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MUTH DISTRIBUTION AND ESTIMATION OF A PARAMETER USING ORDER STATISTICS

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1. Introduction

A continuous random variable X is said to have a Muth distribution with the shape parameter μ, if its probability density function (pdf) is given by

$$f(x;\mu) = (e^{\mu x} - \mu)\, e^{\mu x - \frac{1}{\mu}(e^{\mu x}-1)},\ \mu \in (0,1], x > 0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x;\mu) = 1 - e^{\mu x - \frac{1}{\mu}(e^{\mu x}-1)},\ x > 0. \tag{2}$$

This distribution was first introduced by the French biologist Teissier (1934) by considering a mortality of several domestic animal species dying as a result of pure ageing. Based on the empirical analysis of several species, the author identified that animal mortality does not follow the human one as advocated by the famous Gomperts law used in actuarial practice (see Gompertz, 1825). After a long gap, Laurent (1975) introduced a location version of Teissier's distribution along with its characterization based on the life expectancy and explored its applications to the demographic studies. Muth (1977) studied this distribution and observed that, it displays heavier tail compared to the well known lifetime distributions like gamma, lognormal and Weibull. Later on Rinne (1981) used this model to estimate the lifetime distribution (with lifetime expressed in Kilometers) for a German data set based on prices of used cars. After the work of Rinne (1981), the

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Teissier's distribution and its location version have been forgotten and we did not find any further reference in the available literature. Leemis and McQueston (2008) named this distribution as the "Muth distribution" in their study on schematic representation of various univariate distributional relationships. Pedro *et al.* (2015) reintroduced this distribution and its scaled version and studied its important properties through exponential integral function.

A continuous random variable X is said to have a Muth distribution with shape parameter μ and scale parameter σ (MD(μ, σ)), if its pdf is given by

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\left(e^{\frac{\mu}{\sigma}x}-\mu\right)e^{\frac{\mu}{\sigma}x-\frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)},\ \mu\in(0,1],\sigma>0,x>0. \tag{3}$$

The corresponding cdf is given by

$$F(x;\mu,\sigma)=1-e^{\frac{\mu}{\sigma}x-\frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)},\ x>0. \tag{4}$$

To prove the dominance of the Muth(μ, σ) over some other competing models, Pedro *et al.* (2015) considered a rainfall (in mm) data collected from the rain gauge station of Carrol, located in the State of New South Wales on the east coast of Australia and are given in Appendix A.

The authors proved that its supremacy compared to the well known lifetime distributions like exponential, gamma, lognormal and Weibull based on certain information criterion such as, Akaike information criterion (AIC) and Bayesian information criterion (BIC). For establishing the fitting ability of the MD(μ, σ) compared to other lifetime distributions, we fit the rainfall data using maximum likelihood estimation method. Let $X_1, X_2, ..., X_n$ be a random sample of size n from (3) with unknown parameter vector $\delta=[\mu,\sigma]^T$. Then the likelihood function of δ is given by

$$l(\delta)=\frac{1}{\sigma^n}\prod_{i=1}^{n}\left(e^{\frac{\mu}{\sigma}x_i}-\mu\right)e^{\sum_{i=1}^{n}\left\{\frac{\mu}{\sigma}x_i-\frac{1}{\mu}(e^{\frac{\mu}{\sigma}x_i}-\mu)\right\}}. \tag{5}$$

The log-likelihood function is given by

$$\log l(\delta)=-n\log\sigma+\frac{\mu}{\sigma}\sum_{i=1}^{n}x_i-\frac{1}{\mu}\sum_{i=1}^{n}\left(e^{\frac{\mu}{\sigma}x_i}-1\right)+\sum_{i=1}^{n}\log\left(e^{\frac{\mu}{\sigma}x_i}-1\right). \tag{6}$$

The partial derivatives of $\log l(\delta)$ with respect to the parameters μ and σ are respectively given by

$$\frac{\partial\log l(\delta)}{\partial\mu}=\frac{1}{\sigma}\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\left[\frac{x_i}{\mu\sigma}e^{\frac{\mu}{\sigma}x_i}-\frac{e^{\frac{\mu}{\sigma}x_i}}{\mu^2}+\frac{1}{\mu^2}\right]+\sum_{i=1}^{n}\left[\frac{\frac{x_i}{\sigma}e^{\frac{\mu}{\sigma}x_i}-1}{e^{\frac{\mu}{\sigma}x_i}-\mu}\right] \tag{7}$$

and

$$\frac{\partial \log l(\delta)}{\partial \sigma} = -\frac{n}{\sigma} - \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i e^{\frac{\mu}{\sigma}x_i} - \sum_{i=1}^{n}\left[\frac{\frac{\mu}{\sigma^2}x_i e^{\frac{\mu}{\sigma}x_i}}{e^{\frac{\mu}{\sigma}x_i}-\mu}\right]. \quad (8)$$

The maximum likelihood estimator's of the parameters μ and σ are respectively obtained by solving the equations $\frac{\partial \log l(\delta)}{\partial \mu} = 0$ and $\frac{\partial \log l(\delta)}{\partial \sigma} = 0$. This can only be achieved by numerical optimization techniques such as the Newton-Raphson method and Fisher's scoring algorithm using mathematical packages like `R`, `Mathematica`, etc.

TABLE 1

Parameter estimates, KS statistics and p-values for rainfall data.

Distribution	MLE	KS	p-value
Muth(μ,σ)	μ'=0.46, σ'=33.90	0.0570	0.9501
Gamma(μ,σ) $f(x) = \frac{1}{\sigma^{\mu}\Gamma(\mu)} x^{\mu-1} e^{\frac{-x}{\sigma}}$	μ'=1.52, σ'=22.38	0.0837	0.6065
Lognormal(μ,σ) $f(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-(log x-\mu)^2/2\sigma^2}$	μ'=3.16, σ'=1.03	0.1217	0.1709
Exponentiated lognormal(δ,μ,σ) $f(x) = \frac{\delta}{x\sigma}\phi\left(\frac{log x-\mu}{\sigma}\right)\left[\Phi\left(\frac{log x-\mu}{\sigma}\right)\right]^{\delta-1}$	δ'=0.01, μ'=4.78, σ'=0.13	0.1169	0.2063
Weibull(μ,σ) $f(x) = \frac{\mu}{\sigma}(\frac{x}{\sigma})^{\mu-1} e^{-(x/\sigma)^{\mu}}$	μ'=1.37, σ'=36.91	0.0765	0.7158
Loglogistic(μ,σ) $f(x) = \frac{\mu(x/\sigma)^{\mu}}{x[1+(x/\sigma)^{\mu}]^2}$	μ'=1.80, σ'=26.55	0.0964	0.4229
Exponentiated exponential(μ,σ) $f(x) = \mu\sigma(1-e^{-\sigma x})^{\mu-1} e^{-\sigma x}$	μ'=1.51, σ'=0.04	0.0872	0.5532
Dagum(δ,μ,σ) $f(x) = \frac{\delta\mu}{x}\left(\frac{(x/\sigma)^{\delta\mu}}{((x/\sigma)^{\mu}+1)}\right)$	δ'=0.11, μ'=7.85, σ'=70.71	0.0597	0.9288

For the rainfall data, using maximum likelihood estimation method we have fitted each of the following distributions: 1) gamma distribution 2) lognormal distribution 3) exponentiated lognormal distribution 4) Weibull distribution 5) loglogistic distribution 6) exponentiated exponential distribution and 7) Dagum distribution. The estimated parameters of the fitted distributions and the Kolmogorov Smirnov (KS) goodness-of-fit statistic values are given in Table 1. From the above table we observe that for the rainfall data, MD(μ,σ) performs well (smallest KS value and largest p-value) compared to other two and three parameter distributions considered here. Likewise one can find several practical situations in which MD(μ,σ) performs well. Hence this distribution may be considered as a best alternative for modeling certain real life problems. This immensely motivated us to study certain inferential aspects of MD(μ,σ) using order statistics.

While considering the maximum likelihood method of estimation we come across some demerits. When the sample size is small, maximum likelihood estimator (MLE) is not even unbiased in many cases (see Rohatgi and Ehsanes Saleh, 2009). Though one may obtain asymptotic variance of the MLE's, the exact variance for the small sample cases are generally not obtained in compact form. Thus, for small sample situation, the procedure of obtaining BLUE's of the location and scale parameters of a distribution by order statistics (see Lloyd, 1952) is considered as a very effective method of estimation. However, when the sample size is large the computation of means, variances and covariances of the order statistics of an equivalent sample size arising from the standard form of the given distribution leads the Lloyd's method of estimation very difficult. In such situations, Thomas and Sreekumar (2008) developed a method of estimation by U-statistics.

Hence the main objective of this work is to determine the BLUE of the scale parameter σ of Muth(μ,σ) for some known values of the shape parameter μ and to use them to generate appropriate U-statistics for estimating the scale parameter for any sample size. We have further established the peculiarity of these U-statistics when compared with the MLE of the scale parameter of Muth(μ,σ) in terms of their performance using a real life data set.

2. Best linear unbiased estimator of scale parameter of Muth(μ,σ) using order statistics

The pdf of the scaled Muth distribution is given in (3). Till date, there is no work in the available literature to estimate the scale parameter σ involved in (3). Hence in this section, our aim is to estimate the scale parameter σ included in (3) for some known values of the shape parameter μ.

Let $\mathbf{X}=(X_{1:m},X_{2:m},\cdots,X_{m:m})'$ be the vector of order statistics of a random sample of size m drawn from (3). Define $Y_{r:m}=\frac{X_{r:m}}{\sigma}$, $r=1,2,\cdots,m$. Then $Y_{r:m}$, $r=1,2,\cdots,m$ are distributed as the order statistics of a random sample of size m drawn from the

MD(μ, 1) with pdf given by

$$f_0(y) = (e^{\mu y} - \mu)\, e^{\mu y - \frac{1}{\mu}(e^{\mu y} - 1)},\ \mu \in (0, 1], y > 0. \tag{9}$$

Let $\alpha = (\alpha_{1:m}, \alpha_{2:m}, \cdots, \alpha_{m:m})'$ and $V = ((v_{r,s:m}))$ be the vector of means and dispersion matrix of the vector of order statistics of a random sample of size m drawn from $f_0(y)$. Then the BLUE of σ involved in (3) namely $\tilde{\sigma}$ based on order statistics is given by (see David and Nagaraja, 2003)

$$\tilde{\sigma} = \frac{\alpha' V^{-1}}{(\alpha' V^{-1} \alpha)} \mathbf{X} \tag{10}$$

with variance given by

$$\mathrm{Var}(\tilde{\sigma}) = \frac{\sigma^2}{(\alpha' V^{-1} \alpha)}. \tag{11}$$

From (10), we have $\tilde{\sigma}$ is a linear function of order statistics $X_{r:m}, r = 1, 2, \cdots, m$ and hence $\tilde{\sigma}$ can be written as

$$\tilde{\sigma} = \sum_{r=1}^{m} e_{r:m} X_{r:m}, \tag{12}$$

where $e_{r:m}, r = 1, 2, \cdots, m$ are constants independent of σ. We have computed the values of means of entire order statistics of a sample of size m arising from $f_0(y)$ given in (9) for $m = 2(1)10$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$ by using `Mathcad` software and are presented in Table 2. The variances and co-variances of entire order statistics of a sample of size m arising from (9) for $m = 2(1)10$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$ also have been evaluated using `Mathcad` software and are presented in Table 3. We have included entries for $\mu = 0.43$ in Table 2 and Table 3, in order to illustrate the proposed method of estimation in a real life example. Using the values of means given in Table 2 and values of variances and co-variances given in Table 3, we have evaluated the coefficients of $X_{r:m}$ in the BLUE of σ and its variances for $m = 2(1)10$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$ by using `Mathcad` software and are presented in Table 4.

TABLE 2
Expected values $\alpha_{r:m}$ of order statistics arising from $f_0(y)$ for $m = 2(1)10$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$.

m	r	$\alpha_{r:m}$ for $r = 1$ to m			
		$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
2	1	0.563	0.625	0.688	0.608
	2	1.438	1.375	1.313	1.393
3	1	0.394	0.463	0.542	0.443
	2	0.900	0.949	0.979	0.937
	3	1.706	1.588	1.479	1.620
4	1	0.303	0.370	0.453	0.350
	2	0.665	0.742	0.807	0.721
	3	1.136	1.157	1.151	1.154
	4	1.896	1.732	1.588	1.776
5	1	0.247	0.309	0.393	0.290
	2	0.529	0.613	0.696	0.590
	3	0.868	0.934	0.974	0.918
	4	1.315	1.305	1.270	1.311
	5	2.041	1.838	1.668	1.892
6	1	0.208	0.266	0.348	0.248
	2	0.440	0.525	0.616	0.500
	3	0.707	0.790	0.855	0.768
	4	1.030	1.077	1.092	1.067
	5	1.457	1.419	1.359	1.433
	6	2.158	1.922	1.728	1.983
7	1	0.180	0.234	0.313	0.217
	2	0.377	0.460	0.555	0.435
	3	0.598	0.688	0.768	0.663
	4	0.852	0.926	0.971	0.908
	5	1.162	1.190	1.183	1.187
	6	1.575	1.511	1.429	1.532
	7	2.255	1.990	1.780	2.059

TABLE 2
Continued.

		$\alpha_{r:m}$ for $r = 1$ to m			
m	r	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
8	1	0.158	0.209	0.286	0.192
	2	0.330	0.410	0.507	0.386
	3	0.518	0.611	0.700	0.584
	4	0.730	0.816	0.881	0.794
	5	0.976	1.036	1.062	1.023
	6	1.275	1.283	1.256	1.285
	7	1.675	1.587	1.486	1.614
	8	2.338	2.048	1.822	2.122
9	1	0.142	0.188	0.263	0.173
	2	0.293	0.370	0.468	0.346
	3	0.458	0.550	0.645	0.523
	4	0.639	0.732	0.809	0.707
	5	0.844	0.922	0.970	0.903
	6	1.081	1.127	1.135	1.118
	7	1.372	1.361	1.317	1.368
	8	1.761	1.651	1.535	1.684
	9	2.410	2.098	1.858	2.177
10	1	0.128	0.172	0.244	0.157
	2	0.264	0.338	0.435	0.315
	3	0.410	0.501	0.600	0.474
	4	0.569	0.665	0.751	0.638
	5	0.744	0.833	0.897	0.810
	6	0.943	1.011	1.043	0.995
	7	1.174	1.205	1.197	1.201
	8	1.457	1.428	1.368	1.440
	9	1.837	1.708	1.576	1.745
	10	2.474	2.141	1.889	2.225

TABLE 3

Variances and co-variances $v_{r,s:m}$ of order statistics arising from $f_0(y)$ for $1 \leq r \leq s \leq m$, $m = 2(1)10$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$.

		$v_{r,s:m}$ for $1 \leq r \leq s \leq m$				
m	r	s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
2	1	1	0.240	0.213	0.171	0.221
	1	2	0.191	0.141	0.098	0.154
	2	2	0.068	0.397	0.236	0.460
3	1	1	0.126	0.130	0.118	0.130
	1	2	0.109	0.096	0.076	0.101
	1	3	0.088	0.067	0.048	0.073
	2	2	0.296	0.220	0.150	0.241
	2	3	0.240	0.154	0.095	0.175
	3	3	0.654	0.348	0.196	0.413
4	1	1	0.077	0.088	0.089	0.086
	1	2	0.070	0.070	0.060	0.071
	1	3	0.061	0.054	0.042	0.056
	1	4	0.050	0.039	0.028	0.042
	2	2	0.171	0.148	0.111	0.157
	2	3	0.150	0.115	0.079	0.126
	2	4	0.123	0.083	0.053	0.093
	3	3	0.309	0.206	0.128	0.232
	3	4	0.253	0.149	0.087	0.173
	4	4	0.624	0.313	0.171	0.377
5	1	1	0.052	0.065	0.071	0.062
	1	2	0.048	0.053	0.050	0.052
	1	3	0.044	0.043	0.037	0.044
	1	4	0.039	0.034	0.027	0.036
	1	5	0.032	0.025	0.019	0.027
	2	2	0.113	0.109	0.089	0.112
	2	3	0.102	0.089	0.066	0.094
	2	4	0.090	0.070	0.049	0.076
	2	5	0.074	0.052	0.034	0.057
	3	3	0.190	0.146	0.099	0.159
	3	4	0.168	0.116	0.074	0.130
	3	5	0.138	0.086	0.052	0.098
	4	4	0.309	0.190	0.113	0.219
	4	5	0.255	0.141	0.080	0.166
	5	5	0.597	0.287	0.153	0.348

TABLE 3
Continued.

			$v_{r,s:m}$ for $1 \leq r \leq s \leq m$			
m	r	s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
6	1	1	0.038	0.050	0.058	0.046
	1	2	0.035	0.042	0.042	0.040
	1	3	0.033	0.035	0.032	0.035
	1	4	0.030	0.029	0.025	0.029
	1	5	0.026	0.023	0.019	0.024
	1	6	0.022	0.017	0.013	0.018
	2	2	0.080	0.084	0.074	0.084
	2	3	0.074	0.071	0.056	0.073
	2	4	0.068	0.059	0.044	0.062
	2	5	0.060	0.047	0.033	0.051
	2	6	0.049	0.035	0.024	0.039
	3	3	0.131	0.112	0.081	0.119
	3	4	0.119	0.093	0.063	0.101
	3	5	0.106	0.075	0.049	0.084
	3	6	0.087	0.056	0.035	0.064
	4	4	0.197	0.139	0.088	0.155
	4	5	0.175	0.113	0.068	0.129
	4	6	0.145	0.085	0.049	0.099
	5	5	0.304	0.177	0.102	0.207
	5	6	0.253	0.134	0.074	0.159
	6	6	0.574	0.267	0.141	0.326
7	1	1	0.029	0.040	0.049	0.036
	1	2	0.027	0.034	0.036	0.032
	1	3	0.025	0.029	0.028	0.028
	1	4	0.023	0.024	0.022	0.024
	1	5	0.021	0.021	0.018	0.021
	1	6	0.019	0.017	0.014	0.017
	1	7	0.016	0.013	0.010	0.013
	2	2	0.060	0.067	0.063	0.066
	2	3	0.056	0.058	0.049	0.058
	2	4	0.052	0.049	0.039	0.051
	2	5	0.048	0.041	0.031	0.044
	2	6	0.042	0.034	0.024	0.036
	2	7	0.035	0.026	0.018	0.028

TABLE 3
Continued.

m	r	s	$v_{r,s:m}$ for $1 \leq r \leq s \leq m$ $\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
7	3	3	0.096	0.089	0.069	0.093
	3	4	0.089	0.076	0.055	0.081
	3	5	0.081	0.064	0.044	0.070
	3	6	0.072	0.053	0.035	0.058
	3	7	0.060	0.040	0.025	0.045
	4	4	0.140	0.109	0.074	0.119
	4	5	0.128	0.092	0.059	0.102
	4	6	0.114	0.076	0.047	0.086
	4	7	0.095	0.057	0.034	0.066
	5	5	0.199	0.132	0.080	0.149
	5	6	0.178	0.108	0.064	0.125
	5	7	0.148	0.082	0.047	0.097
	6	6	0.298	0.166	0.093	0.195
	6	7	0.249	0.127	0.069	0.152
	7	7	0.553	0.251	0.131	0.309
8	1	1	0.022	0.032	0.042	0.029
	1	2	0.021	0.028	0.031	0.026
	1	3	0.020	0.024	0.025	0.023
	1	4	0.019	0.021	0.020	0.020
	1	5	0.017	0.018	0.016	0.018
	1	6	0.016	0.015	0.013	0.016
	1	7	0.014	0.013	0.010	0.013
	1	8	0.012	0.010	0.008	0.010
	2	2	0.047	0.055	0.054	0.054
	2	3	0.044	0.048	0.043	0.048
	2	4	0.041	0.042	0.035	0.042
	2	5	0.038	0.036	0.028	0.037
	2	6	0.035	0.031	0.023	0.032
	2	7	0.031	0.025	0.018	0.027
	2	8	0.026	0.019	0.014	0.021
	3	3	0.074	0.073	0.060	0.075
	3	4	0.069	0.064	0.048	0.066
	3	5	0.064	0.055	0.040	0.059
	3	6	0.059	0.047	0.033	0.051
	3	7	0.053	0.039	0.026	0.043
	3	8	0.044	0.030	0.019	0.033

TABLE 3
Continued.

			$v_{r,s:m}$ for $1 \leq r \leq s \leq m$			
m	r	s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
8	4	4	0.105	0.089	0.063	0.095
	4	5	0.098	0.077	0.052	0.084
	4	6	0.090	0.066	0.043	0.073
	4	7	0.080	0.055	0.034	0.061
	4	8	0.067	0.042	0.025	0.048
	5	5	0.144	0.105	0.067	0.116
	5	6	0.132	0.090	0.055	0.101
	5	7	0.118	0.075	0.044	0.086
	5	8	0.099	0.057	0.033	0.067
	6	6	0.199	0.125	0.074	0.143
	6	7	0.178	0.104	0.059	0.121
	6	8	0.149	0.080	0.044	0.095
	7	7	0.291	0.156	0.086	0.186
	7	8	0.244	0.121	0.065	0.146
	8	8	0.536	0.238	0.123	0.294
9	1	1	0.018	0.027	0.036	0.024
	1	2	0.017	0.023	0.028	0.022
	1	3	0.016	0.020	0.022	0.019
	1	4	0.015	0.018	0.018	0.017
	1	5	0.014	0.016	0.015	0.016
	1	6	0.013	0.014	0.012	0.014
	1	7	0.012	0.012	0.010	0.012
	1	8	0.011	0.010	0.008	0.010
	1	9	0.009	0.007	0.006	0.008
	2	2	0.037	0.046	0.048	0.044
	2	3	0.035	0.041	0.038	0.040
	2	4	0.034	0.036	0.031	0.036
	2	5	0.032	0.031	0.026	0.032
	2	6	0.029	0.027	0.022	0.028
	2	7	0.027	0.024	0.018	0.025
	2	8	0.024	0.020	0.015	0.021
	2	9	0.020	0.015	0.011	0.016

TABLE 3
Continued.

		$v_{r,s:m}$ for $1 \leq r \leq s \leq m$				
m	r	s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
9	3	3	0.058	0.062	0.053	0.062
	3	4	0.055	0.054	0.043	0.056
	3	5	0.052	0.048	0.036	0.050
	3	6	0.049	0.042	0.030	0.044
	3	7	0.045	0.036	0.025	0.039
	3	8	0.040	0.030	0.020	0.033
	3	9	0.033	0.023	0.015	0.026
	4	4	0.082	0.075	0.056	0.078
	4	5	0.077	0.066	0.046	0.070
	4	6	0.072	0.057	0.039	0.062
	4	7	0.066	0.049	0.032	0.054
	4	8	0.059	0.041	0.026	0.046
	4	9	0.050	0.032	0.020	0.036
	5	5	0.110	0.087	0.059	0.095
	5	6	0.103	0.076	0.049	0.084
	5	7	0.095	0.066	0.041	0.074
	5	8	0.085	0.055	0.033	0.063
	5	9	0.071	0.042	0.025	0.049
	6	6	0.146	0.101	0.062	0.113
	6	7	0.134	0.087	0.052	0.099
	6	8	0.120	0.073	0.042	0.084
	6	9	0.101	0.056	0.032	0.066
	7	7	0.197	0.118	0.069	0.137
	7	8	0.177	0.100	0.056	0.117
	7	9	0.148	0.077	0.042	0.092
	8	8	0.284	0.148	0.081	0.177
	8	9	0.240	0.116	0.061	0.141
	9	9	0.520	0.227	0.117	0.281
10	1	1	0.015	0.023	0.032	0.020
	1	2	0.014	0.020	0.025	0.018
	1	3	0.013	0.018	0.020	0.016
	1	4	0.013	0.016	0.016	0.015
	1	5	0.012	0.014	0.014	0.013
	1	6	0.011	0.012	0.012	0.012
	1	7	0.011	0.011	0.010	0.011
	1	8	0.010	0.009	0.008	0.009
	1	9	0.009	0.008	0.007	0.008
	1	10	0.007	0.006	0.005	0.037

TABLE 3
Continued.

			$v_{r,s:m}$ for $1 \leq r \leq s \leq m$			
m	r	s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
10	2	2	0.031	0.039	0.043	0.037
	2	3	0.029	0.035	0.034	0.037
	2	4	0.028	0.031	0.028	0.031
	2	5	0.026	0.028	0.024	0.028
	2	6	0.025	0.024	0.020	0.025
	2	7	0.023	0.021	0.017	0.022
	2	8	0.021	0.019	0.014	0.019
	2	9	0.019	0.016	0.012	0.017
	2	10	0.016	0.012	0.009	0.013
	3	3	0.048	0.053	0.047	0.052
	3	4	0.045	0.047	0.039	0.047
	3	5	0.043	0.042	0.033	0.043
	3	6	0.040	0.037	0.028	0.039
	3	7	0.038	0.032	0.024	0.034
	3	8	0.035	0.028	0.020	0.030
	3	9	0.031	0.024	0.016	0.026
	3	10	0.026	0.018	0.012	0.020
	4	4	0.066	0.064	0.050	0.066
	4	5	0.063	0.057	0.042	0.060
	4	6	0.059	0.050	0.036	0.054
	4	7	0.055	0.044	0.030	0.048
	4	8	0.051	0.038	0.025	0.042
	4	9	0.046	0.032	0.021	0.036
	4	10	0.038	0.025	0.016	0.028
	5	5	0.088	0.074	0.052	0.079
	5	6	0.083	0.066	0.044	0.071
	5	7	0.077	0.058	0.038	0.064
	5	8	0.071	0.050	0.032	0.056
	5	9	0.064	0.042	0.026	0.048
	5	10	0.054	0.033	0.020	0.038

TABLE 3
Continued.

m	r	$v_{r,s:m}$ for $1 \leq r \leq s \leq m$				
		s	$\mu = 0.25$	$\mu = 0.50$	$\mu = 0.75$	$\mu = 0.43$
10	6	6	0.113	0.084	0.055	0.093
	6	7	0.106	0.074	0.046	0.083
	6	8	0.098	0.065	0.039	0.073
	6	9	0.088	0.054	0.032	0.063
	6	10	0.074	0.042	0.024	0.050
	7	7	0.146	0.096	0.058	0.110
	7	8	0.135	0.084	0.049	0.097
	7	9	0.121	0.071	0.040	0.083
	7	10	0.102	0.055	0.031	0.066
	8	8	0.194	0.113	0.064	0.132
	8	9	0.175	0.096	0.053	0.113
	8	10	0.147	0.075	0.040	0.090
	9	9	0.278	0.141	0.076	0.170
	9	10	0.235	0.111	0.059	0.136
	10	10	0.507	0.218	0.111	0.271

TABLE 4
Coefficients of $X_{r:m}$ in the BLUE $\tilde{\sigma}$ and $V_1 = \frac{Var(\tilde{\sigma})}{\sigma^2}$.

m	μ	Coefficients										V_1
		$e_{1:m}$	$e_{2:m}$	$e_{3:m}$	$e_{4:m}$	$e_{5:m}$	$e_{6:m}$	$e_{7:m}$	$e_{8:m}$	$e_{9:m}$	$e_{10:m}$	
2		0.27	0.59									0.315
3		0.15	0.22	0.44								0.206
4		0.09	0.13	0.18	0.35							0.153
5		0.07	0.09	0.12	0.16	0.30						0.121
6	0.25	0.06	0.07	0.09	0.11	0.15	0.26					0.100
7		0.05	0.06	0.07	0.08	0.10	0.13	0.23				0.085
8		0.04	0.05	0.06	0.07	0.08	0.12	0.21				0.074
9		0.04	0.04	0.05	0.06	0.15	-0.07	0.15	0.12	0.19		0.066
10		0.03	0.04	0.04	0.46	0.05	0.06	0.07	0.09	0.11	0.18	0.059
2		0.17	0.65									0.205
3		0.07	0.18	0.50								0.131
4		0.04	0.09	0.17	0.41							0.095
5		0.02	0.05	0.09	0.16	0.36						0.075
6	0.50	0.01	0.03	0.06	0.09	0.15	0.31					0.062
7		0.01	0.02	0.04	0.06	0.06	0.20	0.26				0.051
8		0.00	0.02	0.03	0.05	0.07	0.09	0.14	0.26			0.045
9		0.00	0.01	0.02	0.04	0.05	0.07	0.09	0.13	0.24		0.039
10		-0.00	0.01	0.02	0.03	0.04	0.05	0.07	0.22	0.07	0.28	0.036

TABLE 4
Continued.

m	μ	Coefficients										
		$e_{1:m}$	$e_{2:m}$	$e_{3:m}$	$e_{4:m}$	$e_{5:m}$	$e_{6:m}$	$e_{7:m}$	$e_{8:m}$	$e_{9:m}$	$e_{10:m}$	V_1
2		0.15	0.68									0.134
3		0.05	0.19	0.54								0.085
4		0.02	0.09	0.19	0.44							0.061
5		0.00	0.05	0.10	0.18	0.38						0.048
6	0.75	-0.02	0.05	0.07	0.09	0.17	0.34					0.039
7		-0.01	0.02	0.04	0.07	0.11	0.16	0.30				0.033
8		-0.01	-0.04	0.10	0.03	0.07	0.12	0.13	0.27			0.029
9		-0.01	0.01	0.02	0.04	0.05	0.07	0.10	0.14	0.25		0.026
10		-0.01	0.00	0.02	0.03	0.04	0.06	0.07	0.09	0.13	0.23	0.023
2		0.19	0.63									0.231
3		0.09	0.19	0.49								0.148
4		0.05	0.09	0.17	0.40							0.108
5		0.03	0.06	0.10	0.16	0.34						0.085
6	0.43	0.02	0.04	0.07	0.09	0.15	0.30					0.070
7		0.01	0.03	0.05	0.07	0.09	0.14	0.27				0.060
8		0.01	0.02	0.03	0.05	0.07	0.09	0.13	0.25			0.052
9		0.01	0.02	0.03	0.04	0.05	0.07	0.09	0.12	0.23		0.046
10		0.01	0.01	0.02	0.03	0.04	0.05	0.07	0.09	0.12	0.21	0.041

3. ESTIMATION OF SCALE PARAMETER OF MUTH(μ, σ) USING U-STATISTICS

The main disadvantage involved in the Lloyd's BLUE of scale parameter of a distribution by order statistics is that, in order to obtain the estimator, one needed the values of means, variances and co-variances of the entire order statistics arising from the standard form of the original distribution. Explicit expressions does not exist in the means, variances and co-variances for most of the distributions belonging to the scale family. In this case, these values are usually computed numerically for moderate sample and the corresponding estimators and their variances are obtained. However, if one obtains the BLUE of the scale parameter σ by order statistics based on a moderate sample size m and use it as kernel of degree m, then one can construct a suitable U-statistic to estimate σ. Moreover, this estimator is highly preferred as it utilizes the optimal conditions of both BLUE's and U-statistics and is asymptotically distributed as normal (see Hoeffding, 1948). Hence, in this section we estimate the scale parameter σ of MD(μ, σ) using U-statistics based on the best linear functions of order statistics as kernels. Thomas and Sreekumar (2008) developed a method of estimation by U-statistics using BLUEs of the location and scale parameters of a distribution based on order statistics with an appropriate small sample size. Sreekumar and Thomas (2007) have derived the U-statistics estimators for estimating the location and scale parameters of log-gamma distribution. Estimation of the mean of normal distribution with known coefficient of variation by U-statistics are discussed by Sajeevkumar and Irshad (2013). For some recent works in this direction, one can refer Irshad and Sajeevkumar (2016), Thomas and Priya (2015) and Irshad and Maya (2018).

3.1. U-statistics

Let $X_1, X_2, \cdots, X_n$ be independent samples arising from a population with cdf $F(x;\theta)$. The U-statistic for the estimable parameter θ corresponding to the symmetric kernel $h(\cdot)$ of degree m is given by

$$U(X_1, X_2, \cdots, X_n) = \frac{1}{\binom{n}{m}} \sum_{\alpha \in A} h(X_{\alpha_1}, X_{\alpha_2}, \cdots, X_{\alpha_m}), \tag{13}$$

where $A = \{\alpha | \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m), \alpha_1 < \alpha_2 < \cdots < \alpha_m\}$ is one of the $\binom{n}{m}$ combinations of m integers chosen without replacement from the set $(1, 2, \cdots, n)$. Suppose that

$$E[h(X_1, X_2, \cdots, X_m)] = \theta \quad \text{and} \quad E[h^2(X_1, X_2, \cdots, X_m)] < \infty. \tag{14}$$

Let $h(X_1, X_2, \cdots, X_\omega, X_{\omega+1}, \cdots, X_m)$ and $h(X_1, X_2, \cdots, X_\omega, X_{m+1}, \cdots, X_{2m-\omega})$ be two random variables having exactly ω sample observations in common, $\omega = 1, 2, \cdots, m$. Let $\zeta_\omega^{(m)}$ be the co-variance between these two random variables. Then the variance of the U-statistic given in (13) is given by (see Hoeffding, 1948)

$$\text{Var}[U(X_1, X_2, \cdots, X_n)] = \frac{1}{\binom{n}{m}} \sum_{\omega=1}^{m} \binom{m}{\omega} \binom{n-m}{m-\omega} \zeta_\omega^{(m)}. \tag{15}$$

Clearly the U-statistic defined in (13) is an unbiased estimator of θ. For more details about the desirable properties of U- statistics (see Serfling, 1980).

3.2. U-statistics based on BLUE of σ as kernel

Let $X_1, X_2, \cdots, X_m$ be a random sample of size m arising from (3). As a consequence of (12), $\tilde{\sigma}$ can be represented as

$$h(X_1, X_2, \cdots, X_m) = e_{1:m} X_{1:m} + e_{2:m} X_{2:m} + \cdots + e_{m:m} X_{m:m}, \tag{16}$$

where $e_{1:m}, e_{2:m}, \cdots, e_{m:m}$ are suitable constants. Now by considering $h(X_1, X_2, \cdots, X_m)$ as a kernel of degree m, using the results of Thomas and Sreekumar (2008), the U-statistic based on kernel (16) is given by

$$U_n^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} e_{i+1:m} \right] X_{r:n}. \tag{17}$$

In the above Eq. (17), for any two non-negative integers p and q, unlike the usual definition of $\binom{p}{q}$ for $q \leq p$ we also define $\binom{p}{q} = 0$ for $p < q$. Let

$$\begin{aligned} \zeta_d^{(m)} = &\text{Cov}\big[h(X_1, X_2, \cdots, X_d, X_{d+1}, \cdots, X_m), \\ &h(X_1, X_2, \cdots, X_d, X_{m+1}, \cdots, X_{2m-d})\big], d = 1, 2, \cdots, m. \end{aligned}$$

Then the variance of $U_n^{(m)}$ is given by

$$\operatorname{Var}[U_n^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{d=1}^{m} \binom{m}{d} \binom{n-m}{m-d} \zeta_d^{(m)}, \tag{18}$$

clearly $\zeta_m^{(m)} = \operatorname{Var}[h(X_1, X_2, \cdots, X_m)]$ and is given in (11). Now we evaluate the values of $\zeta_d^{(m)}$ for $d = 1, 2, \cdots, m-1$, using the methodology developed by Thomas and Sreekumar (2008) as explained in the following steps.
Define the vector b_{m+k} for $k = 1, 2, \cdots, m-1$ as

$$b'_{m+k} = \left[\frac{\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i} \binom{o}{i} e_{i+1:m}}{\binom{m+k}{m}}, \frac{\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i} \binom{1}{i} e_{i+1:m}}{\binom{m+k}{m}}, \cdots, \frac{\sum_{i=0}^{m-1} \binom{0}{m-1-i} \binom{m+k-1}{i} e_{i+1:m}}{\binom{m+k}{m}} \right]. \tag{19}$$

Then, the expression for $U_{m+k}^{(m)}$ for $k = 1, 2, \cdots, m-1$ is obtained by putting $n = m+k$ in (17).

$$U_{m+k}^{(m)} = b'_{m+k} X_{m+k}, \tag{20}$$

where

$$X_{m+k} = \left(X_{1:m+k}, X_{2:m+k}, \cdots, X_{m+k:m+k} \right)'.$$

Hence,

$$\operatorname{Var}[U_{m+k}^{(m)}] = (b'_{m+k} V_{m+k} b_{m+k}) \sigma^2, \tag{21}$$

where V_{m+k} is the dispersion matrix of the vector of order statistics of random sample of size $m+k$ arising from the distribution with pdf $f_0(y)$ given in (9). Consequently, we can write the following matrix equation (see Thomas and Sreekumar, 2008),

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & \cdots & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & \cdots & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix} \times \begin{bmatrix} \zeta_1^{(m)} \\ \zeta_2^{(m)} \\ \vdots \\ \zeta_{m-1}^{(m)} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{m-1} \end{bmatrix}, \tag{22}$$

where $z_k = \binom{m+k}{m}(b'_{m+k}V_{m+k}b_{m+k})\sigma^2 - \zeta_m^{(m)}, k = 1,2,\cdots,m-1$. If we write H to denote the coefficient matrix of the left side of (22) and W to denote the vector in the right side of (22), then we have

$$\begin{bmatrix} \zeta_1^{(m)} \\ \zeta_2^{(m)} \\ \vdots \\ \zeta_{m-1}^{(m)} \end{bmatrix} = H^{-1}W. \tag{23}$$

Once we obtain the values of $\zeta_d^{(m)}, d = 1,2,\cdots,m-1$, then the actual variances of the U-statistic for estimating σ based on any sample of size n can be obtained by using (18) without any further direct evaluation of moments of order statistics.

One of the peculiarities of this method is that, if one uses the BLUE as kernel based on sample size m, then the evaluation of variances and co-variances of order statistics of sample size up to $2m-1$ arising from (9) alone are necessary to obtain the explicit expression for the variance of the U-statistic $U_n^{(m)}$, whatever is the sample size.

For example if for a given value of the shape parameter μ, we use $\tilde{\sigma}$ as given in (12) for $m = 5$, then with the evaluation of moments of order statistics arising from $f_0(y)$ given in (9) for sample sizes up to 9, one can obtain the explicit form of appropriate U-statistic estimator of σ and their variances for any sample of size n. Using the values of variances and co-variances of order statistics (given in Table 3) and the coefficients $X_{r:m}$ in the BLUE of σ (given in Table 4), we have obtained the values of $\zeta_d^{(m)}$for $d = 1,2,\cdots,m-1$, $m = 2(1)5$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$ and are given in Table 5. For practicing statisticians these table will be helpful to determine the variance of the U-statistic estimator. In real life data, it is unrealistic to always assume that the shape parameter μ involved in the Muth(μ,σ) is known. Through the next theorem, we propose a simple and closed form estimator of μ which is independent of the scale parameter σ.

THEOREM 1. *Suppose $X_1, X_2, \cdots, X_n$ be a random sample of size n arising from (3). Let τ denote the sample correlation between X_i and their ranks. Let $\varkappa$ be the sample coefficient of variation (CV). Then an estimator for the shape parameter, μ, is*

$$\hat{\mu} = 4\left[1 - \frac{\tau\varkappa}{\sqrt{3}}\sqrt{\frac{n+1}{n-1}}\right] - 2. \tag{24}$$

PROOF. Using the result of Stuart (1954), the correlation between X_i and their

TABLE 5
Values of $\zeta_d^{(m)}$, for $d = 1,2,\cdots,m$ and $m = 2(1)5$.

		$\zeta_d^{(m)}$, for $d=1,2,\cdots,m$ and $m=2(1)5$			
m	d	$\mu=0.25$	$\mu=0.50$	$\mu=0.75$	$\mu=0.43$
2	1	0.154	0.097	0.062	0.110
	2	0.315	0.205	0.134	0.231
3	1	0.067	0.041	0.026	0.047
	2	0.135	0.084	0.054	0.096
	3	0.206	0.131	0.085	0.148
4	1	0.037	0.022	0.014	0.026
	2	0.075	0.046	0.029	0.052
	3	0.113	0.070	0.045	0.080
	4	0.153	0.095	0.061	0.108
5	1	0.024	0.014	0.009	0.016
	2	0.047	0.029	0.018	0.033
	3	0.072	0.044	0.028	0.050
	4	0.096	0.059	0.038	0.067
	5	0.121	0.075	0.048	0.085

ranks r_i, $i = 1,2,\cdots,n$ is

$$\tau = \operatorname{corr}(X_i, r_i) = \left(\int_{-\infty}^{+\infty} xF_X(x)dF_X(x) - \frac{\zeta}{2}\right)\sqrt{\frac{12(n-1)}{\varrho^2(n+1)}}, \tag{25}$$

where $\zeta = E(X)$ and $\varrho^2 = \operatorname{Var}(X)$. We have

$$\begin{aligned}\int_{-\infty}^{+\infty} xF_X(x)dF_X(x) = &\int_0^{+\infty} x\left\{1 - e^{\frac{\mu}{\sigma}x - \frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)}\right\}\\ &\left\{\frac{1}{\sigma}\left(e^{\frac{\mu}{\sigma}x} - \mu\right)e^{\frac{\mu}{\sigma}x - \frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)}\right\}dx\\ =&\zeta - \frac{1}{\sigma}\int_0^{+\infty} x\left\{e^{\frac{\mu}{\sigma}x + 2\left[\frac{\mu}{\sigma}x - \frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)\right]}\right\}dx\\ &+\frac{\mu}{\sigma}\int_0^{+\infty} x\left\{e^{2\left[\frac{\mu}{\sigma}x - \frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)\right]}\right\}dx\\ =&\zeta + I_1 + I_2,\end{aligned} \tag{26}$$

where

$$I_1 = -\frac{1}{\sigma}\int_0^{+\infty} x\left\{e^{\frac{\mu}{\sigma}x+2\left[\frac{\mu}{\sigma}x-\frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)\right]}\right\}dx$$

and

$$I_2 = \frac{\mu}{\sigma}\int_0^{+\infty} x\left\{e^{2\left[\frac{\mu}{\sigma}x-\frac{1}{\mu}(e^{\frac{\mu}{\sigma}x}-1)\right]}\right\}dx.$$

After some simple algebra, the term I_1 reduces to

$$I_1 = -\frac{\sigma}{8}\left[2\mu e^{\frac{2}{\mu}}\Gamma(0,2/\mu)+3\mu+2\right]. \tag{27}$$

Similarly the term I_2 reduces to

$$I_2 = \frac{\sigma}{4}\mu\left[e^{\frac{2}{\mu}}\Gamma(0,2/\mu)+1\right]. \tag{28}$$

Substitute (27) and (28) in (26) and after simplifying, we get

$$\int_{-\infty}^{+\infty} xF_X(x)dF_X(x) = \zeta - \frac{\sigma}{8}(\mu+2). \tag{29}$$

Applying (29) in (25) and simplifying, we get

$$\tau = \frac{\sqrt{3}}{\varkappa}\left[1-\frac{\mu+2}{4}\right]\sqrt{\frac{n-1}{n+1}}. \tag{30}$$

Rearranging (30), we get (24). Thus the theorem is proved. □

Since for the rainfall data Muth(μ,σ) provides the best fit, we use this data set to illustrate the better performance of the U-statistic estimator of σ compared to the MLE of σ. For the data, the value of the estimator of the shape parameter μ given in (24) is obtained as 0.43. Thus by taking 0.43 as the known value of the shape parameter μ, we have used the method explained in this paper to obtain U-statistics estimators $U_{83}^{(2)}$, $U_{83}^{(3)}$, $U_{83}^{(4)}$, $U_{83}^{(5)}$ together with KS statistics and p-values and are given in Table 6. From the table, it is to be noted that the KS statistic value computed based on U-statistics estimators for each of the kernel of degree 2,3,4 and 5 are less than that computed based on the MLE method of estimation. This supports our claim that U-statistics estimators are highly efficient as they are unbiased and possess the optimal properties of both BLUE and those of U-statistics.

TABLE 6
U-statistics estimates, KS statistics and p-values for rainfall data.

MD(μ,σ)	Kernel sizes			
	$m=2$	$m=3$	$m=4$	$m=5$
U-statistics	34.003	34.003	33.986	33.966
KS statistics	0.054	0.054	0.054	0.055
p-values	0.969	0.969	0.967	0.966

4. Comparison of U-statistic estimator with standard unbiased estimator

To compare the efficiency of our U-statistic estimator for the scale parameter σ, we take the usual unbiased estimator of σ, namely $\hat{\sigma}$, given by $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i$ and its variance is obtained as

$$\mathrm{Var}(\hat{\sigma})=\frac{1}{n}\left[\frac{2e^{\frac{1}{\mu}}}{\mu}\int_{1}^{\infty}\frac{e^{-\frac{u}{\mu}}}{u}du-1\right]\sigma^2. \tag{31}$$

We have evaluated the numerical values of $\mathrm{Var}(\hat{\sigma})$ for $n=5(5)20(10)40(20)100$, $\mu=0.25(0.25)0.75$ and for $\mu=0.43$ and are presented in Table 7. Using the values of $\zeta_d^{(m)}$ given in Table 5, we have computed the numerical values of $\mathrm{Var}[U_n^{(m)}]$ for $m=2(1)5$, $n=5(5)20(10)40(20)100$, $\mu=0.25(0.25)0.75$ and for $\mu=0.43$ and are given in Table 7. The relative efficiency $RE(U_n^{(m)}|\hat{\sigma})$ of $U_n^{(m)}$ relative to $\hat{\sigma}$ based on a sample of size n is given by

$$RE\left(U_n^{(m)}|\hat{\sigma}\right)=\frac{\mathrm{Var}(\hat{\sigma})}{\mathrm{Var}\left(U_n^{(m)}\right)}. \tag{32}$$

The numerical values of $RE\left(U_n^{(m)}|\hat{\sigma}\right)$ are computed for $n=5(5)20(10)40(20)100$, $\mu=0.25(0.25)0.75$ and for $\mu=0.43$ and are also given in Table 7. From the table, it is clear that the variance of U-statistic decreases with increasing sample size n. We also observe that the variance of the U-statistic decreases as the value of the shape parameter μ increases. In tandem with this, we observe that as the value of the shape parameter μ increases, the distribution tends to be one with a relatively shorter tail. Also from the table, it is evident that the relative efficiency in all cases considered exceed unity and increases with increase in the sample size.

Asymptotic variance of the U-statistic estimator $U_n^{(m)}$ is given by $\frac{m^2\xi_1^{(m)}}{n}$. Therefore the asymptotic relative efficiency $ARE(U_n^{(m)}|\hat{\sigma})$ of $U_n^{(m)}$ relative to $\hat{\sigma}$ is given by

$$ARE(U_n^{(m)}|\hat{\sigma}) = \lim_{n\to\infty}\left[\frac{\text{Var}(\hat{\sigma})}{\text{Var}(U_n^{(m)})}\right] \tag{33}$$

$$= \frac{\left[\frac{2e^{\frac{1}{\mu}}}{\mu}\int_1^{\infty}\frac{e^{-\frac{u}{\mu}}}{u}du - 1\right]}{m^2\xi_1^{(m)}}.$$

TABLE 7
$Var(U_n^{(m)})$, $Var(\hat{\sigma})$ and the relative efficiency (RE) of $U_n^{(m)}$ with respect to the unbiased estimator $\hat{\sigma}$.

		$\mu = 0.25$			$\mu = 0.50$		
m	n	$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE	$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE
	5	0.124	0.130	1.048	0.078	0.089	1.141
	10	0.062	0.065	1.048	0.039	0.045	1.154
	15	0.041	0.043	1.049	0.026	0.030	1.154
	20	0.031	0.033	1.065	0.019	0.022	1.158
2	30	0.021	0.022	1.048	0.013	0.015	1.154
	40	0.015	0.016	1.067	0.009	0.011	1.222
	60	0.010	0.011	1.100	0.006	0.007	1.167
	80	0.007	0.008	1.143	0.005	0.006	1.200
	100	0.006	0.007	1.167	0.003	0.004	1.133
	∞	...	...	1.056	...	...	1.148
	5	0.122	0.130	1.066	0.076	0.089	1.171
	10	0.060	0.065	1.083	0.037	0.045	1.216
	15	0.040	0.043	1.075	0.025	0.030	1.200
	20	0.030	0.033	1.100	0.018	0.022	1.222
3	30	0.020	0.022	1.100	0.012	0.015	1.250
	40	0.015	0.016	1.067	0.009	0.011	1.222
	60	0.010	0.011	1.100	0.006	0.007	1.167
	80	0.007	0.008	1.143	0.005	0.006	1.200
	100	0.006	0.007	1.167	0.003	0.004	1.333
	∞	...	...	1.079	...	...	1.207

We have computed the numerical values of $ARE(U_n^{(m)}|\hat{\sigma})$ for $m = 2(1)5$, $\mu = 0.25(0.25)0.75$ and for $\mu = 0.43$ and are also given in last row of Table 7 corresponding to each cases. From this, it is observed that the asymptotic relative efficiency in all cases considered are greater than unity and increases with kernel size.

TABLE 7
Continued.

m	n	$\mu = 0.75$			$\mu = 0.43$		
		$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE	$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE
2	5	0.051	0.060	1.176	0.089	0.099	1.112
	10	0.025	0.030	1.200	0.044	0.049	1.114
	15	0.017	0.020	1.176	0.029	0.033	1.138
	20	0.013	0.015	1.154	0.022	0.025	1.136
	30	0.008	0.010	1.250	0.015	0.016	1.067
	40	0.006	0.008	1.333	0.011	0.012	1.091
	60	0.004	0.005	1.250	0.007	0.008	1.143
	80	0.003	0.004	1.333	0.005	0.006	1.200
	100	0.002	0.003	1.500	0.004	0.005	1.250
	∞	...	...	1.216	...	...	1.125
3	5	0.049	0.060	1.224	0.086	0.099	1.151
	10	0.024	0.030	1.250	0.043	0.049	1.140
	15	0.016	0.020	1.250	0.028	0.033	1.179
	20	0.012	0.015	1.250	0.021	0.025	1.190
	30	0.008	0.010	1.250	0.014	0.016	1.114
	40	0.006	0.008	1.333	0.011	0.012	1.091
	60	0.004	0.005	1.250	0.007	0.008	1.143
	80	0.003	0.004	1.333	0.005	0.006	1.200
	100	0.002	0.003	1.500	0.004	0.005	1.250
	∞	...	...	1.289	...	...	1.170
4	5	0.121	0.130	1.074	0.075	0.089	1.187
	10	0.060	0.065	1.083	0.037	0.045	1.216
	15	0.040	0.043	1.075	0.024	0.030	1.250
	20	0.030	0.033	1.100	0.018	0.022	1.222
	30	0.020	0.022	1.100	0.012	0.015	1.250
	40	0.015	0.016	1.067	0.009	0.011	1.222
	60	0.010	0.011	1.100	0.006	0.007	1.167
	80	0.007	0.008	1.143	0.004	0.006	1.500
	100	0.006	0.007	1.167	0.003	0.004	1.333
	∞	...	...	1.099	...	...	1.265
5	5	0.121	0.130	1.074	0.075	0.089	1.187
	10	0.060	0.065	1.083	0.036	0.045	1.250
	15	0.040	0.043	1.075	0.024	0.030	1.250
	20	0.030	0.033	1.100	0.018	0.022	1.222
	30	0.020	0.022	1.100	0.012	0.015	1.250
	40	0.015	0.016	1.067	0.009	0.011	1.222
	60	0.009	0.011	1.222	0.006	0.007	1.167
	80	0.007	0.008	1.143	0.004	0.006	1.500
	100	0.006	0.007	1.167	0.003	0.004	1.333
	∞	...	...	1.084	...	...	1.272

TABLE 7
Continued.

m	n	$\mu = 0.75$			$\mu = 0.43$		
		$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE	$\frac{Var(U_n^{(m)})}{\sigma^2}$	$\frac{Var(\hat{\sigma})}{\sigma^2}$	RE
	5	0.048	0.060	1.250	0.085	0.099	1.165
	10	0.023	0.030	1.304	0.042	0.049	1.167
	15	0.015	0.020	1.333	0.028	0.033	1.179
	20	0.012	0.015	1.250	0.021	0.025	1.190
4	30	0.008	0.010	1.250	0.014	0.016	1.143
	40	0.006	0.008	1.250	0.010	0.012	1.200
	60	0.004	0.005	1.250	0.007	0.008	1.143
	80	0.003	0.004	1.333	0.005	0.006	1.200
	100	0.002	0.003	1.500	0.004	0.005	1.250
	∞	...	...	1.347	...	...	1.189
	5	0.048	0.060	1.333	0.085	0.099	1.165
	10	0.023	0.030	1.304	0.041	0.049	1.195
	15	0.015	0.020	1.333	0.027	0.033	1.222
	20	0.011	0.015	1.364	0.020	0.025	1.250
5	30	0.008	0.010	1.250	0.014	0.016	1.143
	40	0.006	0.008	1.333	0.010	0.012	1.200
	60	0.004	0.005	1.250	0.007	0.008	1.143
	80	0.003	0.004	1.333	0.005	0.006	1.200
	100	0.002	0.003	1.500	0.004	0.005	1.250
	∞	...	...	1.341	...	...	1.237

5. Conclusion

Through this article, we conclude that one may choose MD(μ, σ) as a best alternative for well known lifetime distributions like gamma, log-normal and Weibull etc. for certain real life data sets. Here, first we propose the BLUE of the scale parameter σ of MD(μ, σ) using order statistics for some given values of the shape parameter. Then we provide a U-statistic estimator by utilizing the BLUE of σ as the kernel of appropriate size. For practicing statisticians this estimator may be very useful, because in this estimator, the mean and its variance are obtained without any direct evaluation of moments of order statistics for any sample size (say $n = 1000$ or more). Also, this estimator competes with the classical estimators in the efficiency point of view.

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References

H. A. David, H. N. Nagaraja (2003). *Order Statistics*. John Wiley and Sons, New York, third ed.

B. Gompertz (1825). *On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies*. Philosophical Transactions of the Royal Society of London, 115, pp. 513–583.

W. Hoeffding (1948). *A class of statistics with asymptotically normal distribution*. The Annals of Mathematical Statistics, 19, no. 3, pp. 293–325.

M. R. Irshad, R. Maya (2018). *On a less cumbersome method of estimation of parameters of Lindley distribution by order statistics*. Statistics in Transition New Series, 19, pp. 597–620.

M. R. Irshad, N. K. Sajeevkumar (2016). *Estimating the mean of the logistic distribution with known coefficient of variation by U-statistics*. Thailand Statistician, 14, pp. 117–128.

A. G. Laurent (1975). *Failure and mortality from wear and ageing - The Teissier model*. In G. Patil, S. Kotz, J. Ord (eds.), *A Modern Course on Statistical Distributions in Scientific Work*, Springer, Heidelberg, vol. 17, pp. 301–320.

L. M. Leemis, J. T. McQueston (2008). *Univariate distribution relationships*. The American Statistician, 62, no. 1, pp. 45–53.

E. H. Lloyd (1952). *Least-squares estimation of location and scale parameters using order statistics*. Biometrika, 39, pp. 88–95.

E. J. MUTH (1977). *Reliability models with positive memory derived from the mean residual life function*. The Theory and Applications of Reliability, 2, pp. 401–435.

J. PEDRO, M. D. JIMÉNEZ GAMERO, V. ALBA-FERNÁNDEZ (2015). *On the Muth distribution*. Mathematical Modelling and Analysis, 20, pp. 291–310.

H. RINNE (1981). *Estimating the lifetime distribution of private motor-cars using prices of used cars-the Teissier model*. Statistiks Zwischen Theorie und Praxis, pp. 172–184.

V. K. ROHATGI, A. K. EHSANES SALEH (2009). *An Introduction to Probability and Statistics*. Wiley.

N. K. SAJEEVKUMAR, M. R. IRSHAD (2013). *Estimation of the mean of the Normal distribution with known coefficient of variation by U-statistics*. IAPQR Transactions, 38, pp. 51–65.

R. J. SERFLING (1980). *Approximation Theorems of Mathematical Statistics*. John Wiley and Sons, New York.

N. V. SREEKUMAR, P. Y. THOMAS (2007). *Estimation of the parameters of log-gamma distribution using order statistics*. Metrika, 66, pp. 115–127.

A. STUART (1954). *The correlation between variate-values and ranks in samples from a continuous distribution*. British Journal of Statistical Psychology, 7, no. 1, pp. 37–44.

G. TEISSIER (1934). *Recherches sur le vieillissement et sur les lois de la mortalité. II. essai d'interprétation genérale des courbes de survie*. Annales de Physiologie et de Physicochimie Biologique, 10, pp. 260–284.

P. Y. THOMAS, R. PRIYA (2015). *On a less cumbersome method of estimation of parameters of type III generalized logistic distribution by order statistics*. Statistica, 75, pp. 291–312.

P. Y. THOMAS, N. V. SREEKUMAR (2008). *Estimation of location and scale parameters of a distribution by U-statistics based on best linear functions of order statistics*. Journal of Statistical Planning and Inference, 138, pp. 2190–2200.

SUMMARY

In this work, we have considered a lifetime distribution namely Muth distribution and pointed out instances where it appears as a good model to study the stochastic nature of the variable under consideration. We have derived the best linear unbiased estimator (BLUE) of the scale parameter of the Muth distribution based on order statistics for some known values of the shape parameter. We have further estimated the scale parameter of Muth distribution by U-statistics based on best linear functions of order statistics as kernels. The efficiency of the BLUE relative to the usual unbiased estimator has been also evaluated. An illustration describing the performance of U-statistics estimation method when compared with the classical maximum likelihood method is also given.

Keywords: Order statistics; Muth distribution; Best linear unbiased estimator; U-statistics.

Appendix

A. Data

12.0, 22.7, 75.5, 28.6, 65.8, 39.4, 33.1, 84.0, 41.6, 62.3,52.5, 13.9, 15.4, 31.9, 32.5, 37.7, 9.5, 49.9, 31.8, 32.2, 50.2, 55.8, 20.4, 5.9, 10.1, 44.5, 19.7, 6.4, 29.2, 42.5, 19.4, 23.8, 55.2, 7.7, 0.8, 6.7, 4.8, 73.8, 5.1, 7.6, 25.7, 50.7, 59.7, 57.2, 29.7, 32.0, 24.5, 71.6, 15.0, 17.7, 8.2, 23.8, 46.3, 36.5, 55.2, 37.2, 33.9, 53.9, 51.6, 17.3, 85.7, 6.6, 4.7, 1.8, 98.7, 62.8, 59.0, 76.1, 67.9, 73.7, 27.2, 39.5, 6.9, 14.0, 3.0, 41.6, 49.5, 11.2, 17.9, 12.7, 0.8, 21.1, 24.5.