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THE RANGE OF DERIVATIVE'S ARBITRAGE PRICES IN A GENERAL INCOMPLETE MARKET

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1. INTRODUCTION

In valuing financial derivatives, is crucial the completeness of the market that assure the uniqueness of the price and the replication, only by the hypothesis of the absence of arbitrage opportunities. In this contest, it is possible also represent this arbitrage price, in term of the expected value of the derivative's final payoff under the unique martingale measure relative to the choosen numeraire by the martingale representations and by the Girsanov Theorem.

This arguments failed if we consider an incomplete market where, by the absence of arbitrage opportunities hypothesis, we can't define an unique derivative's price and where the perfect hedging is not possible. En effect, we have several equivalent martingale measures relative to such a numeraire and this set is strictly linked by the replication problem.

In this contest we have different approach like the one by utility function hypothesis proposed by (Davis, 1994), the one of risk-minimizing strathegy by (Föllmer and Schweizer, 1991) and the super-replication approach proposed by (El Karoui and Quenez, 1995). In a general incomplete market driven by a mixed diffusion of finite dimension, we follow the super-replication approach to characterize the range of derivative's arbitrage prices in term of solution of the related dynamic control problem and of the dependence of the price's bounds by the relevant parameters.

In the section 2, we propose a model in the hypothesis of deterministic interest rate and present some applications in different incomplete situations, like one in a standard stochastic volatility model (by the other we remember the model of Hull and White, 1987, the one of Stein and Stein, 1991 and the one of Heston, 1993). The other application of DIRH model is to Gas market, where we consider a model like the one proposed by Marzo and Romagnoli (2005) but in an incomplete contest; we observe that here we may apply the DIRH model also if the interest rate is stochastic, because we consider an exponential affine structure of contingent claim's final payoff and so we may insert this variable like a component of the director process of the market.

By the other hand, if we don't have this particular function of final payoff we have to consider separatly the stochasticity of interest rate and we have to apply the model in the hypothesis of stochastic interest rate (SIRH) presented in section 5. We can see that, in this case, we define a different set of equivalent martingale measure but after we can characterize the range of contingent claim's arbitrage prices in the same way of in DIRH model.

2. THE DIRH MODEL

We consider a financial market where the incertainty is represented by a stochastic variable $(X)_t \in \mathbb{R}^n$, that is the price of risky assets, which dynamic is the following jump diffusion:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t + \mathrm{d}Z_t$$

where, $\mu(t, X_t) \in \mathbb{R}^n$, $\sigma(t, X_t) \in \mathbb{R}^{n\times n}$, $(W)_t$ is a $\mathbb{P}$-Brownian motion in $\mathbb{R}^n$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$[1] and $(Z)_t$ is a pure $\mathbb{P}$-jump process related to $\mathbb{P}$-martingale $(M)_t$ with stochastic intensity $\Lambda(t, X)$, whose jump size distribution is ν, a probability distribution on $\mathbb{R}^n$. We define the jump transform vector $\theta(c)$ that determines the probability distribution of each jump measure ν, so that for any $c \in \mathbb{R}^n$, we have:

$$\theta(c) \quad = \quad \int_{\mathbb{R}^n} \exp(c^{\mathsf{T}} z)\mathrm{d}\nu(z)$$

where $c \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$.

3. THE SET OF EQUIVALENT MARTINGALE MEASURE AND THE RANGE OF ARBITRAGE PRICES IN THE DIRH

If we suppose that the market is incomplete, to determine the range of prices of a contingent claim on X, we have to consider the set of equivalent martingale measures $\mathbb{Q}^\gamma$ for X, so that:

$$\mathbb{E}_{\mathbb{Q}^\gamma}\left[\exp\left(-\int_t^T r_u \mathrm{d}u\right) X_T \mid \mathcal{F}_t\right] = X_t$$

where the interest rate r_t is a deterministic function of the time[2]. In the following

[1] This probability is the conventional so called hystorical or objective probability measure.
[2] In the section 5 we consider the case of SIRH.

we suppose, for computational semplicity and without loosing in generality, that the interest rate is null (NIRH).

We have the following;

Proposition 1. In a general incomplete market in the NIRH, the set of equivalent martingale measures $\mathbb{Q}^{\gamma}$ for X is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{\mathrm{d}\mathbb{Q}^{\gamma}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{\gamma} \tag{1}$$

where:

$$L_t^{\gamma} = \exp\left[\int_0^t \beta_u^{\mathsf{T}} \mathrm{d}W_u - \frac{1}{2}\int_0^t |\beta_u^{\mathsf{T}}\beta|\,\mathrm{d}u + \int_0^t [\ln(1+\gamma_u)]^{\mathsf{T}}\,\mathrm{d}M_u - \int_0^t \theta(\gamma_u)\Lambda(u,X_u)\mathrm{d}u\right] \tag{2}$$

assumed to be a $\mathbb{P}$-square integrable strictly positive martingales and where the process $(\gamma)_t$ is so that $(1+\gamma_t) > 0$.

In particular $(\beta,\gamma)_t$ are two predictable process in $\mathbb{R}^n$ linked by the following relation[3]:

$$\mu(t,X_t) + \sigma(t,X_t)\beta_t + \theta(\gamma_t)\Lambda(t,X_t)\underline{1} = \underline{0} \tag{3}$$

where $\underline{1}$ is the n-vector of units and $\underline{0}$ is the n-vector of zero components.

Proof. The dynamic of $(X)_t$ under the probability $\mathbb{Q}^{\gamma}$, defined by the Radon-Nycodim derivative (2) is the following:

$$\mathrm{d}X_t = (\mu(t,X_t) + \sigma(t,X_t)\beta_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}W_t^{\mathbb{Q}^{\gamma}} + \mathrm{d}Z_t^{\mathbb{Q}^{\gamma}} + \theta(\gamma_t)\Lambda(t,X_t)\mathrm{d}t$$

that is a $\mathbb{Q}^{\gamma}$-martingale under condition (3) and where, by Girsanov theorem, we define the $\mathbb{Q}^{\gamma}$-Brownian motion $(W^{\mathbb{Q}^{\gamma}})_t$ and the $\mathbb{Q}^{\gamma}$-pure jump process $(Z^{\mathbb{Q}^{\gamma}})_t$, as follows:

$$\begin{aligned} W_t^{\mathbb{Q}^{\gamma}} &= W_t - \int_0^t \beta_u \mathrm{d}u \\ M_t^{\mathbb{Q}^{\gamma}} &= Z_t^{\mathbb{Q}^{\gamma}} - \int_0^t \theta(\gamma_u)\Lambda(u,X_u)\mathrm{d}u \end{aligned}$$

where $(M^{\mathbb{Q}^{\gamma}})_t$ is a $\mathbb{Q}^{\gamma}$-martingale. □

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[3] If the interest rate is a deterministic function of the time but not null, equation (3) is equal to $r_t\underline{1}$ where $\underline{1}$ is a n-vector with unit components.

Proposition 2. In a general incomplete market in the NIRH, the arbitrage prices bounds of a contingent claim which final payoff is a function $f(t,x) \in \mathbb{C}^{1,2}$, is $[f^d, f^u[$, where f^u is the vischiosity solution of the following:

$$-\partial_t f^u(t,x) - \frac{1}{2} tr(\sigma(t,x)^\intercal \partial_{xx} f^u(t,x) \sigma(t,x)) \geq 0 \tag{4}$$

subject to the final condition $f^u(T^-,x) \geq f(x)$, where $\partial_t f^u(t,x) \in \mathbb{R}^n$, $\partial_{xx} f^u(t,x) \in \mathbb{R}^{n \times n}$ and where f^d is the vischiosity solution of the following:

$$\partial_t f^d(t,x) - \frac{1}{2} tr(\sigma(t,x)^\intercal \partial_{xx} f^d(t,x) \sigma(t,x)) \leq 0$$

subject to the final condition $f^d(T^-,x) \leq f(x)$.

Proof. We define the upper bound f^u of contingent claim's price interval, as follows:

$$\begin{aligned} f^u &= \sup_{\gamma \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\gamma}[f(X_T)] \\ &= -\inf_{\gamma \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\gamma}[-f(X_T)] \end{aligned}$$

where $\mathcal{D}$ is the set of $\mathcal{F}$-adapted and limited process with values on $\mathbb{R}^n$, and consider the related dynamic problem:

$$f^u(t,x) = \sup_{\gamma \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\gamma}[f(X_T) \mid X_t = x]$$

so that $f^u = f^u(0,x_0)$. If we consider the Bellman's equation for this stochastic control, we can characterize $f^u(t,x)$ like as the vischiosity super-solution of the following:

$$\begin{aligned} &-\partial_t f^u(t,x) + \inf_{\gamma \in \mathcal{D}} \Big\{ -[\mu(t,x) + \sigma(t,x)\beta_t + \theta(\gamma_t)\Lambda(t,x)\underline{1}]^\intercal \partial_x f^u(t,x) + \\ &-\frac{1}{2} tr(\sigma(t,x)^\intercal \partial_{xx} f^u(t,x) \sigma(t,x)) \Big\} \geq 0 \end{aligned} \tag{5}$$

with the final condition $f^u(T^-,x) \geq f(x)$. We observe that, if the condition (3) is satisfied, so (5) begun the (4) in the thesis of proposition.

For the bound f^d we proceed in the same way.□

Proposition 3. If we suppose that the volatility $\sigma(t,x)$ is not bounded, so the upper bound f^u is not increasing on t and is concave on X and by the same argument f^d is not decreasing on t and is convex on X.

Proof. If we suppose that $\sigma(t,x)$ is a matrix with null components in the principal diagonal, so equation (4) prove the not increasing property of $f^u(t,x)$ respect to t.

By the other hand, if we suppose the limit case in which the principal diagonal's components of the matrix $\sigma(t,x)\to+\infty$, so to respect (4) we must have $\partial_{xx}f^u(t,x)\le 0$.

By the same arguments we find the characterization of $f^d(t,x)$.□

In the following Proposition we make the explicit characterization of the bounds of derivative's prices;

Proposition 4. If $f^{cv}(x)$ is the t-price of the concave envelopment of function $f(x)$, i.e. the minimal concave function that is not minor of f, in NIRH model we have that:

$$f^u(t,x)=f^{cv}(x)$$

and in particular $f^u=f^{cv}(x_0)$.

In the same way, if $f^{cx}(x)$ is the t-price of the convex envelopment of $f(x)$, i.e. the maximal convex function that is minor of f, we have that:

$$f^d(t,x)=f^{cx}(x)$$

and in particular $f^d=f^{cx}(x_0)$.

Proof. By the final condition on $f^u(t,x)$ and his not increasing property on t, we have that:

$$f^u(t,x)\ge f(x) \qquad \forall(t,x)\in[0,T[\times\mathbb{R}^n$$

and, by the concave on x characterization of $f^u(t,x)$, we have that:

$$f^u(t,x)\ge f^{cv}(x) \qquad \forall(t,x)\in[0,T[\times\mathbb{R}^n$$

So, if we remember the definition of the super-replication price like as the minor initial investment to super-hedge the contingent claim, follows that:

$$f^{u}(t,x) \le f^{cv}(x) \qquad \forall (t,x) \in [0,T[\times \mathbb{R}^{n}$$

and finally we have the thesis of proposition. To prove the explicit characterization of f^{d}, we proceed in the same way. □

For particular function of payoff $f(x)$ it may be possible to define explicitly the bounds of derivative's prices. For example in (Bellamy and JeanBlanc, 1997) is shown that the range of the European call option's prices $[f_c^{d}, f_c^{u}[$, is:

$$f_c^{u}(t,x) = x$$

and in particular $f_c^{u} = x_0$, and as:

$$f_c^{d}(t,x) = BSC(t,x)$$

where $BSC(t,x)$ is the Black-Scholes price of the call option written on $\bar{X}$:

$$\mathrm{d}\bar{X} = \sigma(t,\bar{X})\mathrm{d}W_t \qquad \bar{X}_t = x$$

while the range of the European put option's prices, $[f_p^{d}, f_p^{u}[$, is explicitly defined as follow:

$$f_p^{u}(t,x) = K$$

where K is the option's strike price, and that:

$$f_p^{d}(t,x) = BSP(t,x)$$

where $BSP(t,x)$ is the t-Black-Scholes price of the put option.

4. SOME EXAMPLES IN THE DIRH

4.1. *The future price of gas in an affine model*

In this example we consider a particular incomplete market driven by an affine jump diffusion where, we may follows the transform method proposed by Duffie *et al.* (1999) to determine explicitly the derivative's price. This constraint of the model structure, applied to the market of gas derivatives, make the problem more tractable, mathematically specking (see for the same application, but in the contest of complete market, the paper by (Marzo and Romagnoli, 2005).

In this contest, we consider the future price of gas, as a function of the actual spot price of gas G, the price of petrol P^{P} which is modellized by a diffusion with jump, the convenience yield δ, the stochastic volatility v and the the stochastic interest rate r [4]:

$$FG = f(G, P^P, \delta, v, r)$$

The price of petrol follows a diffusion with jump[5]:

$$dP_t^P = P_t^P(r - \delta_t)dt + e^v \sqrt{v} dW_{1,t}^{\mathbb{Q}} + dZ_t^{\mathbb{Q}}$$

where $(W_1^{\mathbb{Q}})_t$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and $(Z^{\mathbb{Q}})_t$ is a pure $\mathbb{Q}$-jump process with stochastic intensity $\lambda(t, P^P)$, whose jump size distribution is ν, a probability distribution on $\mathbb{R}$. We suppose that $\lambda(t, P^P)$ is affine on P^P, so that:

$$\lambda(t, P^P) = l_0 + l_1 P_t^P$$

and we define the jump transform $\theta(c)$ that determines the probability distribution of each jump measure ν, so that for any $c \in \mathbb{R}$, we have:

$$\theta(c) = \int_{\mathbb{R}} \exp(cz) \mathrm{d}\nu(z)$$

The convenience yield $(\delta)_t$ follows the diffusion:

$$d\delta_t = (\overline{\alpha} - \overline{\beta}\delta_t)dt + e^v \sigma_\delta \sqrt{v} dW_{2,t}^{\mathbb{Q}}$$

and the volatility is stochastic and follows the diffusion:

$$dv_t = (\theta - kv_t)dt + \sigma_v \sqrt{v} dW_{3,t}^{\mathbb{Q}}$$

where $(W_2^{\mathbb{Q}})_t$ and $(W_3^{\mathbb{Q}})_t$ are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

The instantaneous interest rate $(r)_t$ follows an Ornstein-Uhlenbeck mean-reverting process:

$$dr_t = a(b - r_t)dt - \sigma_r dW_{4,t}^{\mathbb{Q}}$$

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[4] Here we consider the problem as a case of DIRH model because we have a particular function of final payoff of contingent claim, like an affine exponential function of vector *X* and we may reverse the incertainty of *r* in the vector *X*, that is in the exponential of the final payoff of contingent claim.

[5] Here we suppose to work under the risk-neutral probability measure for P^P.

where the parameters a, b and σ_r are constants, and where $(W_4^{\mathbb{Q}})_t$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

We note that all the processes are constructed on the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ filtered by $(\mathcal{F}_t)$ which is generated by the processes $(W_{i,i=1,\ldots,4}^{\mathbb{Q}}, Z^{\mathbb{Q}})_t$. It is well known that the Brownian motion $(W_1^{\mathbb{Q}})_t$ is independent by $(Z^{\mathbb{Q}})_t$, and we suppose that also the others Brownian motions of this model are independent by $(Z^{\mathbb{Q}})_t$. We suppose also that the Brownian motions $(W_{i,i=1,\ldots,4}^{\mathbb{Q}})_t$ are incorrelated.

If we introduce a matrix notation, the vector X_t, defined as:

$$X_t = \begin{bmatrix} P^P \\ \delta_t \\ v_t \\ r_t \end{bmatrix}$$

follows the diffusion:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}} + I^{\mathsf{T}}\mathrm{d}Z_t^{\mathbb{Q}}$$

where $I^{\mathsf{T}} = [1 \quad 0 \quad 0 \quad 0]$ and $(W^{\mathbb{Q}})_t$ is a standard Brownian vector in $\mathbb{R}^4$ like that:

$$W_t^{\mathbb{Q}} = \begin{bmatrix} W_{1,t}^{\mathbb{Q}} \\ W_{2,t}^{\mathbb{Q}} \\ W_{3,t}^{\mathbb{Q}} \\ W_{4,t}^{\mathbb{Q}} \end{bmatrix}$$

and we may write $\mu(t, X_t)$ and $\sigma(t, X_t)$ like as two affine functions of X_t:

$$\begin{aligned} \mu(t, X_t) &= K_0 + K_1 X_t \\ \sigma(t, X_t)\sigma(t, X_t)^{\mathsf{T}} &= H_0 + H_1 X_t^{\mathsf{T}} \end{aligned}$$

where:

$$K_0 = \begin{bmatrix} 0 \\ \overline{\alpha} \\ \theta \\ ab \end{bmatrix}; K_0 \in \mathbb{R}^4$$

$$K_1(t) = \begin{bmatrix} -\delta_t & 0 & 0 & 0 \\ 0 & -\overline{\beta} & 0 & 0 \\ 0 & 0 & -k & 0 \\ 0 & 0 & 0 & -a \end{bmatrix}; K_1(t) \in \mathbb{R}^{4\times 4}$$

$$H_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_r^2 \end{bmatrix}; H_0 \in \mathbb{R}^{4\times 4}$$

$$H_1 = \begin{bmatrix} H_1^{(1)} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & H_1^{(2)} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & H_1^{(3)} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & H_1^{(3)} \end{bmatrix}; H_1 \in \mathbb{R}^{4\times 4\times 4}$$

where $\underline{0}$ is the null vector in $\mathbb{R}^4$ and where:

$$H_1^{(1)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{2\nu} \end{bmatrix}; H_1^{(1)} \in \mathbb{R}^4$$

$$H_1^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ e^{2\nu}\sigma_\delta^2 \end{bmatrix}; H_1^{(2)} \in \mathbb{R}^4$$

$$H_1^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma_\nu^2 \end{bmatrix}; H_1^{(3)} \in \mathbb{R}^4$$

$$H_1^{(4)} = \underline{0} \in \mathbb{R}^4$$

We may write also r_t like as an affine function of X_t:

$$r(X_t, t) = \rho_0 + \rho_1^\top X_t$$

where:

$$\rho_0 = 0$$

$$\rho_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In the matrix-notation we have:

$$\begin{aligned} \Lambda(t, X_t) &= L_0 + L_1 X_t \\ &= \begin{bmatrix} l_0 + l_1 P_t^P \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

where:

$$L_0 = \begin{bmatrix} l_0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; L_0 \in \mathbb{R}^4$$

$$L_1 = \begin{bmatrix} l_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; L_1 \in \mathbb{R}^{4\times 4}$$

and also:

$$\begin{aligned} \theta(c) &= \int_{\mathbb{R}^4} \exp(c^\top z) \mathrm{d}\nu(z) \\ &= \theta_P(c); \quad \theta(c) \in \mathbb{R} \end{aligned}$$

In this contest we have to change probability, considering that only the first component of X is the price of risky asset.

Corollary 5. In an incomplete market driven by an affine jump diffusion, like explicitly before, the set of equivalent martingale measures $\mathbb{Q}^{\gamma}$ for the risky asset's price, that is the first component of the vector X, is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{\mathrm{d}\mathbb{Q}^{\gamma}}{\mathrm{d}\mathbb{Q}}\right|_{\mathcal{F}_t} = L_t^{\gamma} \tag{6}$$

where L_t^{γ} has the same expression presented in (2).

In particular $(\beta,\gamma)_t$ are two predictable process in $\mathbb{R}^4$ linked by the following relation:

$$\mu(t,X_t)+\sigma(t,X_t)\beta_t+\theta(\gamma_t)\Lambda(t,X_t)\underline{1}=Y_t \tag{7}$$

where $\underline{1}$ is the 4-dimensional unit vector and Y_t is a 4-dimensional vector where the first component is null.

Proof. We proceed like in proof of proposition 1, but here we consider a martingale measure only respect the first component of vector X, unlike in the previous section where all the vector X represent the risky asset's prices.□

So we have the following corollary, like as an application of the proposition 2;

Corollary 6. If we suppose that the future price of gas is the actual price of gas G, times the expectation of an exponential affine function on X_T, $g(t,x)\in\mathbb{C}^{1,2}$, in an incomplete market driven by an affine jump diffusion, the range of his arbitrage future prices is the gas price G, times $[FG^d, FG^u[$, where FG^u is the vischiosity solution of the following:

$$-\partial_t FG^u(t,x)-\frac{1}{2}(\sigma^{(1,1)}(t,x))^2\partial_{p^P p^P}FG^u(t,x)\geq 0 \tag{8}$$

subject to the final condition $FG^u(T^-,x)\geq g(x)$, and where $FG^d(t,x)$ is the vischiosity solution of the following:

$$\partial_t FG^d(t,x)+\frac{1}{2}(\sigma^{(1,1)}(t,x))^2\partial_{p^P p^P}FG^d(t,x)\leq 0$$

subject to the final condition $FG^d(T^-,x)\leq g(x)$, with $g(x)=\exp(u^{\mathsf{T}}x)$ for a given $u\in\mathbb{R}^4$ and where $\partial_{p^P p^P}FG^d(t,x)$ is the first component in the diagonal of matrix $\partial_{xx}FG^d(t,x)$ and $\sigma^{(1,1)}$ is the component in place $(1,1)$ of the matrix $\sigma(t,x)$ [6].

Proof. We proceed like as in proof of proposition 2, but when we consider the Bellman's equation (5), and suppose that the condition (7) is satisfied, $FG^u(t,x)$ is the vischiosity super-solution of the following:

$$-\partial_t FG^u(t,x)+\inf_{\gamma\in\mathcal{D}}\left\{-Y_t^{\intercal}\partial_x FG^u(t,x)-\frac{1}{2}tr(\sigma(t,x)^{\intercal}\partial_{xx}FG^u(t,x)\sigma(t,x))\right\}\geq 0 \quad (9)$$

with the final condition $FG^u(T^-,x)\geq g(x)$, where $\mathcal{D}$ is the set of $\mathcal{F}$-adapted and limited process with values on $\mathbb{R}^4$.

If we suppose that $\beta,\gamma\to\mp\infty$ in equation (9), we have to impose that $\partial_x FG^u(t,x)$ is a vector where the components, except the first, are null and from this follows that also the second derivatives respect these variables, in the principal diagonal of matrix $\partial_{xx}FG^u(t,x)$, are null. So the only variable that is not null is the first component in the principal diagonal of matrix $\partial_{xx}FG^u(t,x)$, that is the second derivative respect to the first component of vector X, like in the thesis of proposition.

For the bound $FG^d(t,x)$, we proceed in the same way. □

We observe that the bounds FG^d and FG^u are function only of time t and of the price of petrol P^P, so in the following we may write $FG^{d,u}(t,P^P)$. Also we may characterize these extremes of the interval of prices, like in the following;

Corollary 7. If we suppose that the volatility of the price of petrol P^P, $\sigma^{(1,1)}$, is not bounded, so the upper bound FG^u not increase on t and is a concave function on the first component P^P of X and by the same argument FG^d is not decreasing on t and is a convex function on P^P.

Proof. If we suppose that $\sigma^{(1,1)}$ is null, so equation (8) prove the not increasing property of $FG^u(t,P^P)$ respect to t.

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[6] Here we have only the first component of the diagonal of $\sigma(t,x)$, because the change of probability from $\mathbb{Q}$ to $\mathbb{Q}^{\gamma}$, make null only the drift of the first component of the vector X, that is the drift of the risky asset's price.

By the other hand, if we suppose that $\sigma^{(1,1)} \to +\infty$, so to respect (8) we must have $\partial_{P^P P^P} FG^u(t, P^P) \leq 0$.

By the same arguments we find the characterization of $FG^d(t, P^P)$.□

In this example of DIRH model, we consider an exponential function $g(x)$ for the final payoff of contingent claim and so, applying the proposition 4, we have the following characterization of the Future Gas prices bounds;

Corollary 8. In this exponential affine model, we have that $FG^d(T, P^P)$ is the convex envelopment of $g(P^P)$, while the upper bound of prices is not defined.

Proof. This is an application of proposition 4 to the case of an affine exponential function of contingent claim's final payoff.□

4.2. *Stochastic volatility model*

Here we consider a particular application of our model to the case of incomplete market where the incertainty is characterize by a 2-dimensional Brownian motion, one which drive the price of risky asset and one which drive his volatility. In this standard case, we doesn't have a mixed diffusion so is not necessary to impose a particular restriction on the structure of parameters and final payoff of contingent claim.

We suppose that under the probability measure $\mathbb{P}$, the dynamics are the following:

$$\mathrm{d}X_t \quad = \quad \mu(X_t)\mathrm{d}t + \underline{\sigma}\mathrm{d}W_t$$

where $\mu(X_t) \in \mathbb{R}^2$, $\underline{\sigma}(X_t) \in \mathbb{R}^{2\times 2}$ and $(W)_t$ is a 2-dimensional $\mathbb{P}$-Brownian motion in $\mathbb{R}^2$.

Explicitly the vector X has the following expression:

$$\mathrm{d}\begin{bmatrix} S_t \\ Y_t \end{bmatrix} \quad = \quad \begin{bmatrix} \mu S_t \\ \eta \end{bmatrix}\mathrm{d}t + \begin{bmatrix} \sigma(Y_t) & 0 \\ 0 & \gamma \end{bmatrix}\begin{bmatrix} \mathrm{d}W_{1,t} \\ \mathrm{d}W_{2,t} \end{bmatrix}$$

Like as in the previous example, we have to change probability from measure $\mathbb{P}$ to $\mathbb{Q}^\beta$, that is a martingale measure respect to the first component of X, the risky asset price;

Corollary 9. In a standard stochastic volatility model in NIRH, like explicitly before, the set of equivalent martingale measures $\mathbb{Q}^\beta$ for the risky asset's price, that is the first component of the vector X, is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{\mathrm{d}\mathbb{Q}^{\beta}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = L_t^{\beta} \tag{10}$$

where L_t^{β} has the following expression:

$$L_t^{\beta} = \exp\left[\int_0^t \beta_u^{\mathsf{T}} \mathrm{d}W_u - \frac{1}{2}\int_0^t |\beta_u^{\mathsf{T}}\beta_u| \,\mathrm{d}u\right]$$

assumed to be a square integrable strictly positive martingale.

In particular $(\beta)_t$ is a predictable process in $\mathbb{R}^2$ satisfying the following relation:

$$\mu(X_t) + \sigma\beta_t = Y_t \tag{11}$$

where Y_t is a 2-dimensional vector where the first component is null.

Proof. We proceed like in proof of proposition 1, but here we consider a martingale measure only respect to the first component of vector X and we don't have a jump process, unlike in the section 2 where all the vector X, that is a mixed diffusion, represent the risky asset's prices.□

Like as an application of the proposition 2 we have the following;

Corollary 10. In a standard stochastic volatility model in NIRH, the range of the arbitrage prices of a contingent claim which final payoff is a function $f(t,x) \in \mathbb{C}^{1,2}$, is $[f^d, f^u[$, where f^u is the vischiosity solution of the following:

$$-\partial_t f^u(t,x) - \frac{1}{2}\sigma^2 \partial_{SS} f^u(t,x) \geq 0 \tag{12}$$

subject to the final condition $f^u(T^-, x) \geq f(x)$, and where f^d is the vischiosity solution of the following:

$$\partial_t f^d(t,x) + \frac{1}{2}\sigma^2 \partial_{SS} f^u(t,x) \leq 0$$

subject to the final condition $f^d(T^-, x) \leq f(x)$ and where $\partial_{SS} f^u(t,x)$ is the first component in the diagonal of matrix $\partial_{xx} f^u(t,x)$ and σ is the component in place $(1,1)$ of the matrix $\underline{\sigma}$.

Proof. We proceed like as in proof of proposition 2, but when we consider the Bellman's equation (5), and suppose that the condition (11) is satisfied, $f^u(t,x)$ is the vischiosity super-solution of the following:

$$-\partial_t f^u(t,x) + \inf_{\beta \in \mathcal{D}} \left\{ -Y_t^\top \partial_x f^u(t,x) - \frac{1}{2} tr(\underline{\sigma}^\top \partial_{xx} f^u(t,x) \underline{\sigma}) \right\} \geq 0 \tag{13}$$

with the final condition $f^u(T^-, x) \geq f(x)$, where $\mathcal{D}$ is the set of $\mathcal{F}$-adapted and limited process with values on $\mathbb{R}^2$.

If we suppose that $\beta \to \mp\infty$ in equation (13), we have to impose that $\partial_x f^u(t,x)$ is a vector where the second component are null and from this follows that also the second derivatives respect this variable, in the principal diagonal of matrix $\partial_{xx} f^u(t,x)$, is null. So the only variable that is not null is the first component in the principal diagonal of matrix $\partial_{xx} f^u(t,x)$, that is the second derivative respect to S, the first component of vector X, like in the thesis of proposition.

For the bound $f^d(t,x)$, we procede in the same way.□

We observe that the range of derivative's prices depend only on t and on the price S, so in the following we may write $f^{d,u}(t,S)$. Also we may characterize these extremes of the interval of contingent claims prices, like in the following;

Corollary 11. If we suppose that the volatility of the price of risky asset S, σ, is not bounded, in NIRH model, so the upper bound f^u not increase on t and is a concave function on the first component S of X and by the same argument f^d is not decreasing on t and is a convex function on S.

Proof. If we suppose that σ is null, so equation (12) prove the not increasing property of $f^u(t,S)$ respect to t.

By the other hand, if we suppose that $\sigma \to +\infty$, so to respect (12) we must have $\partial_{SS} f^u(t,S) \leq 0$.

By the same arguments we find the characterization of $f^d(t,S)$.□

An explicitly characterization of the derivative's prices bounds, is possible only for particular derivative's final payoff functions, like as it is pointed in proposition 4.

5. THE MODEL IN THE SIRH

We consider a financial market where the incertainty is represented by a stochastic variable $(X)_t \in \mathbb{R}^n$, that is the price of risky assets, which dynamic is the following jump diffusion[7]:

[7] Here we consider the proportional drift and diffusion on X_t to semplify the calculation.

$$\frac{\mathrm{d}X_t}{X_t} = r_t \mathrm{d}t + \sigma(t, X_t)^{\mathsf{T}} \mathrm{d}W_t^{\mathbb{Q}^\gamma} + \underline{1}^{\mathsf{T}} \mathrm{d}Z_t^{\mathbb{Q}^\gamma}$$

where, r_t is the spot interest rate, $\underline{1} \in \mathbb{R}^n$, $\sigma(t, X_t) \in \mathbb{R}^n$, and the parameters are defined like in section 2 and section 3.

We suppose that the interest rate is stochastic and in particular we suppose that the dynamic of the zero coupon price $(B(.,T))_t$, is the following:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r_t \mathrm{d}t + \Gamma(t,T)^{\mathsf{T}} \mathrm{d}W_t^{\mathbb{Q}} \tag{14}$$

or equivalently, on the measure $\mathbb{Q}^\gamma$:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = (r_t + \Gamma(t,T)^{\mathsf{T}} \beta_t)\mathrm{d}t + \Gamma(t,T)^{\mathsf{T}} \mathrm{d}W_t^{\mathbb{Q}^\gamma} \tag{15}$$

where, $B(t,T) \in \mathbb{R}$ and $\Gamma(t,T) \in \mathbb{R}^n$.

6. THE SET OF EQUIVALENT MARTINGALE MEASURE AND THE INTERVAL OF ARBITRAGE PRICES IN THE SIRH

If we suppose that the market is incomplete and we make the SIRH, to determine the range of prices of a contingent claim of X, we have to consider the set of equivalent martingale measures $\mathbb{Q}^\eta$ for X, so that:

$$\mathbb{E}_{\mathbb{Q}^\eta}\left[X_T \mid \mathcal{F}_t\right] = \frac{X_t}{B(t,T)}$$

where the zcb price follows the SDE (14) or (15).

We have the following;

Proposition 12. In a general incomplete market with SIRH, the set of equivalent martingale measures $\mathbb{Q}^\eta$ for X is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{\mathrm{d}\mathbb{Q}^\eta}{\mathrm{d}\mathbb{Q}^\gamma}\right|_{\mathcal{F}_t} = L_t^\eta \tag{16}$$

where:

$$L_t^\eta = \exp\left[\int_0^t \xi_u^\intercal \mathrm{d}W_u^{\mathbb{Q}^\gamma} - \frac{1}{2}\int_0^t |\xi_u^\intercal \xi_u| \mathrm{d}u + \int_0^t [\ln(1+\eta_u)]^\intercal \mathrm{d}M_u^{\mathbb{Q}^\gamma} + \right. \\ \left. - \int_0^t \theta(\eta_u)\Lambda(u, X_u)\mathrm{d}u\right] \tag{17}$$

assumed to be a $\mathbb{Q}^\gamma$-square integrable strictly positive martingale and where the process $(\eta)_t$ is so that $(1+\eta_t) > 0$.

In particular $(\xi, \eta)_t$ are two predictable process in $\mathbb{R}^n$ linked by the following relation:

$$\left(\frac{X_t}{B(t,T)}\right)\Big[|\Gamma(t,T)^\intercal \Gamma(t,T)| - \Gamma(t,T)^\intercal \beta_t - \sigma(t,X_t)^\intercal \Gamma(t,T) + \\ + |\sigma(t,X_t) - \Gamma(t,T)|^\intercal \xi_t + \theta(\eta_t)\Lambda(t,X_t)\Big] = \underline{0} \tag{18}$$

where $\underline{0}$ is the n-dimensional vector of zeros.

Proof. The dynamic of $\left(\frac{X}{B(.,T)}\right)_t$ under the probability $\mathbb{Q}^\eta$, defined by the Radon-Nycodim derivative (17) is the following[8]

$$\frac{\mathrm{d}\left(\frac{X_t}{B(t,T)}\right)}{\frac{X_t}{B(t,T)}} = (|\Gamma(t,T)^\intercal \Gamma(t,T)| - \Gamma(t,T)^\intercal \beta_t - \sigma(t,X_t)^\intercal \Gamma(t,T))\mathrm{d}t + \\ + |\sigma(t,X_t) - \Gamma(t,T)|^\intercal \mathrm{d}W_t^{\mathbb{Q}^\gamma} + \underline{1}^\intercal \mathrm{d}Z_t^{\mathbb{Q}^\gamma} \\ = (|\Gamma(t,T)^\intercal \Gamma(t,T)| - \Gamma(t,T)^\intercal \beta_t - \sigma(t,X_t)^\intercal \Gamma(t,T) + |\sigma(t,X_t) - \Gamma(t,T)|^\intercal \xi_t + \\ + \theta(\eta_t)\Lambda(t,X_t))\mathrm{d}t + |\sigma(t,X_t) - \Gamma(t,T)|^\intercal \mathrm{d}W_t^{\mathbb{Q}^\eta} + \underline{1}^\intercal \mathrm{d}Z_t^{\mathbb{Q}^\eta}$$

that is a $\mathbb{Q}^\eta$-martingale under condition (18) and where, by Girsanov theorem, we define the $\mathbb{Q}^\eta$-Brownian motion $(W^{\mathbb{Q}^\eta})_t$ and the $\mathbb{Q}^\eta$-pure jump process $(Z^{\mathbb{Q}^\eta})_t$, as follows:

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[8] By applying the It ô lemma, we find the dynamic of $\left(\frac{1}{B(.,T)}\right)_t$:

$$\frac{\mathrm{d}\left(\frac{1}{B(t,T)}\right)}{\frac{1}{B(t,T)}} = -[(r_t + \Gamma(t,T)^\intercal \beta_t)\mathrm{d}t + \Gamma(t,T)^\intercal \mathrm{d}W_t^{\mathbb{Q}^\gamma} - |\Gamma(t,T)^\intercal \Gamma(t,T)| \mathrm{d}t]$$

$$\begin{aligned} W_t^{\mathbb{Q}^\eta} &= W_t^{\mathbb{Q}^\gamma} - \int_0^t \xi_u \mathrm{d}u \\ M_t^{\mathbb{Q}^\eta} &= Z_t^{\mathbb{Q}^\eta} - \int_0^t \theta(\eta_u)\Lambda(u, X_u)\mathrm{d}u \end{aligned}$$

where $(M^{\mathbb{Q}^\eta})_t$ is a $\mathbb{Q}^\eta$-martingale.

Proposition 13. In a general incomplete market with SIRH, the arbitrage prices range of a contingent claim which final payoff is a function $F(t,x) \in \mathbb{C}^{1,2}$, is $[F^d, F^u[$, where F^u is the vischiosity solution of the following:

$$\begin{aligned} - \quad & \partial_t F^u(t,x) - \frac{1}{2} tr\left[\left(\frac{x}{B(t,T)}\right)\left|\sigma(t,x) - \Gamma(t,T)\right|^\intercal \partial_{xx} F^u(t,x)\right. \\ & \left.\left|\sigma(t,x) - \Gamma(t,T)\right|\left(\frac{x}{B(t,T)}\right)^\intercal\right] \geq 0 \end{aligned} \tag{19}$$

subject to the final condition $F^u(T^-, x) \geq F(x)$, where $\partial_t F^u(t,x) \in \mathbb{R}^n$, $\partial_{xx} F^u(t,x) \in \mathbb{R}^{n \times n}$ and where F^d is the vischiosity solution of the following:

$$\begin{aligned} & \partial_t F^d(t,x) + \frac{1}{2} tr\left[\left(\frac{x}{B(t,T)}\right)\left|\sigma(t,x) - \Gamma(t,T)\right|^\intercal \partial_{xx} F^u(t,x)\right. \\ & \left.\left|\sigma(t,x) - \Gamma(t,T)\right|\left(\frac{x}{B(t,T)}\right)^\intercal\right] \leq 0 \end{aligned} \tag{20}$$

subject to the final condition $F^d(T^-, x) \leq F(X_T)$.

Proof. We define the upper bound F^u of contingent claim's prices range, as follows:

$$\begin{aligned} F^u &= \sup_{\eta \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\eta}[F(X_T)] \\ &= -\inf_{\eta \in \mathcal{D}} \mathbb{E}_{\mathbb{Q}^\eta}[-F(X_T)] \end{aligned}$$

where $\mathcal{D}$ is the set of $\mathcal{F}$-adapted and limited process with values on $\mathbb{R}^n$, and consider the related dynamic problem:

$$F^{u}(t,x)=\sup_{\eta\in\mathcal{D}}\mathbb{E}_{\mathbb{Q}^{\eta}}[F(X_T)\,|\,X_t=x]$$

so that $F^{u}=F^{u}(0,x_0)$. If we consider the Bellman's equation for this stochastic control, we can characterize $F^{u}(t,x)$ like as the vischiosity super-solution of the following:

$$-\partial_t F^{u}(t,x)+\inf_{\eta\in\mathcal{D}}\left\{-\left(\frac{X_t}{B(t,T)}\right)^{\intercal}\Big[|\Gamma(t,T)\Gamma(t,T)^{\intercal}|-\Gamma(t,T)\beta_t^{\intercal}+\right.$$
$$-\sigma(t,X_t)\Gamma(t,T)^{\intercal}+|\sigma(t,X_t)-\Gamma(t,T)|\,\xi_t^{\intercal}+\theta(\eta_t)\Lambda(t,X_t)\Big]\partial_x F^{u}(t,x)+$$
$$\left.-\frac{1}{2}tr\left[\left(\frac{x}{B(t,T)}\right)\left|\sigma(t,x)-\Gamma(t,T)\right|^{\intercal}\partial_{xx}F^{u}(t,x)\left|\sigma(t,x)-\Gamma(t,T)\right|\left(\frac{x}{B(t,T)}\right)^{\intercal}\right]\right\}\geq 0 \qquad (21)$$

with the final condition $F^{u}(T^{-},x)\geq F(x)$. We observe that, if the condition (18) is satisfied, so (21) begun the (19) in the thesis of proposition.

For the bound F^{d} we proceed in the same way.□

Proposition 14. If we suppose that $|\sigma(t,x)-\Gamma(t,T)|$ is a not bounded vector, so the upper bound F^{u} is not increasing on t and is concave on X and by the same argument F^{d} is not decreasing on t and is convex on X.

Proof. If we suppose that $|\sigma(t,x)-\Gamma(t,T)|$ is a vector with null components, so equation (19) prove the not increasing property of $F^{u}(t,x)$ respect to t.

By the other hand, if we suppose the limit case in which all the components of the vector $|\sigma(t,x)-\Gamma(t,T)|\to+\infty$, so to respect (19) we must have $\partial_{xx}F^{u}(t,x)\leq 0$. By the same arguments we find the characterization of $F^{d}(t,x)$.□

Like as in DIRH model, we have the following explicitly characterization of the bounds of derivative's prices in SIRH model;

Proposition 15. If $F^{cv}(x)$ is the t-price of the concave envelopment of function $F(x)$, in SIRH model we have that:

$$F^{u}(t,x)=F^{cv}(x)$$

and in particular $F^{u} = F^{cv}(x_0)$.

In the same way, if $F^{cx}(x)$ is the t-price of the convex envelopment of $F(x)$, we have that:

$$F^{d}(t,x) = F^{cx}(x)$$

and in particular $F^{d} = F^{cx}(x_0)$.

Proof. We proceed like as in proof of proposition 4. □

7. SOME EXAMPLES IN THE SIRH

7.1. *The future price of gas in an affine model*

In this example we consider a particular incomplete market driven by an affine jump diffusion like this presented in subsection 4.1, applied to the market of gas derivatives where we suppose that the final payoff function of future contract is not exponential affine and the interest rate is stochastic.

We consider the future price of gas, as a function of the actual spot price of gas G, the price of petrol P^{P} which is modellized by a diffusion with jump, the convenience yield δ and the stochastic volatility v:

$$\overline{F}G = f(G, P^{P}, \delta, v)$$

where the variables follows a process like this proposed in subsection 4.1., and the instantaneous interest rate $(r)_t$ follows an Ornstein-Uhlenbeck mean-reverting process:

$$dr_t = a(b - r_t)dt - \sigma_r dW_{4,t}^{\mathbb{Q}}$$

where the parameters a, b and σ_r are constants, and where $(W_4^{\mathbb{Q}})_t$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. From this hypothesis follow that the zcb price $B(t,T)$ is:

$$B(t,T) = \exp -\left[TR_{\infty} + (R_{\infty} - r_t)C(T-t) + \frac{\sigma_r^2}{4a} C(T-t)^2 \right]$$

where $R_{\infty} = b - \dfrac{\sigma^2}{2a^2}$ and the zcb's price volatility is:

$$\Gamma(t,T) = \sigma_r C(T-t)$$

where $C(T-t) = \dfrac{1-e^{-a(T-t)}}{a}$.

So we define the vector $\overline{X}_t$, as:

$$\overline{X}_t = \begin{bmatrix} P^P \\ \delta_t \\ v_t \end{bmatrix}$$

that follows the following diffusion:

$$\frac{\mathrm{d}\overline{X}_t}{\overline{X}_t} = \mu(t,\overline{X}_t)\mathrm{d}t + \sigma(t,\overline{X}_t)^{\intercal}\mathrm{d}W_t^{\mathbb{Q}} + I^{\intercal}\mathrm{d}Z_t^{\mathbb{Q}}$$

where $I^{\intercal} = [1 \quad 0 \quad 0]$, $\mu(t,\overline{X}_t) \in \mathbb{R}$, $\sigma(t,\overline{X}_t) \in \mathbb{R}^3$ and $(W^{\mathbb{Q}})_t$ is a standard Brownian vector in $\mathbb{R}^3$ and where we may write $\mu(t,\overline{X}_t)$ and $\sigma(t,\overline{X}_t)$ like as two affine functions of $\overline{X}_t$:

$$\begin{aligned} \mu(t,\overline{X}_t) &= K_0 + K_1\overline{X}_t \\ \sigma(t,\overline{X}_t)\sigma(t,\overline{X}_t)^{\intercal} &= H_0 + H_1\overline{X}_t^{\intercal} \end{aligned}$$

where:

$$K_0 = \begin{bmatrix} 0 \\ \overline{\alpha} \\ \theta \end{bmatrix}; K_0 \in \mathbb{R}^3$$

$$K_1(t) = \begin{bmatrix} -\delta_t & 0 & 0 \\ 0 & -\overline{\beta} & 0 \\ 0 & 0 & -k \end{bmatrix}; K_1(t) \in \mathbb{R}^{3\times 3}$$

$$H_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; H_0 \in \mathbb{R}^{3\times 3}$$

$$H_1 = \begin{bmatrix} H_1^{(1)} & \underline{0} & \underline{0} \\ \underline{0} & H_1^{(2)} & \underline{0} \\ \underline{0} & \underline{0} & H_1^{(3)} \end{bmatrix}; H_1 \in \mathbb{R}^{3\times 3\times 3}$$

where $\underline{0}$ is the null vector in $\mathbb{R}^3$ and where:

$$H_1^{(1)} = \begin{bmatrix} 0 \\ 0 \\ e^{2\nu} \end{bmatrix}; H_1^{(1)} \in \mathbb{R}^3$$

$$H_1^{(2)} = \begin{bmatrix} 0 \\ 0 \\ e^{2\nu}\sigma_\delta^2 \end{bmatrix}; H_1^{(2)} \in \mathbb{R}^3$$

$$H_1^{(3)} = \begin{bmatrix} 0 \\ 0 \\ \sigma_\nu^2 \end{bmatrix}; H_1^{(3)} \in \mathbb{R}^3$$

In this contest we have to change probability, considering that only the first component of $\overline{X}$ is the price of risky asset.

Corollary 16. In an incomplete market driven by an affine jump diffusion and in the SIRH, like explicitly before, the set of equivalent martingale measures $\mathbb{Q}^\eta$ for the risky asset's price, that is the first component of the vector $\overline{X}$, is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{d\mathbb{Q}^\eta}{d\mathbb{Q}^\gamma}\right|_{\mathcal{F}_t} = L_t^\eta \tag{22}$$

where L_t^η has the same expression presented in (17) of proposition 12.

In particular $(\xi,\eta)_t$ are two predictable process in $\mathbb{R}^3$ linked by the relation (18) equated to a vector $Y_t \in \mathbb{R}^3$ where the first component is null, and where $\Gamma(t,T)$ is zcb's price volatility in the Ornstein-Uhlenbeck hypothesis.

Proof. We proceed like in proof of proposition 12, but here we consider a martingale measure only respect the first component of vector $\overline{X}$, unlike in the previous section where all the vector $\overline{X}$ represent the risky asset's prices.□

So we have the following corollary, like as an application of the Proposition 13;

Corollary 17. If we suppose that the future price of gas is the actual price of gas G, times a function on $\overline{X}_T$, $g(t,\overline{x}) \in \mathbb{C}^{1,2}$, in an incomplete market driven by an affine jump diffusion in SIRH, the range of his arbitrage future prices is the gas price G, times $[\overline{F}G^d, \overline{F}G^u[$, where $\overline{F}G^u$ is the vischiosity solution of the following:

$$-\partial_t \overline{F}G^u(t,\overline{x}) - \frac{1}{2}\left(\frac{P^P}{B(t,T)}\right)^2 (\sigma^{(1)}(t,\overline{x}) - \Gamma(t,T))^2 \partial_{P^P P^P} \overline{F}G^u(t,\overline{x}) \geq 0 \qquad (23)$$

subject to the final condition $\overline{F}G^u(T^-,\overline{x}) \geq g(\overline{x})$, and where $FG^d(t,\overline{x})$ is the vischiosity solution of the following:

$$\partial_t \overline{F}G^d(t,\overline{x}) + \frac{1}{2}\left(\frac{P^P}{B(t,T)}\right)^2 (\sigma^{(1)}(t,\overline{x}) - \Gamma(t,T))^2 \partial_{P^P P^P} \overline{F}G^d(t,\overline{x}) \leq 0$$

subject to the final condition $\overline{F}G^d(T^-,\overline{x}) \leq g(\overline{x})$ and where $\partial_{P^P P^P} \overline{F}G^{u,d}(t,\overline{x})$ is the first component in the diagonal of matrix $\partial_{xx} \overline{F}G^{u,d}(t,\overline{x})$ and $\sigma^{(1)}$ is the first component of the vector $\sigma(t,\overline{x})$ [9].

Proof. We proceed like as in proof of proposition 2 and proposition 12, but when we consider the Bellman's equation (21), if we suppose that the condition (18) equated to Y_t is satisfied, and that $\xi,\eta \to \mp\infty$ in equation (21), we have to impose that $\partial_x \overline{F}G^u(t,\overline{x})$ is a vector where the components, except the first, are null and from this follows that also the second derivatives respect these variables, in the principal diagonal of matrix $\partial_{xx} \overline{F}G^u(t,\overline{x})$, are null. So the only variable that is not null is the first component in the principal diagonal of matrix $\partial_{xx} \overline{F}G^u(t,\overline{x})$, that is the second derivative respect to the first component of vector $\overline{X}$, like in the thesis of proposition.

For the bound $\overline{F}G^d(t,\overline{x})$, we proceed in the same way.□

Also in the contest of a SIRH model, we may characterize the bounds of the range of future gas prices, like in the proposition 14 and proposition 15 in the case of $\sigma^{(1,1)}(t,\overline{x}) - \Gamma(t,T) \in [0,+\infty[$.

7.2. *Stochastic volatility model*

Here we consider the example proposed in subsection 4.2 in the contest of SIRH model.

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[9] Here we have only the first component of the vector $\sigma(t,\overline{x})$, because the change of probability from $\mathbb{Q}^\gamma$ to $\mathbb{Q}^\eta$, make null only the drift of the first component of the vector $\overline{X}$, that is the drift of the risky asset's price.

We suppose that under the probability measure $\mathbb{P}$, the dynamics of the vector X, with component S and Y, is the following:

$$\frac{\mathrm{d}\overline{X}_t}{\overline{X}_t} = \mu(\overline{X}_t)\mathrm{d}t + \tilde{\sigma}^{\mathsf{T}}\mathrm{d}W_t$$

where $\mu(\overline{X}_t) \in \mathbb{R}$, $\tilde{\sigma} \in \mathbb{R}^2$ and $(W)_t$ is a 2-dimensional $\mathbb{P}$-Brownian motion. Like as in the previous example, we have to change probability from measure $\mathbb{P}$ to $\mathbb{Q}^{\xi}$, that is a martingale measure respect to the first component of X, the risky asset price;

Corollary 18. In a standard stochastic volatility model in SIRH, like explicitly before, the set of equivalent martingale measures $\mathbb{Q}^{\xi}$ for the risky asset's price, that is the first component of the vector $\overline{X}$, is defined by the following Radon-Nycodim derivatives:

$$\left.\frac{\mathrm{d}\mathbb{Q}^{\xi}}{\mathrm{d}\mathbb{Q}^{\beta}}\right|_{\mathcal{F}_t} = L_t^{\xi} \tag{24}$$

where L_t^{ξ} has the following expression:

$$L_t^{\xi} = \exp\left[\int_0^t \xi_u^{\mathsf{T}}\mathrm{d}W_u - \frac{1}{2}\int_0^t |\,\xi_u^{\mathsf{T}}\xi_u\,|\,\mathrm{d}u\right]$$

assumed to be a square integrable strictly positive martingale.

In particular $(\xi)_t$ is a predictable process in $\mathbb{R}^2$ satisfying the relation (18) but equated to Y_t, a 2-dimensional vector where the first component is null.

Proof. We proceed like in proof of proposition 12, but here we consider a martingale measure only respect to the first component of vector $\overline{X}$ and we don't have a jump process, unlike in the section 6 where all the vector X, that is a mixed diffusion, represent the risky asset's prices.□

Like as an application of the proposition 12 we have the following;

Corollary 19. In a standard stochastic volatility model in SIRH, the range of the arbitrage prices of a contingent claim which final payoff is a function $f(\overline{X}_T) \in \mathbb{C}^{1,2}$, is $[\overline{f}^d, \overline{f}^u[$, where $\overline{f}^u$ is the vischiosity solution of the following:

$$-\partial_t \overline{f}^u(t,\overline{x}) - \frac{1}{2}\left(\frac{S}{B(t,T)}\right)^2 (\tilde{\sigma}^{(1)} - \Gamma(t,T))^2 \partial_{SS} \overline{f}^u(t,\overline{x}) \geq 0 \tag{25}$$

subject to the final condition $\overline{f}^u(T^-,\overline{x}) \geq f(\overline{x})$, and where $\overline{f}^d$ is the vischiosity solution of the following:

$$\partial_t \overline{f}^d(t,\overline{x}) + \frac{1}{2}\left(\frac{S}{B(t,T)}\right)^2 (\tilde{\sigma}^{(1)} - \Gamma(t,T))^2 \partial_{SS} \overline{f}^d(t,\overline{x}) \leq 0$$

subject to the final condition $\overline{f}^d(T^-,\overline{x}) \leq f(\overline{x})$ and where $\partial_{SS} \overline{f}^{u,d}(t,\overline{x})$ is the first component in the diagonal of matrix $\partial_{xx} \overline{f}^{u,d}(t,\overline{x})$ and $\tilde{\sigma}^{(1)}$ is the first component of the vector $\tilde{\sigma}$.

Proof. We proceed like as in proof of proposition 13, but when we consider the Bellman's equaton (21), and suppose that the condition (18)) equated to vector Y_t is satisfied, $\overline{f}^u(t,x)$ is the vischiosity super-solution of the following:

$$\begin{aligned} &-\partial_t \overline{f}^u(t,\overline{x}) + \inf_{\xi \in \mathcal{D}} \Big\{ -Y_t^{\intercal} \partial_{\overline{x}} \overline{f}^u(t,\overline{x}) + \\ &-\frac{1}{2} tr\left[\left(\frac{\overline{x}}{B(t,T)}\right) \left|\tilde{\sigma} - \Gamma(t,T)\right|^{\intercal} \partial_{\overline{xx}} \overline{f}^u(t,\overline{x}) \left|\tilde{\sigma} - \Gamma(t,T)\right| \left(\frac{\overline{x}}{B(t,T)}\right)^{\intercal} \right] \Big\} \geq 0 \end{aligned} \tag{26}$$

with the final condition $\overline{f}^u(T^-,\overline{x}) \geq f(\overline{x})$, where $\mathcal{D}$ is the set of $\mathcal{F}$-adapted and limited process with values on $\mathbb{R}^2$.

If we suppose that $\xi \to \mp\infty$ in equation (26), we have to impose that $\partial_{\overline{x}} \overline{f}^u(t,\overline{x})$ is a vector where the second component are null and from this follows that also the second derivatives respect this variable, in the principal diagonal of matrix $\partial_{\overline{xx}} \overline{f}^u(t,\overline{x})$, is null. So the only variable that is not null is the first component in the principal diagonal of matrix $\partial_{\overline{xx}} \overline{f}^u(t,\overline{x})$, that is the second derivative respect to S, the first component of vector $\overline{X}$, like in the thesis of proposition.

For the bound $\overline{f}^d(t,\overline{x})$, we proceed in the same way. □

We may characterize these extremes of the interval of contingent claims prices, like in proposition 15 in the case of $(\tilde{\sigma}^{(1)} - \Gamma(t,T)) \in [0,+\infty[$.

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RIASSUNTO

L'intervallo dei prezzi d'arbitraggio dei derivati in un mercato incompleto generale

In questo lavoro si considera un mercato incompleto generale guidato da una diffusione mista di dimensione finita e si caratterizza l'intervallo dei prezzi d'arbitraggio dei derivati attraverso un approccio di super-replicazione in ipotesi di tasso d'interesse deterministico (DIRH) ed in ipotesi di tasso d'interesse stocastico (SIRH). Si presentano esempi di applicazione di tali modelli a particolari situazioni d'incompletezza.

SUMMARY

The range of derivative's arbitrage prices in a general incomplete market

In this paper we work in a general incomplete market driven by a mixed diffusion of finite dimension and we characterize the range of derivative's arbitrage prices by the super-replication approach in the deterministic interest rate hypothesis (DIRH) and in the stochastic interest rate hypothesis (SIRH). We give some examples of applications of this models in particular incomplete situations.