.489^{6}	-0.044^{1}	-0.182^{3}	0.556^{7}	-0.122^{2}
	RMSE($\hat{\alpha}$)	7.137^{8}	0.217^{4}	0.252^{5}	0.490^{6}	0.053^{1}	0.201^{3}	1.254^{7}	0.151^{2}
	Bias($\hat{\theta}$)	7.952^{8}	-0.113^{4}	-0.134^{5}	-0.520^{6}	-0.700^{7}	-0.106^{3}	0.080^{1}	-0.082^{2}
	RMSE($\hat{\theta}$)	144.676^{8}	0.118^{3}	0.138^{4}	0.521^{6}	0.700^{7}	0.111^{2}	0.148^{5}	0.089^{1}
	D_{abs}	0.999^{8}	0.111^{3}	0.119^{4}	0.513^{6}	0.600^{7}	0.109^{2}	0.266^{5}	0.105^{1}
	$D_{\max}$	0.999^{8}	0.255^{3}	0.293^{4}	0.996^{6}	0.998^{7}	0.243^{2}	0.423^{5}	0.207^{1}
	$\sum$Ranks	48^{8}	21^{3}	27^{4}	36^{7}	$30^{5.5}$	15^{2}	$30^{5.5}$	9^{1}
75	Bias($\hat{\alpha}$)	7.036^{8}	-0.199^{4}	-0.250^{5}	-0.631^{7}	-0.043^{1}	-0.187^{3}	0.511^{6}	-0.125^{2}
	RMSE($\hat{\alpha}$)	7.078^{8}	0.210^{4}	0.255^{5}	0.631^{6}	0.055^{1}	0.199^{3}	1.181^{7}	0.144^{2}
	Bias($\hat{\theta}$)	5.420^{8}	-0.113^{4}	-0.138^{5}	-0.217^{6}	-0.700^{7}	-0.108^{3}	0.071^{1}	-0.083^{2}
	RMSE($\hat{\theta}$)	87.865^{8}	0.116^{3}	0.140^{4}	0.217^{6}	0.700^{7}	0.111^{2}	0.142^{5}	0.088^{1}
	D_{abs}	0.999^{8}	0.109^{3}	0.119^{4}	0.214^{5}	0.601^{7}	0.108^{2}	0.248^{6}	0.104^{1}
	$D_{\max}$	0.999^{8}	0.255^{3}	0.301^{4}	0.419^{6}	0.998^{7}	0.247^{2}	0.394^{5}	0.208^{1}
	$\sum$Ranks	48^{8}	21^{3}	27^{4}	36^{7}	$30^{5.5}$	15^{2}	$30^{5.5}$	9^{1}
100	Bias($\hat{\alpha}$)	7.637^{8}	-0.199^{4}	-0.254^{5}	-0.631^{7}	-0.038^{1}	-0.190^{3}	0.450^{6}	-0.126^{2}
	RMSE($\hat{\alpha}$)	7.675^{8}	0.207^{4}	0.257^{5}	0.631^{6}	0.054^{1}	0.199^{3}	1.101^{7}	0.141^{2}
	Bias($\hat{\theta}$)	6.581^{8}	-0.112^{4}	-0.140^{5}	-0.217^{6}	-0.700^{7}	-0.109^{3}	0.059^{1}	-0.083^{2}
	RMSE($\hat{\theta}$)	114.718^{8}	0.115^{3}	0.141^{5}	0.217^{6}	0.700^{7}	0.111^{2}	0.132^{4}	0.087^{1}
	D_{abs}	0.999^{8}	0.107^{3}	0.119^{4}	0.214^{5}	0.601^{7}	0.106^{2}	0.224^{6}	0.103^{1}
	$D_{\max}$	0.998^{8}	0.255^{3}	0.304^{4}	0.420^{6}	0.995^{7}	0.248^{2}	0.353^{5}	0.208^{1}
	$\sum$Ranks	48^{8}	21^{3}	28^{4}	36^{7}	30^{6}	15^{2}	29^{5}	9^{1}

TABLE 4
Simulation results for $\alpha = 5$, $\theta = 0.1$ and $\lambda = 1$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RAD
10	Bias($\hat{\alpha}$)	0.357^{1}	-4.490^{5}	-4.402^{2}	-4.658^{6}	-5.000^{7}	-4.404^{3}	$4.773e+51^{8}$	-4.410^{4}
	RMSE($\hat{\alpha}$)	2.174^{1}	4.490^{5}	4.402^{2}	4.663^{6}	5.000^{7}	4.404^{3}	$3.116e+52^{8}$	4.410^{4}
	Bias($\hat{\theta}$)	-1215.273^{7}	-0.034^{1}	-0.043^{2}	-0.059^{5}	0.120^{6}	-0.045^{4}	$-5.023e+51^{8}$	-0.045^{3}
	RMSE($\hat{\theta}$)	6284.950^{7}	0.041^{1}	0.047^{2}	0.060^{5}	0.120^{6}	0.053^{4}	$3.267e+52^{8}$	0.053^{3}
	D_{abs}	0.185^{6}	0.021^{1}	0.028^{2}	0.041^{5}	0.987^{7}	0.029^{4}	0.999^{8}	0.029^{3}
	D_{max}	0.485^{6}	0.097^{1}	0.117^{2}	0.151^{5}	0.998^{7}	0.127^{4}	0.999^{8}	0.127^{3}
	$\sum$Ranks	28^{5}	14^{2}	12^{1}	32^{6}	40^{7}	22^{4}	48^{8}	20^{3}
20	Bias($\hat{\alpha}$)	4.031^{1}	-4.490^{5}	-4.380^{2}	-4.556^{6}	-5.000^{7}	-4.398^{4}	$3.370e+50^{8}$	-4.382^{3}
	RMSE($\hat{\alpha}$)	5.249^{7}	4.490^{4}	4.380^{1}	4.558^{5}	5.000^{6}	4.398^{3}	$9.111e+51^{8}$	4.382^{2}
	Bias($\hat{\theta}$)	-109.895^{7}	-0.043^{2}	-0.057^{5}	-0.071^{6}	0.020^{1}	-0.050^{4}	$-3.451e+50^{8}$	-0.050^{3}
	RMSE($\hat{\theta}$)	2594.334^{7}	0.047^{1}	0.058^{5}	0.071^{6}	0.020^{1}	0.053^{4}	$9.296e+51^{8}$	0.053^{3}
	D_{abs}	0.156^{6}	0.028^{1}	0.039^{4}	0.051^{5}	$.^{7}$	0.032^{3}	0.999^{8}	0.032^{2}
	D_{max}	0.474^{6}	0.118^{1}	0.155^{4}	0.192^{5}	0.998^{7}	0.135^{3}	0.999^{8}	0.135^{2}
	$\sum$Ranks	34^{7}	$15^{1.5}$	$21^{3.5}$	33^{6}	29^{5}	$21^{3.5}$	48^{8}	$15^{1.5}$
50	Bias($\hat{\alpha}$)	4.543^{6}	-4.490^{5}	-4.388^{2}	-4.469^{4}	-5.000^{8}	-4.392^{3}	-4.857^{7}	-4.382^{1}
	RMSE($\hat{\alpha}$)	7.986^{8}	4.490^{5}	4.388^{2}	4.470^{4}	5.000^{7}	4.392^{3}	4.857^{6}	4.382^{1}
	Bias($\hat{\theta}$)	0.173^{8}	-0.047^{2}	-0.067^{6}	0.079^{7}	0.020^{1}	-0.050^{4}	-0.054^{5}	-0.050^{3}
	RMSE($\hat{\theta}$)	0.184^{8}	0.048^{2}	0.067^{6}	0.080^{7}	0.020^{1}	0.051^{4}	0.056^{5}	0.051^{3}
	D_{abs}	0.140^{7}	0.033^{1}	0.049^{5}	0.058^{6}	0.999^{8}	0.035^{3}	0.039^{4}	0.035^{2}
	D_{max}	0.471^{7}	0.128^{1}	0.183^{5}	0.216^{6}	0.999^{8}	0.135^{3}	0.147^{4}	0.135^{2}
	$\sum$Ranks	44^{8}	16^{2}	26^{4}	34^{7}	33^{6}	20^{3}	31^{5}	12^{1}
75	Bias($\hat{\alpha}$)	4.745^{6}	-4.490^{5}	-4.395^{3}	-4.443^{4}	-5.000^{8}	-4.383^{2}	-4.853^{7}	-4.380^{1}
	RMSE($\hat{\alpha}$)	8.424^{8}	4.490^{5}	4.395^{3}	4.444^{4}	5.000^{7}	4.383^{2}	4.853^{6}	4.380^{1}
	Bias($\hat{\theta}$)	0.172^{8}	-0.048^{2}	-0.070^{6}	-0.082^{7}	0.020^{1}	-0.050^{3}	-0.056^{5}	-0.050^{4}
	RMSE($\hat{\theta}$)	0.180^{8}	0.049^{2}	0.071^{6}	0.082^{7}	0.020^{1}	0.051^{3}	0.058^{5}	0.051^{4}
	D_{abs}	0.135^{7}	0.034^{1}	0.052^{5}	0.060^{6}	0.999^{8}	0.036^{2}	0.041^{4}	0.036^{3}
	D_{max}	0.468^{7}	0.131^{1}	0.192^{5}	0.222^{6}	0.999^{8}	0.136^{2}	0.153^{4}	0.136^{3}
	$\sum$Ranks	44^{8}	$16^{2.5}$	28^{4}	34^{7}	33^{6}	14^{1}	31^{5}	$16^{2.5}$
100	Bias($\hat{\alpha}$)	4.745^{6}	-4.490^{5}	-4.395^{3}	-4.443^{4}	-5.000^{8}	-4.383^{2}	-4.853^{7}	-4.380^{1}
	RMSE($\hat{\alpha}$)	8.424^{8}	4.490^{5}	4.395^{3}	4.444^{4}	5.000^{7}	4.383^{2}	4.853^{6}	4.380^{1}
	Bias($\hat{\theta}$)	0.172^{8}	-0.048^{2}	-0.070^{6}	-0.082^{7}	0.020^{1}	-0.050^{3}	-0.056^{5}	-0.050^{4}
	RMSE($\hat{\theta}$)	0.180^{8}	0.049^{2}	0.071^{6}	0.082^{7}	0.020^{1}	0.051^{3}	0.058^{5}	0.051^{4}
	D_{abs}	0.135^{7}	0.034^{1}	0.052^{5}	0.060^{6}	0.999^{8}	0.036^{2}	0.041^{4}	0.036^{3}
	D_{max}	0.468^{7}	0.131^{1}	0.192^{5}	0.222^{6}	0.999^{8}	0.136^{2}	0.153^{4}	0.136^{3}
	$\sum$Ranks	44^{8}	$16^{2.5}$	28^{4}	34^{7}	33^{6}	14^{1}	31^{5}	$16^{2.5}$

TABLE 5
Simulation results for $\alpha = 5$, $\theta = 0.3$ and $\lambda = 1$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RTAD
10	Bias($\hat{\alpha}$)	1.716^{6}	-0.010^{1}	0.608^{3}	5259.747^{8}	-4.999^{7}	0.623^{5}	0.430^{2}	0.619^{4}
	RMSE($\hat{\alpha}$)	2.733^{6}	0.010^{1}	0.613^{3}	15005.501^{8}	4.999^{7}	0.626^{4}	0.523^{2}	0.650^{5}
	Bias($\hat{\theta}$)	0.514^{8}	-0.147^{2}	-0.166^{6}	-0.205^{7}	0.041^{1}	-0.150^{3}	-0.152^{5}	-0.150^{4}
	RMSE($\hat{\theta}$)	0.529^{8}	0.148^{2}	0.168^{6}	0.210^{7}	0.042^{1}	0.152^{3}	0.157^{5}	0.152^{4}
	D_{abs}	0.999^{8}	0.319^{1}	0.359^{5}	0.444^{7}	0.424^{6}	0.325^{3}	0.325^{2}	0.325^{4}
	D_{max}	0.997^{7}	0.400^{1}	0.451^{5}	0.558^{6}	0.999^{8}	0.408^{2}	0.414^{4}	0.408^{3}
	$\sum$Ranks	$43^{7.5}$	8^{1}	28^{5}	$43^{7.5}$	30^{6}	$20^{2.5}$	$20^{2.5}$	24^{4}
20	Bias($\hat{\alpha}$)	1.306^{6}	-0.010^{1}	0.575^{4}	407.438^{8}	-5.000^{7}	0.570^{3}	0.469^{2}	0.782^{5}
	RMSE($\hat{\alpha}$)	1.683^{6}	0.010^{1}	0.589^{4}	2900.798^{8}	5.000^{7}	0.575^{3}	0.561^{2}	0.829^{5}
	Bias($\hat{\theta}$)	0.516^{8}	-0.148^{2}	-0.180^{6}	-0.228^{7}	0.041^{1}	-0.150^{4}	-0.149^{3}	-0.150^{5}
	RMSE($\hat{\theta}$)	0.523^{8}	0.149^{2}	0.181^{6}	0.228^{7}	0.041^{1}	0.151^{3}	0.151^{5}	0.151^{4}
	D_{abs}	0.999^{8}	0.325^{2}	0.395^{5}	0.503^{7}	0.421^{6}	0.329^{3}	0.324^{1}	0.329^{4}
	D_{max}	0.998^{7}	0.403^{1}	0.489^{5}	0.620^{6}	0.999^{8}	0.408^{3}	0.404^{2}	0.408^{4}
	$\sum$Ranks	$43^{7.5}$	9^{1}	$30^{5.5}$	$43^{7.5}$	$30^{5.5}$	19^{3}	15^{2}	27^{4}
50	Bias($\hat{\alpha}$)	1.129^{6}	-0.010^{1}	0.450^{2}	8.135^{8}	-5.000^{7}	0.600^{4}	0.566^{3}	0.732^{5}
	RMSE($\hat{\alpha}$)	1.256^{5}	0.010^{1}	0.492^{2}	779.642^{8}	5.000^{7}	0.605^{3}	1.939^{6}	0.789^{4}
	Bias($\hat{\theta}$)	0.515^{8}	-0.149^{3}	-0.197^{6}	-0.236^{7}	0.041^{1}	-0.150^{5}	-0.147^{2}	-0.150^{4}
	RMSE($\hat{\theta}$)	0.518^{8}	0.150^{3}	0.197^{6}	0.236^{7}	0.041^{1}	0.150^{5}	0.148^{2}	0.150^{4}
	D_{abs}	0.999^{8}	0.330^{2}	0.435^{6}	0.523^{7}	0.422^{5}	0.331^{4}	0.323^{1}	0.331^{3}
	D_{max}	0.999^{7}	0.406^{2}	0.535^{5}	0.642^{6}	0.999^{8}	0.408^{4}	0.399^{1}	0.408^{3}
	$\sum$Ranks	42^{7}	12^{1}	27^{5}	43^{8}	29^{6}	25^{4}	15^{2}	23^{3}
75	Bias($\hat{\alpha}$)	1.131^{6}	-0.010^{1}	0.635^{3}	-2.908^{7}	-5.000^{8}	0.610^{2}	0.682^{4}	0.764^{5}
	RMSE($\hat{\alpha}$)	1.249^{5}	0.010^{1}	0.711^{3}	2.949^{6}	5.000^{8}	0.614^{2}	3.376^{7}	0.834^{4}
	Bias($\hat{\theta}$)	0.515^{8}	-0.150^{3}	-0.203^{6}	-0.239^{7}	0.041^{1}	-0.150^{5}	-0.146^{2}	-0.150^{4}
	RMSE($\hat{\theta}$)	0.517^{8}	0.150^{3}	0.203^{6}	0.239^{7}	0.041^{1}	0.150^{5}	0.147^{2}	0.150^{4}
	D_{abs}	0.999^{8}	0.331^{2}	0.449^{6}	0.528^{7}	0.422^{5}	0.332^{4}	0.323^{1}	0.332^{3}
	D_{max}	0.997^{7}	0.407^{2}	0.552^{5}	0.649^{6}	0.999^{8}	0.408^{4}	0.398^{1}	0.408^{3}
	$\sum$Ranks	42^{8}	12^{1}	29^{5}	40^{7}	31^{6}	22^{3}	17^{2}	23^{4}
100	Bias($\hat{\alpha}$)	1.175^{6}	-0.010^{1}	0.692^{4}	-2.904^{7}	-5.000^{8}	0.598^{2}	1.127^{5}	0.644^{3}
	RMSE($\hat{\alpha}$)	1.299^{5}	0.010^{1}	0.744^{4}	2.931^{6}	5.000^{7}	0.602^{2}	13.048^{8}	0.686^{3}
	Bias($\hat{\theta}$)	0.516^{8}	-0.150^{3}	-0.207^{6}	-0.240^{7}	0.041^{1}	-0.150^{5}	-0.145^{2}	-0.150^{4}
	RMSE($\hat{\theta}$)	0.517^{8}	0.150^{3}	0.207^{6}	0.240^{7}	0.041^{1}	0.150^{5}	0.147^{2}	0.150^{4}
	D_{abs}	0.999^{8}	0.331^{2}	0.458^{6}	0.531^{7}	0.421^{5}	0.332^{4}	0.324^{1}	0.332^{3}
	D_{max}	0.999^{7}	0.407^{2}	0.563^{5}	0.652^{6}	0.999^{8}	0.408^{4}	0.398^{1}	0.408^{3}
	$\sum$Ranks	42^{8}	12^{1}	31^{6}	40^{7}	30^{5}	22^{4}	19^{2}	20^{3}

TABLE 6
Simulation results for $\alpha = 5$, $\theta = 0.7$ and $\lambda = 1$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RTAD
10	Bias($\hat{\alpha}$)	6.184^{7}	0.100^{1}	1.117^{3}	116.572^{8}	0.923^{2}	1.118^{4}	1.139^{5}	1.220^{6}
	RMSE($\hat{\alpha}$)	6.187^{7}	0.100^{1}	1.119^{4}	116.733^{8}	0.972^{2}	1.118^{3}	1.169^{5}	1.222^{6}
	Bias($\hat{\theta}$)	0.451^{2}	-0.516^{5}	-0.516^{6}	-0.602^{8}	-0.507^{3}	-0.516^{7}	-0.404^{1}	-0.516^{4}
	RMSE($\hat{\theta}$)	10.923^{8}	0.516^{3}	0.516^{4}	0.602^{7}	0.526^{6}	0.516^{5}	0.404^{1}	0.516^{2}
20	Bias($\hat{\alpha}$)	6.384^{7}	0.100^{1}	1.117^{3}	194.567^{8}	0.915^{2}	1.117^{4}	1.214^{5}	1.216^{6}
	RMSE($\hat{\alpha}$)	6.396^{7}	0.100^{1}	1.117^{3}	194.286^{8}	0.974^{2}	1.118^{4}	1.820^{6}	1.225^{5}
	Bias($\hat{\theta}$)	1.047^{8}	-0.516^{4}	-0.516^{3}	-0.595^{7}	-0.571^{6}	-0.516^{5}	-0.404^{1}	-0.516^{1}
	RMSE($\hat{\theta}$)	23.730^{8}	0.516^{4}	0.516^{3}	0.595^{7}	0.589^{6}	0.516^{5}	0.405^{1}	0.516^{1}
50	Bias($\hat{\alpha}$)	6.532^{7}	0.100^{1}	1.116^{4}	203.260^{8}	0.877^{2}	1.114^{3}	1.642^{6}	1.208^{5}
	RMSE($\hat{\alpha}$)	6.565^{7}	0.100^{1}	1.117^{4}	204.856^{8}	0.955^{2}	1.117^{3}	5.635^{6}	1.231^{5}
	Bias($\hat{\theta}$)	5.936^{8}	-0.516^{5}	-0.516^{3}	-0.490^{2}	-0.641^{7}	-0.516^{6}	-0.404^{1}	-0.516^{4}
	RMSE($\hat{\theta}$)	106.568^{8}	0.516^{5}	0.516^{3}	0.490^{2}	0.652^{7}	0.516^{6}	0.404^{1}	0.516^{4}
75	Bias($\hat{\alpha}$)	7.282^{7}	0.100^{1}	1.115^{4}	267.416^{8}	0.844^{2}	1.114^{3}	2.054^{6}	1.206^{5}
	RMSE($\hat{\alpha}$)	7.325^{6}	0.100^{1}	1.116^{4}	268.029^{8}	0.929^{2}	1.116^{3}	8.017^{7}	1.233^{5}
	Bias($\hat{\theta}$)	3.024^{8}	-0.516^{5}	-0.516^{3}	-0.425^{2}	-0.661^{7}	-0.516^{6}	-0.403^{1}	-0.516^{4}
	RMSE($\hat{\theta}$)	46.766^{8}	0.516^{5}	0.516^{3}	0.425^{2}	0.669^{7}	0.516^{6}	0.403^{1}	0.516^{4}
100	Bias($\hat{\alpha}$)	8.348^{7}	0.100^{1}	1.115^{4}	267.170^{8}	0.799^{2}	1.114^{3}	3.653^{6}	1.208^{5}
	RMSE($\hat{\alpha}$)	8.385^{6}	0.100^{1}	1.116^{3}	268.411^{8}	0.891^{2}	1.116^{4}	15.418^{7}	1.231^{5}
	Bias($\hat{\theta}$)	57.789^{8}	-0.516^{5}	-0.516^{3}	-0.425^{2}	-0.675^{7}	-0.516^{6}	-0.400^{1}	-0.516^{4}
	RMSE($\hat{\theta}$)	104.632^{8}	0.516^{5}	0.516^{3}	0.425^{2}	0.680^{7}	0.516^{6}	0.401^{1}	0.516^{4}

TABLE 7
Simulation results for $\alpha = 0.5$, $\theta = 0.5$ and $\lambda = 0.5$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RTAD
10	Bias($\hat{\alpha}$)	-0.461^{7}	0.129^{3}	0.155^{4}	0.124^{2}	-0.091^{1}	0.240^{5}	0.550^{8}	0.249^{6}
	RMSE($\hat{\alpha}$)	0.653^{7}	0.355^{2}	0.431^{4}	0.403^{3}	0.150^{1}	0.434^{5}	1.710^{8}	0.515^{6}
	Bias($\hat{\theta}$)	987.176^{8}	0.077^{4}	0.073^{3}	0.067^{1}	-0.071^{2}	0.134^{7}	0.096^{5}	0.116^{6}
	RMSE($\hat{\theta}$)	1723.588^{8}	0.242^{3}	0.252^{4}	0.275^{6}	0.103^{1}	0.268^{5}	0.233^{2}	0.276^{7}
	Bias($\hat{\lambda}$)	91.851^{5}	0.976^{2}	93000.046^{7}	10.936^{4}	6.671^{3}	0.627^{1}	20499.052^{6}	169552.321^{8}
	RMSE($\hat{\lambda}$)	122.754^{4}	2.752^{2}	470922.796^{7}	25.066^{3}	464.056^{5}	2.377^{1}	266742.689^{6}	655496.490^{8}
	D_{abs}	0.998^{7}	0.112^{5}	0.106^{4}	0.999^{8}	0.092^{1}	0.105^{3}	0.280^{6}	0.105^{2}
	D_{max}	0.998^{7}	0.175^{5}	0.165^{4}	0.999^{8}	0.153^{1}	0.164^{2}	0.462^{6}	0.165^{3}
	$\sum$Ranks	53^{8}	26^{2}	37^{5}	35^{4}	15^{1}	29^{3}	47^{7}	46^{6}
20	Bias($\hat{\alpha}$)	-0.523^{8}	0.091^{2}	0.092^{3}	0.054^{1}	-0.099^{4}	0.151^{6}	0.204^{7}	0.127^{5}
	RMSE($\hat{\alpha}$)	0.601^{7}	0.276^{2}	0.312^{4}	0.304^{3}	0.107^{1}	0.318^{5}	1.045^{8}	0.337^{6}
	Bias($\hat{\theta}$)	1010.825^{8}	0.060^{4}	0.048^{3}	0.025^{2}	-0.076^{6}	0.094^{7}	0.024^{1}	0.069^{5}
	RMSE($\hat{\theta}$)	1963.272^{8}	0.202^{3}	0.204^{4}	0.241^{7}	0.079^{1}	0.220^{6}	0.169^{2}	0.215^{5}
	Bias($\hat{\lambda}$)	91.279^{6}	0.402^{3}	3652.375^{7}	13.470^{5}	0.130^{1}	0.250^{2}	1.551^{4}	7398.233^{8}
	RMSE($\hat{\lambda}$)	126.194^{6}	1.842^{3}	23610.394^{7}	31.000^{4}	0.144^{1}	1.695^{2}	50.814^{5}	3876.927^{8}
	D_{abs}	0.998^{7}	0.086^{5}	0.079^{2}	0.999^{8}	0.071^{1}	0.082^{4}	0.178^{6}	0.082^{3}
	D_{max}	0.998^{7}	0.143^{5}	0.128^{1}	0.999^{8}	0.129^{2}	0.133^{3}	0.293^{6}	0.135^{4}
	$\sum$Ranks	57^{8}	27^{2}	31^{3}	38^{5}	17^{1}	35^{4}	39^{6}	44^{7}
50	Bias($\hat{\alpha}$)	-0.585^{8}	0.051^{5}	0.020^{3}	0.001^{1}	-0.094^{7}	0.075^{6}	-0.006^{2}	0.047^{4}
	RMSE($\hat{\alpha}$)	0.585^{8}	0.183^{4}	0.164^{2}	0.200^{6}	0.100^{1}	0.200^{5}	0.273^{7}	0.174^{3}
	Bias($\hat{\theta}$)	2342.443^{8}	0.039^{5}	0.007^{1}	-0.008^{2}	-0.066^{7}	0.056^{6}	-0.032^{3}	0.034^{4}
	RMSE($\hat{\theta}$)	3848.028^{8}	0.149^{5}	0.132^{3}	0.189^{7}	0.068^{1}	0.158^{6}	0.107^{2}	0.142^{4}
	Bias($\hat{\lambda}$)	167.993^{6}	0.214^{3}	8097.553^{7}	19.331^{5}	0.127^{2}	0.104^{1}	0.225^{4}	9906.849^{8}
	RMSE($\hat{\lambda}$)	207.356^{6}	1.974^{4}	90739.361^{7}	40.669^{5}	0.142^{1}	1.744^{3}	1.259^{2}	112496.280^{8}
	D_{abs}	0.999^{7}	0.065^{5}	0.057^{1}	0.999^{8}	0.058^{2}	0.064^{4}	0.093^{6}	0.063^{3}
	D_{max}	0.999^{7}	0.121^{5}	0.096^{1}	0.999^{8}	0.109^{2}	0.115^{4}	0.157^{6}	0.115^{3}
	$\sum$Ranks	58^{8}	36^{5}	25^{2}	42^{7}	23^{1}	35^{4}	32^{3}	37^{6}
75	Bias($\hat{\alpha}$)	-0.585^{8}	0.036^{5}	-0.001^{1}	-0.009^{2}	-0.090^{7}	0.052^{6}	-0.031^{4}	0.030^{3}
	RMSE($\hat{\alpha}$)	0.586^{8}	0.145^{4}	0.116^{2}	0.168^{6}	0.096^{1}	0.155^{5}	0.224^{7}	0.129^{3}
	Bias($\hat{\theta}$)	1718.612^{8}	0.029^{4}	-0.009^{1}	-0.017^{2}	-0.062^{7}	0.040^{5}	-0.046^{6}	0.024^{3}
	RMSE($\hat{\theta}$)	3159.931^{8}	0.124^{5}	0.104^{3}	0.171^{7}	0.065^{1}	0.129^{6}	0.089^{2}	0.115^{4}
	Bias($\hat{\lambda}$)	132.417^{6}	-0.005^{1}	1710.164^{7}	19.299^{5}	0.127^{3}	-0.057^{2}	0.207^{4}	3176.807^{8}
	RMSE($\hat{\lambda}$)	173.835^{6}	1.392^{4}	25716.268^{7}	44.044^{5}	0.142^{1}	1.313^{3}	0.995^{2}	52977.800^{8}
	D_{abs}	0.999^{7}	0.060^{5}	0.051^{1}	0.999^{8}	0.054^{2}	0.059^{2}	0.073^{6}	0.058^{3}
	D_{max}	0.999^{7}	0.116^{5}	0.087^{1}	0.999^{8}	0.102^{2}	0.112^{2}	0.127^{6}	0.111^{3}
	$\sum$Ranks	58^{8}	33^{3}	23^{1}	43^{7}	24^{2}	$35^{4.5}$	37^{6}	$35^{4.5}$
100	Bias($\hat{\alpha}$)	-0.585^{8}	0.028^{4}	-0.011^{1}	-0.015^{2}	-0.087^{7}	0.040^{5}	-0.045^{6}	0.023^{3}
	RMSE($\hat{\alpha}$)	0.589^{8}	0.120^{5}	0.092^{1}	0.151^{7}	0.093^{2}	0.127^{6}	0.118^{4}	0.103^{3}
	Bias($\hat{\theta}$)	2693.603^{8}	0.024^{4}	-0.017^{1}	-0.021^{3}	-0.059^{7}	0.033^{5}	-0.051^{6}	0.019^{2}
	RMSE($\hat{\theta}$)	5072.659^{8}	0.106^{5}	0.088^{3}	0.159^{7}	0.061^{1}	0.111^{6}	0.080^{2}	0.098^{4}
	Bias($\hat{\lambda}$)	168.672^{6}	-0.137^{2}	971.249^{8}	23.158^{5}	0.126^{1}	-0.173^{3}	0.173^{4}	438.540^{7}
	RMSE($\hat{\lambda}$)	232.431^{6}	0.964^{4}	17087.004^{8}	49.297^{5}	0.142^{1}	0.875^{3}	0.432^{2}	13122.398^{7}
	D_{abs}	0.998^{7}	0.056^{5}	0.047^{1}	0.999^{8}	0.052^{2}	0.055^{4}	0.065^{6}	0.055^{3}
	D_{max}	0.998^{7}	0.114^{5}	0.081^{1}	0.999^{8}	0.096^{2}	0.110^{4}	0.115^{6}	0.110^{3}
	$\sum$Ranks	58^{8}	34^{4}	24^{2}	45^{7}	23^{1}	$36^{5.5}$	$36^{5.5}$	32^{3}

TABLE 8
Simulation results for $\alpha = 0.5$, $\theta = 0.7$ and $\lambda = 0.7$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RTAD
10	Bias($\hat{\alpha}$)	0.771^{8}	0.281^{3}	0.365^{5}	0.164^{1}	-0.349^{4}	0.44^{6}	0.175^{2}	0.47^{7}
	RMSE($\hat{\alpha}$)	1.046^{7}	0.516^{2}	0.663^{5}	0.554^{3}	0.355^{1}	0.626^{4}	1.619^{8}	0.78^{6}
	Bias($\hat{\theta}$)	0.341^{8}	0.173^{3}	0.181^{4}	0.069^{2}	-0.196^{6}	0.237^{7}	-0.039^{1}	0.19^{5}
	RMSE($\hat{\theta}$)	0.352^{8}	0.296^{3}	0.318^{5}	0.304^{4}	0.201^{2}	0.326^{6}	0.157^{1}	0.33^{7}
	Bias($\hat{\lambda}$)	$8.903e07^{8}$	-0.209^{1}	16887.02^{5}	10.668^{4}	0.214^{2}	-0.351^{3}	$5.599e07^{7}$	48334.62^{6}
	RMSE($\hat{\lambda}$)	$2.889e09^{7}$	1.343^{3}	22504.09^{5}	23.898^{4}	0.231^{1}	1.195^{2}	$2.978e09^{8}$	48296.09^{6}
	D_{abs}	0.999^{8}	0.164^{4}	0.159^{3}	0.166^{5}	0.195^{6}	0.156^{2}	0.998^{7}	0.155^{1}
	D_{max}	0.999^{8}	0.366^{5}	0.363^{4}	0.342^{3}	0.429^{6}	0.336^{2}	0.998^{7}	0.326^{1}
	$\sum$Ranks	62^{8}	24^{1}	36^{5}	26^{2}	28^{3}	32^{4}	41^{7}	39^{6}
20	Bias($\hat{\alpha}$)	0.563^{8}	0.363^{4}	0.498^{7}	0.085^{2}	-0.338^{3}	0.460^{6}	-0.031^{1}	0.369^{5}
	RMSE($\hat{\alpha}$)	0.631^{7}	0.514^{3}	0.704^{8}	0.441^{2}	0.343^{1}	0.587^{5}	0.579^{4}	0.614^{6}
	Bias($\hat{\theta}$)	0.327^{8}	0.232^{5}	0.261^{6}	0.032^{1}	-0.196^{4}	0.270^{7}	-0.073^{2}	0.187^{3}
	RMSE($\hat{\theta}$)	0.332^{7}	0.300^{5}	0.338^{8}	0.269^{3}	0.199^{2}	0.324^{6}	0.123^{1}	0.289^{4}
	Bias($\hat{\lambda}$)	467478.567^{8}	-0.578^{2}	498.891^{6}	13.745^{4}	0.210^{1}	-0.617^{3}	227.945^{5}	5173.632^{7}
	RMSE($\hat{\lambda}$)	684114.356^{8}	0.865^{3}	18904.195^{6}	31.243^{4}	0.224^{1}	0.845^{2}	2806.788^{5}	11367.590^{7}
	D_{abs}	0.954^{7}	0.142^{5}	0.137^{1}	0.999^{8}	0.187^{6}	0.138^{3}	0.148^{5}	0.137^{2}
	D_{max}	0.999^{7}	0.386^{3}	0.387^{4}	0.999^{8}	0.457^{6}	0.365^{2}	0.388^{5}	0.350^{1}
	$\sum$Ranks	60^{8}	29^{3}	46^{7}	32^{4}	24^{1}	34^{5}	28^{2}	35^{6}
50	Bias($\hat{\alpha}$)	0.479^{7}	0.420^{5}	0.556^{8}	0.015^{1}	-0.313^{4}	0.465^{6}	-0.082^{2}	0.278^{3}
	RMSE($\hat{\alpha}$)	0.502^{6}	0.498^{5}	0.678^{8}	0.309^{2}	0.318^{3}	0.535^{7}	0.226^{1}	0.432^{4}
	Bias($\hat{\theta}$)	0.319^{8}	0.280^{5}	0.315^{7}	-0.003^{1}	-0.190^{4}	0.298^{6}	-0.089^{2}	0.181^{3}
	RMSE($\hat{\theta}$)	0.321^{6}	0.309^{5}	0.351^{8}	0.217^{3}	0.191^{2}	0.323^{7}	0.130^{1}	0.245^{4}
	Bias($\hat{\lambda}$)	1398.828^{8}	-0.724^{4}	-0.598^{2}	15.990^{6}	0.201^{1}	-0.734^{5}	19.718^{7}	-0.603^{3}
	RMSE($\hat{\lambda}$)	8789.764^{8}	0.762^{3}	10.508^{5}	40.036^{6}	0.217^{1}	0.750^{2}	53.653^{7}	0.910^{4}
	D_{abs}	0.945^{7}	0.126^{4}	0.120^{1}	0.999^{8}	0.178^{6}	0.124^{3}	0.139^{5}	0.124^{2}
	D_{max}	0.998^{7}	0.396^{4}	0.395^{3}	0.999^{8}	0.447^{5}	0.387^{2}	0.456^{6}	0.370^{1}
	$\sum$Ranks	57^{8}	$35^{4.5}$	42^{7}	$35^{4.5}$	26^{2}	38^{6}	31^{3}	24^{1}
75	Bias($\hat{\alpha}$)	0.464^{7}	0.431^{5}	0.542^{8}	0.001^{1}	-0.300^{4}	0.462^{6} -	0.095^{2}	0.254^{3}
	RMSE($\hat{\alpha}$)	0.479^{5}	0.488^{6}	0.639^{8}	0.263^{2}	0.305^{3}	0.515^{7}	0.157^{1}	0.374^{4}
	Bias($\hat{\theta}$)	0.317^{7}	0.291^{5}	0.319^{8}	-0.010^{1}	-0.185^{4}	0.303^{6}	-0.096^{2}	0.180^{3}
	RMSE($\hat{\theta}$)	0.318^{6}	0.311^{5}	0.346^{8}	0.195^{3}	0.186^{2}	0.321^{7}	0.139^{1}	0.229^{4}
	Bias($\hat{\lambda}$)	1306.167^{8}	-0.739^{3}	-0.758^{5}	16.325^{6}	0.194^{1}	-0.744^{4}	31.021^{7}	-0.625^{2}
	RMSE($\hat{\lambda}$)	3659.366^{8}	0.743^{3}	0.765^{5}	44.457^{6}	0.211^{1}	0.748^{4}	72.954^{7}	0.643^{2}
	D_{abs}	0.943^{8}	0.122^{4}	0.116^{1}	0.130^{5}	0.174^{7}	0.121^{3}	0.146^{6}	0.121^{2}
	D_{max}	0.999^{8}	0.396^{5}	0.393^{4}	0.390^{2}	0.435^{6}	0.390^{3}	0.529^{7}	0.374^{1}
	$\sum$Ranks	57^{8}	36^{5}	47^{7}	26^{2}	28^{3}	40^{6}	33^{4}	21^{1}
100	Bias($\hat{\alpha}$)	0.455^{6}	0.435^{5}	0.529^{8}	-0.007^{1}	-0.287^{4}	0.460^{7}	-0.103^{2}	0.240^{3}
	RMSE($\hat{\alpha}$)	0.467^{5}	0.482^{6}	0.614^{8}	0.235^{2}	0.293^{3}	0.504^{7}	0.162^{1}	0.337^{4}
	Bias($\hat{\theta}$)	0.316^{7}	0.296^{5}	0.320^{8}	-0.014^{1}	-0.181^{4}	0.306^{6}	-0.104^{2}	0.178^{3}
	RMSE($\hat{\theta}$)	0.317^{6}	0.312^{5}	0.343^{8}	0.182^{3}	0.182^{2}	0.321^{7}	0.147^{1}	0.220^{4}
	Bias($\hat{\lambda}$)	9870.329^{8}	-0.744^{3}	-0.758^{5}	14.992^{6}	0.187^{1}	-0.748^{4}	39.415^{7}	-0.631^{2}
	RMSE($\hat{\lambda}$)	1817.763^{8}	0.747^{3}	0.764^{5}	42.873^{6}	0.207^{1}	0.751^{4}	84.683^{7}	0.644^{2}
	D_{abs}	0.942^{8}	0.120^{4}	0.113^{1}	0.126^{5}	0.172^{7}	0.120^{3}	0.152^{6}	0.119^{2}
	D_{max}	0.999^{8}	0.396^{5}	0.390^{3}	0.380^{2}	0.425^{6}	0.392^{4}	0.579^{7}	0.375^{1}
	$\sum$Ranks	56^{8}	36^{5}	46^{7}	26^{2}	28^{3}	42^{6}	33^{4}	21^{1}

TABLE 9
Simulation results for $\alpha = 0.7$, $\theta = 0.9$ and $\lambda = 0.3$.

n	Est.	MLE	LSE	WLS	PCE	MPS	CVM	AD	RTAD
10	Bias($\hat{\alpha}$)	-0.211^{3}	0.144^{2}	0.293^{4}	0.075^{1}	0.600^{7}	0.300^{5}	1.021^{8}	0.428^{6}
	RMSE($\hat{\alpha}$)	1.311^{7}	0.405^{1}	0.611^{4}	0.514^{3}	0.880^{6}	0.480^{2}	2.773^{8}	0.755^{5}
	Bias($\hat{\theta}$)	746.298^{8}	0.052^{2}	0.070^{4}	-0.026^{1}	-0.290^{7}	0.104^{6}	0.058^{3}	0.084^{5}
	RMSE($\hat{\theta}$)	149.152^{8}	0.197^{2}	0.228^{4}	0.246^{6}	0.378^{7}	0.197^{3}	0.156^{1}	0.238^{5}
	Bias($\hat{\lambda}$)	73.258^{6}	0.188^{2}	9101.656^{7}	11.413^{5}	1.816^{4}	-0.063^{1}	1.315^{3}	4215.683^{8}
	RMSE($\hat{\lambda}$)	116.493^{6}	1.608^{2}	1237.868^{7}	23.070^{5}	19.321^{4}	1.298^{1}	8.110^{3}	253.486^{8}
	D_{abs}	0.999^{8}	0.145^{4}	0.139^{3}	0.148^{5}	0.409^{7}	0.137^{2}	0.313^{6}	0.137^{1}
	D_{max}	0.999^{8}	0.277^{5}	0.273^{4}	0.260^{3}	0.741^{7}	0.250^{2}	0.487^{6}	0.243^{1}
	$\sum$Ranks	54^{8}	20^{1}	37^{4}	29^{3}	49^{7}	22^{2}	38^{5}	39^{6}
20	Bias($\hat{\alpha}$)	-0.434^{5}	0.248^{2}	0.484^{7}	0.026^{1}	0.503^{8}	0.337^{3}	0.442^{6}	0.377^{4}
	RMSE($\hat{\alpha}$)	1.127^{6}	0.402^{1}	0.671^{5}	0.437^{2}	5.625^{8}	0.457^{3}	1.680^{7}	0.645^{4}
	Bias($\hat{\theta}$)	106.642^{8}	0.113^{4}	0.156^{6}	-0.042^{2}	-0.243^{7}	0.140^{5}	0.026^{1}	0.098^{3}
	RMSE($\hat{\theta}$)	285.449^{8}	0.186^{2}	0.230^{6}	0.229^{5}	0.322^{7}	0.194^{3}	0.123^{1}	0.214^{4}
	Bias($\hat{\lambda}$)	91.241^{6}	-0.326^{1}	133.275^{7}	14.518^{5}	1.915^{4}	-0.393^{2}	0.401^{3}	129.039^{8}
	RMSE($\hat{\lambda}$)	140.674^{6}	1.094^{2}	254.649^{7}	28.829^{4}	53.592^{5}	1.034^{1}	3.671^{3}	1946.017^{8}
	D_{abs}	0.999^{8}	0.124^{4}	0.118^{1}	0.126^{5}	0.373^{7}	0.120^{3}	0.203^{6}	0.119^{2}
	D_{max}	0.999^{8}	0.288^{4}	0.294^{5}	0.243^{1}	0.670^{7}	0.267^{3}	0.317^{6}	0.254^{2}
	$\sum$Ranks	55^{8}	20^{1}	44^{6}	25^{3}	53^{7}	23^{2}	33^{4}	35^{5}
50	Bias($\hat{\alpha}$)	-0.932^{8}	0.350^{4}	0.694^{7}	-0.012^{1}	0.094^{2}	0.388^{6}	0.169^{3}	0.350^{5}
	RMSE($\hat{\alpha}$)	1.092^{8}	0.409^{2}	0.752^{7}	0.347^{1}	0.417^{3}	0.439^{4}	0.722^{6}	0.533^{5}
	Bias($\hat{\theta}$)	2547.841^{8}	0.170^{4}	0.242^{7}	-0.049^{2}	-0.207^{6}	0.180^{5}	0.002^{1}	0.125^{3}
	RMSE($\hat{\theta}$)	4299.216^{8}	0.189^{2}	0.254^{7}	0.200^{5}	0.236^{6}	0.195^{4}	0.086^{1}	0.191^{3}
	Bias($\hat{\lambda}$)	188.037^{8}	-0.665^{2}	83.237^{6}	16.616^{5}	1.002^{4}	-0.675^{3}	0.141^{1}	135.600^{7}
	RMSE($\hat{\lambda}$)	239.744^{6}	0.814^{3}	546.800^{8}	32.818^{5}	1.066^{4}	0.806^{1}	0.813^{2}	4374.072^{7}
	D_{abs}	0.999^{8}	0.109^{5}	0.103^{1}	0.108^{4}	0.332^{7}	0.107^{3}	0.128^{6}	0.107^{2}
	D_{max}	0.999^{8}	0.303^{5}	0.324^{6}	0.224^{2}	0.539^{7}	0.292^{4}	0.198^{1}	0.276^{3}
	$\sum$Ranks	62^{8}	27^{3}	49^{7}	25^{2}	39^{6}	30^{4}	21^{1}	35^{5}
75	Bias($\hat{\alpha}$)	-1.018^{8}	0.377^{5}	0.729^{7}	-0.017^{1}	0.089^{2}	0.403^{6}	0.111^{3}	0.346^{4}
	RMSE($\hat{\alpha}$)	1.094^{8}	0.414^{3}	0.760^{7}	0.312^{1}	0.362^{2}	0.435^{4}	0.454^{5}	0.495^{6}
	Bias($\hat{\theta}$)	342.421^{8}	0.184^{4}	0.257^{7}	-0.047^{2}	-0.196^{6}	0.190^{5}	-0.007^{1}	0.136^{3}
	RMSE($\hat{\theta}$)	648.897^{8}	0.193^{4}	0.260^{7}	0.186^{3}	0.214^{6}	0.198^{5}	0.073^{1}	0.185^{2}
	Bias($\hat{\lambda}$)	208.067^{8}	-0.716^{3}	0.147^{1}	16.745^{6}	1.057^{5}	-0.720^{4}	0.174^{2}	99.342^{7}
	RMSE($\hat{\lambda}$)	273.353^{7}	0.731^{1}	65.498^{6}	35.883^{5}	1.115^{3}	0.731^{2}	1.176^{4}	473.660^{8}
	D_{abs}	0.999^{8}	0.106^{5}	0.101^{1}	0.105^{2}	0.337^{7}	0.105^{4}	0.113^{6}	0.105^{3}
	D_{max}	0.999^{8}	0.306^{5}	0.330^{6}	0.219^{2}	0.520^{7}	0.299^{4}	0.171^{1}	0.283^{3}
	$\sum$Ranks	63^{8}	30^{3}	42^{7}	22^{1}	38^{6}	34^{4}	23^{2}	36^{5}
100	Bias($\hat{\alpha}$)	-1.063^{8}	0.392^{5}	0.734^{7}	-0.022^{1}	0.110^{3}	0.411^{6}	0.089^{2}	0.344^{4}
	RMSE($\hat{\alpha}$)	1.309^{8}	0.417^{3}	0.758^{7}	0.290^{1}	0.373^{2}	0.434^{5}	0.425^{4}	0.471^{6}
	Bias($\hat{\theta}$)	2846.790^{8}	0.190^{5}	0.260^{7}	-0.046^{2}	-0.183^{4}	0.195^{6}	-0.013^{1}	0.143^{3}
	RMSE($\hat{\theta}$)	5139.224^{8}	0.196^{4}	0.262^{7}	0.176^{2}	0.201^{6}	0.200^{5}	0.063^{1}	0.181^{3}
	Bias($\hat{\lambda}$)	230.887^{8}	-0.728^{3}	-0.738^{5}	15.681^{7}	1.022^{6}	-0.730^{4}	0.151^{1}	-0.537^{2}
	RMSE($\hat{\lambda}$)	305.767^{8}	0.735^{2}	4.954^{6}	35.345^{7}	1.122^{4}	0.738^{3}	0.683^{1}	4.258^{5}
	D_{abs}	0.999^{8}	0.104^{6}	0.099^{1}	0.102^{3}	0.338^{7}	0.104^{4}	0.102^{2}	0.104^{5}
	D_{max}	0.999^{8}	0.307^{5}	0.332^{6}	0.214^{2}	0.517^{7}	0.301^{4}	0.150^{1}	0.287^{3}
	$\sum$Ranks	64^{8}	33^{4}	46^{7}	25^{2}	39^{6}	37^{5}	13^{1}	31^{3}

5. Real data analysis

In this section, we compare the fits of the DNH distribution with Poisson (P), negative binomial (NB), and zero inflated Poisson (ZIP) distributions by means of two data sets to illustrate the potentiality of the DNH model. The PMFs of these distributions are given as:

$$P(X=x)=\frac{\lambda^x \exp^{-\lambda}}{x!}, X=0,1,2\cdots;\lambda>0, \tag{65}$$

$$P(X=x)=\binom{x+r-1}{k}p^x(1-p)^r, 0<p\leq 1; k,r>0, \tag{66}$$

$$P(X=x)=\{\pi+(1-\pi)\exp(-\lambda)\}\mathbf{1}_{x=0}+(1-\pi)\frac{\lambda^x \exp^{-\lambda}}{x!}\mathbf{1}_{x>0},$$
$$0<\pi\leq 1, \lambda>0, x=0,1,2,\cdots \tag{67}$$

To fit the distributions, the `fitdistrplus` `R` package is used. The fits of these four distributions are presented in the form of comparative density plot and the plot of the distribution functions. The histogram is the best tool for showing the classical goodness of fit of a theoretical model to observed data. For model comparison, we have taken into account the Anderson Darling test and the likelihood-based statistics (Akaike's information criterion (AIC) and the Bayesian information criterion (BIC)). A distribution with lowest AIC and BIC is said to be better fit to the data. Note that $AIC=2m-2LL$ and $BIC=m\log(n)-2LL$, where m is the number of parameters in the statistical model, n the sample size, and LL is the maximized value of the logarithmic likelihood function for the estimated model.

The first data set (Table 10) is taken from Bakouch *et al.* (2014). This data is about the numbers of fire in Greece forest districts for the period from 1st July 1998 to 31st August 1998. Only fires in forest districts are considered. There are 123 observation recorded in this data set. The parameter estimates along with goodness of fit measures are given in Table 11. The graphical depiction is given in Figure 3 and Figure 4. The second data set is taken from Bakouch *et al.* (2014) which is given in Table 12, consists of the 2003 final examination marks of 48 slow pace students in mathematics in the Indian Institute of Technology at Kanpur. The parameter estimates along with goodness of fit measures are given in Table 13. From these tables, on the basis of AIC, BIC and log-likelihood, the DNH is the best as compared to other models like Poisson, negative-binomial and zero-inflated Poisson, since DNH model has the lowest AIC and BIC as compared to other models. The graphical depiction for this data set is given in Figure 5 and Figure 6.

TABLE 10
Forest fire in Greece.

2, 4, 4, 3, 3, 1, 2, 4, 3, 1, 1, 0, 5, 5, 0, 3, 1, 1, 0, 1, 0, 2, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0 0, 1, 4, 2, 2, 1, 2, 1, 2, 0, 2, 2, 1, 0, 3, 2, 1, 2, 2, 7, 3, 5, 2, 5, 4, 5, 6, 5, 4, 3, 8, 43, 8, 4, 4, 3, 10, 5, 4, 5, 12, 3, 8, 12, 10, 11, 6, 1, 8, 9, 12, 9, 4, 8, 12, 11, 8, 6, 4, 7, 9, 15, 12, 15, 15, 12, 9, 16, 7, 11, 9, 11, 6, 5, 20, 9, 8

TABLE 11
Parameter estimation and comparison with some existing distributions for forest fire data.

Model	Estimates	Loglik.	AIC	BIC	AD	p-value
P	$\hat{\lambda}$=5.802 (0.219)	−598.971	1199.942	1202.738	1334.5	< 0.001
NB	$\hat{\mu}$=5.801, $\hat{k}$=0.994 (0.572, 0.151)	−343.610	691.221	696.812	735.88	< 0.001
ZIP	$\hat{\mu}$=6.677, $\hat{\sigma}$=0.131 (91.55, 0.000)	−546.538	1097.076	1102.668	632.86	< 0.001
DNH	$\hat{\theta}$=0.912, $\hat{\alpha}$=1.165 (0.017, 0.071)	−237.703	479.406	484.998	4.363	0.15

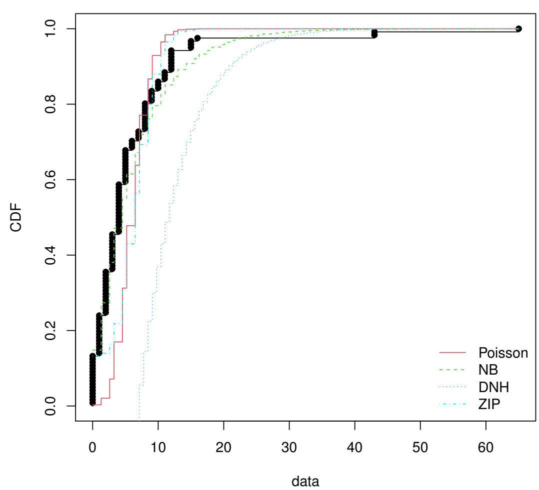


Figure 3 – Empirical CDF plot for the forest fire data.

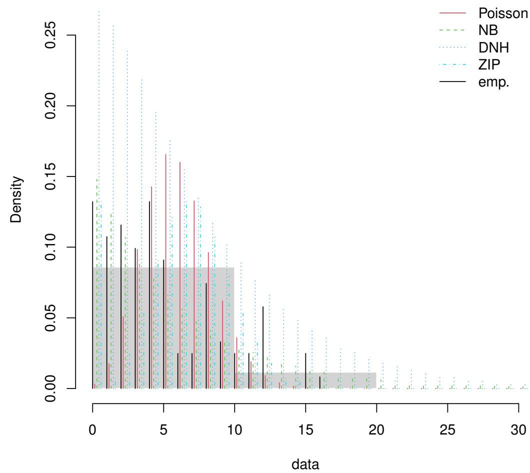


Figure 4 – Density plot for the forest fire data.

TABLE 12
Data on examination marks of mathematics students.

29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, 31

TABLE 13
Model summaries for data on examination marks in mathematics.

Model	Estimates	Loglik.	AIC	BIC	AD	p-value
P	$\hat{\lambda}$=25.896 (0.735)	−396.589	795.178	797.049	63761.86	<0.001
NB	$\hat{\mu}$=25.896, $\hat{k}$=2.444 (2.501, 0.522)	−197.522	399.044	402.786	19.141	<0.001
ZIP	$\hat{\mu}$=25.897, $\hat{\sigma}$=0.001 (23.62, 0.048)	−396.637	797.275	801.017	10019.141	<0.001
DNH	$\hat{\theta}$=0.996, $\hat{\alpha}$=1.610 (0.996, 1.610)	−151.229	306.458	310.201	4.346	0.12

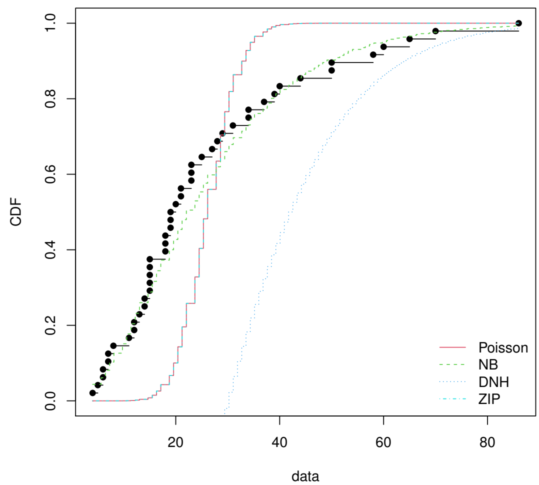


Figure 5 – Empirical CDF plot for the data on examination marks in mathematics.

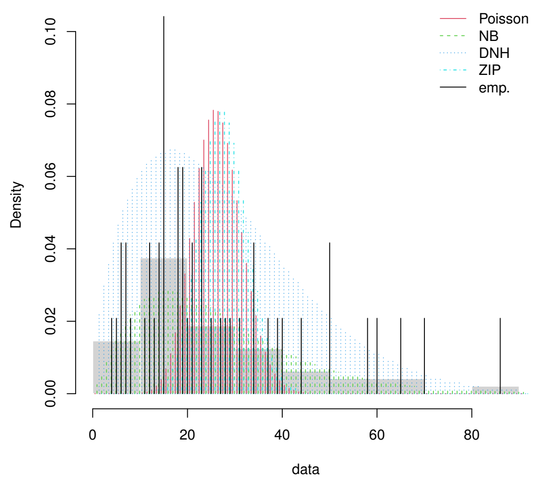


Figure 6 – Density plot for the data on examination marks in mathematics.

6. Conclusion

In this article, a new discrete probability distribution known as DNH distribution is proposed which can serve as an alternatives to Poisson (P), negative binomial (NB), and zero inflated Poisson (ZIP) distributions. Some mathematical properties of the proposed model are presented. The estimation of model parameters are obtained by eight different methods of estimation, namely, maximum likelihood, the least squares, weighted least squares, percentile, maximum product of spacing, Cramer-von Mises, Anderson-Darling, and right tail Anderson-Darling. We have carried out a comprehensive Monte Carlo simulation experiment to compare these methods. The estimators are compared on the basis of biases, root mean-squared errors, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. We provide two applications to real data sets. We illustrate the usefulness of the proposed distribution for modeling these kind of data and also prove empirically that the DNH distribution is quite competitive to Poisson, negative binomial, and zero inflated Poisson distributions and is expected that in some situations it might work better (in terms of model fitting) than the models stated, although it cannot be always guaranteed. We hope that the DNH distribution attract wider sets of applications such as survival analysis.

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SUMMARY

An extension of the exponential distribution due to Nadarajah and Haghighi referred to as Nadarajah and Haghighi (NH) distribution is an alternative that always provides better fits than the gamma, Weibull, and the generalized exponential distributions whenever the data contains zero values. However, in practice, discrete data is easy to collect as compared to continuous data. Thus, keeping in mind the utility of discrete data, we introduce the discrete analogue of NH distribution. Our main focus is the estimation from the frequentist point of view of the unknown parameters along with deriving some mathematical properties of the new model. We briefly describe different frequentist approaches, namely, maximum likelihood, percentile based, least squares, weighted least squares, maximum product of spacings, Cramèr-von-Mises, Anderson-Darling, and right-tail Anderson-Darling estimators, and compare them using extensive numerical simulations. Monte Carlo simulations are performed to compare the performances of the proposed methods of estimation for both small and large samples. The potentiality of the distribution is analyzed by means of two real data sets.

Keywords: Maximum likelihood estimator; Least square estimator; Percentile estimator; Anderson Darling estimator; Nadarajah and Haghighi distribution.