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THE INVERTED EXPONENTIATED GAMMA DISTRIBUTION: A HEAVY-TAILED MODEL WITH UPSIDE DOWN BATHTUB SHAPED HAZARD RATE

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1. Introduction

In lifetime analysis, monotone failure rate models play a vital role to study the reliability characteristics. The most popular monotone hazard rate models are Weibull, gamma, exponentiated exponential, etc. The gamma distribution did not receive the desired attention of the analyst due to its non-closed form of the cumulative distribution function (CDF). Therefore, several generalizations based on gamma distribution have been developed, viz generalized gamma distribution, exponentiated generalized gamma distribution, etc. The exponentiated gamma distribution (EGD) is one of these. The genesis of exponentiated gamma distribution is as follows. Let a variable $Z \sim \text{Gamma}(\alpha,\beta)$ and $F(z,\alpha,\beta)$ is the CDF of random variable Z with shape parameter α and scale parameters β. If we consider $\alpha = 2$ and $\beta = 1$, then the new variable $Y = [F(z,2,1)]^{\theta}$ has CDF of exponentiated gamma distribution with shape parameter θ. The resulting probability density function (PDF) and CDF of the EGD are written by

$$f(y,\theta) = \theta y e^{-y}\left\{1 - e^{-y}(1+y)\right\}^{\theta-1}; \quad y > 0, \theta > 0 \tag{1}$$

and

$$F(y) = \left\{1 - e^{-y}(1+y)\right\}^{\theta}; \qquad y, \theta > 0 \tag{2}$$

respectively.

The EGD was firstly studied in detail by Shawky and Bakoban (2006). It may be noted here that, the EGD is parsimonious in parameters and hence, simple to use. The other advantage is that it has various shapes of hazard function for different values of the shape parameter θ. It has an increasing hazard rate when $\theta > \frac{1}{2}$ and takes bathtub shape

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for $\theta < \frac{1}{2}$. The applicability and work related to the EGD have been discussed by several authors, see Shawky and Bakoban (2008), Singh *et al.* (2011), Shawky and Bakoban (2012), etc.

The inverse transformation method of baseline variable is one of the most popular way to obtain the inverted version of any lifetime distribution. Generally, the generalized inverted family of distribution possesses the property of upside down bathtub (UBT) hazard rate. Another remark with such distributions is parsimonious in parameter and simple to use. For example, inverse exponential distribution (IED) (Lin *et al.*, 1989), inverse Lindley distribution (ILD) (Sharma *et al.*, 2015). Recently, Yadav *et al.* (2019) introduced the inverted version of xgamma distribution, etc. All the above-cited models exhibit the property of UBT hazard rate. The counterpart to bathtub hazard rate is the three-phase situation in which the failure rate initially increases, then becomes essentially constant, and ultimately decreases. This failure rate function, which we will term UBT shaped. Also, the detailed description of different types of hazard rate including UBT may be seen in Domma *et al.* (2017). Here, the same transformation has been used to introduce the inverted form of EGD. If a random variable Y has $EGD(\theta)$ with PDF given in (1), then the random variable $X = \frac{1}{Y}$ is said to follow the inverted exponentiated gamma distribution (IEGD). The CDF (3) and PDF (4) of IEGD are given by the following equations

$$F(x,\theta) = 1 - \left[1 - e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right)\right]^{\theta} \quad x > 0, \theta > 0 \tag{3}$$

and

$$f(x,\theta) = \frac{\theta}{x^3} e^{-\frac{1}{x}} \left[1 - e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right)\right]^{\theta-1} \quad x > 0, \theta > 0. \tag{4}$$

The primary objective of this article is to propose a new lifetime distribution called as IEGD and study its related distributional characteristics. The proposed distribution belongs to the inverted family of distributions and shows the UBT type hazard rate and is the member of family of heavy-tailed distribution. Further, the classical and Bayesian estimation procedures have also been discussed to estimate the parameter of IEGD. In classical estimation, the maximum likelihood estimation (MLE), the maximum product spacing estimation (MPSE), least square and weighted least square estimation (LSE & WLSE), Cramer-von-Mises estimation (CVME) have been used. The Bayes estimate of the parameter is computed by assuming gamma prior under squared error loss function. To the best of our knowledge thus so far, no attempt has been made to introduce the inverted version of EGD. The present study is desired to fill-up this gap.

The rest of the article is organized as follows. Section 1 describes the introduction part related to the considered problem. In Section 2, we have studied different statistical properties and associated measures of IEGD. Different classical methods of estimation of the parameter of IEGD have been considered in Section 3. The Bayes estimation has been discussed in Section 4. The simulation study has been carried out to compare the different obtained estimators in Section 5. Empirical application based on a real data set is discussed in Section 6. Finally, concluding remarks are given in Section 7.

2. Some statistical properties

In this section, different statistical properties related to the new distribution have been derived in the following subsections.

2.1. *Reliability and hazard functions*

The reliability and hazard functions are the two fundamental elements to study the characteristics of the time to event data. The term reliability is used in engineering field and it gives the probability that equipment survives at least time t. The hazard function of particular phenomenon describes the nature of failure rate associated with the lifetime of the specific equipment. The reliability and hazard functions of IEGD are given in the following.

- The reliability function $R(x)$ can be easily obtained by using the relation $R(x,\theta)=1-F(x,\theta)$. Hence,
$$R(x,\theta)=\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta}. \tag{5}$$
- The hazard function $h(x)$ is the instantaneous failure rate and is obtained as
$$h(x)=\frac{f(x,\theta)}{R(x,\theta)}=\frac{\theta}{x^2\left(xe^{\frac{1}{x}}-x-1\right)}. \tag{6}$$
- The reverse hazard function $H(x)$ is obtained as
$$H(x)=\frac{f(x,\theta)}{F(x,\theta)}=\frac{\theta e^{-\frac{1}{x}}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta-1}}{x^3\left\{1-\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta}\right\}}. \tag{7}$$
- The cumulative hazard function for IEGD is defined by
$$Ch(x)=-\log R(x,\theta)=-\theta\log\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]. \tag{8}$$

See Figure 1 for the reliability function plot and Figure 2 for the hazard function plot.

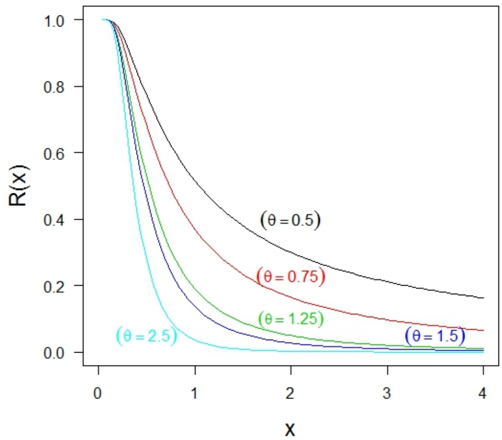

Figure 1 – Reliability function plot of IEGD for different values of θ.

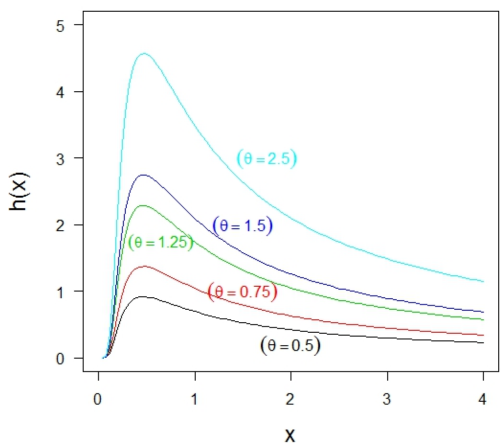

Figure 2 – Hazard function plot of IEGD for different values of θ.

2.2. *Shape of the distribution*

The shape of the distribution can be traced either graphically or mathematically. It meditate the idea whether the distribution is symmetric or skewed. The graphical investigation for the proposed PDF is presented in Figure 3.

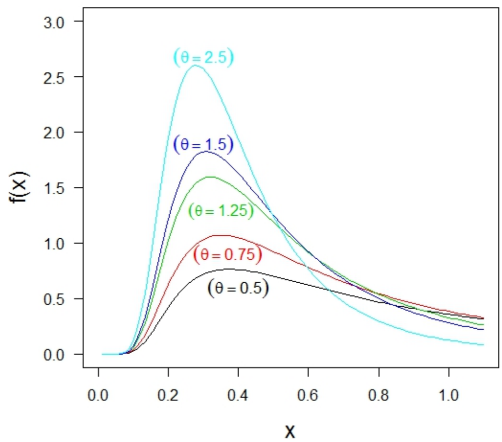

Figure 3 – Density plot of IEGD for different values of θ.

From the figure, it is clear that the proposed distribution shows the pattern of the right-skewed model. Also, using Equation (6), we can have an idea about the nature of its hazard rate, we have plotted the shapes of hazard rate for the various value of the parameter θ in Figure 2. This figure shows that the proposed distribution has UBT type hazard rate. This can also be verified mathematically by using the result of Glaser (1980). Glaser proved that if $\eta'(t) > 0$ for all $t \in (0, t_0)$, and $\eta'(t_0) = 0$ and $\eta'(t) < 0$ for all $t > t_0$ and satisfying $\lim_{t \to 0} f(t) = 0$ then distribution has upside down bathtub failure rate (UBT), where $\eta(t)$ is equal to $-f'(t)/f(t)$ and $f'(t)$ is the first order derivative of density function $f(t)$ with respect to t. In case of our proposed distribution, we see that

$$\eta(t) = \frac{3}{t} - \frac{1}{t^2} + \frac{(\theta - 1)e^{-\frac{1}{t}}}{t^3\left[1 - e^{-\frac{1}{t}}\left(1 + \frac{1}{t}\right)\right]} \tag{9}$$

and

$$\eta'(t) = \frac{-3}{t^2} + \frac{2}{t^3} + (\theta - 1)\frac{te^{\frac{1}{t}} - 3t^2 e^{\frac{1}{t}} + 3t^2 + 2t}{(t^3 e^{\frac{1}{t}} - t^3 - t^2)^2}. \tag{10}$$

Since, the above equation is not in explicit form to get the solution, so we use a simulation study and it was seen that for $t_0 = 0.562\theta$ approx, $\eta'(t) > 0$ for all $t \in (0, t_0)$, and $\eta'(t_0) = 0$ and $\eta'(t) < 0$ for all $t > t_0$. Also, from Equation (4) we can verify that $\lim_{x\to 0} f(t) = 0$ (because rate of convergence of exponential function is more than the algebraic function). Hence, the proposed distribution is right skewed distribution having UBT shape of hazard rate and it appears to be more useful for medical and reliability data.

2.3. *Tail area property*

The heavy tail area property of any probability distribution can be detected by the criterion suggested by Klugman *et al.* (2012). It is to be noted that the distributions which do not have all their power moments finite; and some others to those distributions that do not have a finite variance. Consequently, the arithmetic mean of the distribution is obtained by solving the following integration

$$E(x^r) = \int_\theta x^r f(x,\theta)\,dx = \int_0^\infty \frac{\theta}{x^{2-r}} e^{-\frac{1}{x}} \left[1 - e^{-\frac{1}{x}}\left(1 + \frac{1}{x}\right)\right]^{\theta-1} dx.$$

It has been verified that the above integral is the divergent integral, hence the moments of the proposed distribution do not exist consequently proposed distribution is heavy-tailed distribution. Also, following the ideas of Nair *et al.* (2013), Foss *et al.* (2011) and Marshall and Olkin (2007) that the ratio of the hazard function of the proposed distribution to the another heavy tailed distribution (usually exponential distribution taken) tend to infinite when $x \to 0$. Thus, the ratio obtained is

$$\gamma(x) = \frac{1}{x^2\left(xe^{\frac{1}{x}} - x - 1\right)} \tag{11}$$

$$\lim_{x\to 0} \gamma(x) = \lim_{x\to 0} \frac{1}{x^2\left(xe^{\frac{1}{x}} - x - 1\right)} \to \infty.$$

Hence, the tail of the proposed distribution is heavier than the exponential distribution and in this way it possesses heavy tailed property.

2.4. *Quantile function*

Suppose X is a random variable observed from (4). The quantile function $Q(p)$ is defined by; $F[Q(p)] = p$ is the root of equation

$$\left(1 + \frac{1}{Q(p)}\right) e^{-\frac{1}{Q(p)}} = 1 - (1-p)^{\frac{1}{\theta}} \tag{12}$$

after some simplification; the above equation can be written in the following form

$$-\left(1+\frac{1}{Q(p)}\right)e^{-\left(1+\frac{1}{Q(p)}\right)}=\left[(1-p)^{\frac{1}{\theta}}-1\right]e^{-1}. \tag{13}$$

From the above equation, we see that $-\left(1+\frac{1}{Q(p)}\right)e^{-\left(1+\frac{1}{Q(p)}\right)}$ is the Lambert W function with real argument $\left[(1-p)^{\frac{1}{\theta}}-1\right]e^{-1}$. The Lambert W function is the multi-valued complex function defined as the solution of equation $W(z)\exp(W(z))=z$ where z is the complex number. Thus, we have

$$W\left\{\left[(1-p)^{\frac{1}{\theta}}-1\right]e^{-1}\right\}=-\left(1+\frac{1}{Q(p)}\right). \tag{14}$$

After simplification and using the property of negative Lambert W function,

$$W_{-1}\left\{\left[(1-p)^{\frac{1}{\theta}}-1\right]e^{-1}\right\}=-\left(1+\frac{1}{Q(p)}\right) \tag{15}$$

it is computed as

$$Q(p)=\left[-W_{-1}\left\{\left[(1-p)^{\frac{1}{\theta}}-1\right]e^{-1}\right\}-1\right]^{-1}. \tag{16}$$

If $p=0.5$, we get the median of IEGD, given below

$$Md=\left[-W_{-1}\left\{\left[(0.5)^{\frac{1}{\theta}}-1\right]e^{-1}\right\}-1\right]^{-1}. \tag{17}$$

2.5. *Coefficient of skewness and kurtosis*

The different quantile can be computed by Equation (16). Therefore, the quantile based coefficients of skewness and kurtosis can be defined by using Bowley (1920) and Moors (1988) results. Thus, Bowley's coefficient of skewness is defined as

$$B=\frac{Q(3/4)+Q(1/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}$$

and the Moors coefficient of kurtosis is given as

$$M=\frac{Q(3/8)-Q(1/8)+Q(7/8)-Q(5/8)}{Q(6/8)-Q(2/8)}.$$

The graph of the skewness and kurtosis for the variation of the parameter are given in Figure 4.

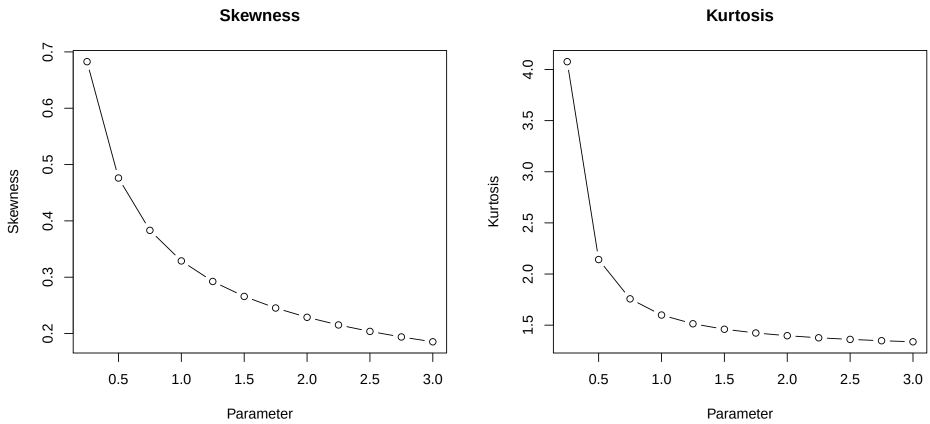

Figure 4 – Skewness and kurtosis for different values of θ.

2.6. *Stochastic ordering*

The comparative behaviour of two random variables are shown by using the theory of stochastic ordering. At first the concept of stochastic ordering was proposed by Shaked and Shanthikumar (1994). Let X and Y are the two random variables with CDFs F_X and F_Y respectively, then then X is said to be smaller than Y in

- stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x;
- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x;
- mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x;
- likelihood ratio order $(X \leq_{lr} Y)$ if $\left(\frac{f_X(x)}{f_Y(x)}\right)$ decreases in x.

From the above relations, we analyzed that

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Downarrow (X \leq_{st} Y) \Rightarrow (X \leq_{mrl} Y).$$

The proposed distribution is also ordered with respect to the strongest likelihood ratio ordering as shown in the following Theorem.

THEOREM 1. *Let $X \sim f(\theta_1)$ and $Y \sim f(\theta_2)$. If $\theta_1 > \theta_2$, then $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y)$, $(X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$.*

PROOF. According to the definition of likelihood ratio order first we obtain the ratio $\left[\frac{f_X(x)}{f_Y(x)}\right]$ i.e.

$$\Psi = \frac{f_X(x)}{f_Y(x)} = \frac{\frac{\theta_1}{x^3}e^{-\frac{1}{x}}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_1-1}}{\frac{\theta_2}{x^3}e^{-\frac{1}{x}}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_2-1}}$$
$$= \frac{\theta_1}{\theta_2}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_1-\theta_2}.$$

Therefore,

$$\Psi' = \frac{\theta_1(\theta_1-\theta_2)}{\theta_2 y^3}e^{-\frac{1}{x}}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_1+\theta_2-1}. \tag{18}$$

If $\theta_1 > \theta_2$, then $\Psi' > 0$, hence $(X \leq_{lr} Y)$ i.e the ratio decreasing in x. The remaining statements can be established by same way. □

2.7. *Stress-strength reliability*

Here, the expression of stress-strength reliability parameter $R = P[Y < X]$ is computed for IEGD. It measure the component reliability when it is subjected to random stress Y and has strength X. In this context, R can be considered as a measure of system performance and naturally arise in electrical and electronic systems. For more details about stress-strength reliability the readers may see, Kotz *et al.* (2003). Let $X \sim IEGD(\theta_1)$ and $Y \sim IEGD(\theta_2)$ denotes the strength-stress random variable with parameter θ_1, θ_2 respectively. The parameter R for IEGD is obtained as

$$R = P[Y < X] = \int_{x=0}^{\infty}\int_{y=0}^{x} f(x,\theta_1)f(x,\theta_2)\,dx\,dy$$
$$= \int_{x=0}^{\infty}\frac{\theta_1}{x^3}e^{-\frac{1}{x}}\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_1-1}\left\{1-\left[1-e^{-\frac{1}{x}}\left(1+\frac{1}{x}\right)\right]^{\theta_2}\right\}dx.$$

After simplification, the expression for R is obtained as

$$R = P[Y < X] = \frac{\theta_2}{\theta_1+\theta_1}. \tag{19}$$

2.8. *Order statistics*

Suppose $x_1, x_2, \ldots, x_n$ be the independent random sample from (4) and $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ denotes the corresponding order statistics. Then the PDF $f_r(x_{r:n})$ and CDF

$F_r(x_{r:n})$ of r^{th} order statistics $x_{r:n}$ is computed by using the following equations

$$f_r(x_{r:n}) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x_{r:n})[1-F(x_{r:n})]^{n-r} f(x_{r:n}). \tag{20}$$

Using Equations (3) and (4) in above equation, we get

$$\begin{aligned} f_r(x_{r:n}) = &\frac{n!}{(r-1)!(n-r)!} \frac{\theta e^{-\frac{1}{x_{r:n}}}}{x_{r:n}^3} \left\{1-\left[1-e^{-\frac{1}{x_{r:n}}}\left(1+\frac{1}{x_{r:n}}\right)\right]^{\theta}\right\}^{r-1} \\ &\times \left[1-e^{-\frac{1}{x_{r:n}}}\left(1+\frac{1}{x_{r:n}}\right)\right]^{\theta(n-r)+\theta-1} \end{aligned} \tag{21}$$

and

$$F_r(x_{r:n}) = \sum_{i=r}^{n} \binom{n}{i} F^i(x_{r:n})[1-F(x_{r:n})]^{n-i} \sum_{i=r}^{n}\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j}(-1)^j F^{i+j}(x_{r:n}). \tag{22}$$

Using Equation (3) in Equation (22) we have

$$F_r(x_{r:n}) = \sum_{i=r}^{n}\sum_{j=0}^{n-i} \binom{n}{i}\binom{n-i}{j}(-1)^j \left\{1-\left[1-e^{-\frac{1}{x_{r:n}}}\left(1+\frac{1}{x_{r:n}}\right),\right]^{\theta}\right\}^{i+j} \tag{23}$$

respectively.

Similarly the distribution of $x_{(1)} = \min(x_1, x_2, \cdots x_n)$ and $x_{(n)} = \max(x_1, x_2, \cdots x_n)$ are obtained by assuming $r = 1$ & $r = n$, respectively.

3. Different classical methods of estimation

In this section, we discuss different estimation procedure for estimating the unknown model parameters of the proposed model. These methods are presented below.

3.1. *Maximum likelihood estimation*

Let $x_1, x_2, \cdots, x_n$ be the random sample of size n from density function (4). The likelihood function for density function (4) is written as

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} f(x_i, \theta) \\ &= \frac{\theta^n}{\prod_{i=1}^{n} x_i^3} e^{-\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n}\left[1-e^{-\frac{1}{x_i}}\left(1+\frac{1}{x_i}\right)\right]^{\theta-1}. \end{aligned} \tag{24}$$

Hence, log-likelihood by ignoring constant is written as

$$\ln L(\theta) = n\ln\theta - 3\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{1}{x_i} + (\theta-1)\sum_{i=1}^{n}\ln\left[1-e^{-\frac{1}{x_i}}\left(1+\frac{1}{x_i}\right)\right]. \tag{25}$$

Thus, for MLE of the parameter $\hat{\theta}_{ml}$ is obtained by solving the equation $L'(\theta)=0$ under the condition $L''(\theta)|_{\hat{\theta}_{ml}} < 0$, where $L'(\theta)$ is the first derivative of $L(\theta)$ w. r. t. θ. After simplification, the MLE of the parameter is obtained as

$$\hat{\theta}_{ml} = -\frac{n}{\sum_{i=1}^{n}\ln\left[1-e^{-\frac{1}{x_i}}\left(1+\frac{1}{x_i}\right)\right]}. \tag{26}$$

Therefore, using above equation, we see that

$$L''(\theta)|_{\hat{\theta}_{ml}} = -\frac{1}{n}\left(\sum_{i=1}^{n}\ln\left[1-e^{-\frac{1}{x_i}}\left(1+\frac{1}{x_i}\right)\right]\right)^2 < 0,$$

which insure the existence of MLE of the parameter for IEGD.

3.2. *Maximum product spacing method of estimation*

An alternative method of estimation of MLE is maximum product spacing estimation (MPSE) method. At first, the MPSE method was used and is discussed by Cheng and Amin (1983). Coolen and Newby (1990) mentioned that MPS method possess similar properties as MLE. Recently, Singh *et al.* (2014) used this method and illustrated its beauty. In this method, the likelihood function is defined as the differences between two consecutive CDFs and is given by

$$L'(\alpha,\theta) = \sqrt[n+1]{\prod_{i=1}^{n+1}\zeta_i} \tag{27}$$

such that $\sum_{i=1}^{n}\zeta_i = 1$ and $\zeta_i = \left[1-e^{-\frac{1}{x_{i-1}}}\left(1+\frac{1}{x_{i-1}}\right)\right]^{\theta} - \left[1-e^{-\frac{1}{x_i}}\left(1+\frac{1}{x_i}\right)\right]^{\theta}$

The MPS estimates of the parameter θ are obtained by maximizing the above equation w. r. t. the parameter.

3.3. *Ordinary and weighted least square estimation*

In the theory of classical estimation method, least square estimation (LSE) and weighted least square estimation (WLSE) method are a conventional method to obtain the estimate of the parameter as is introduced by Swain *et al.* (1988). They used LSE and WLSE method to estimate the parameters of a Beta distribution. The LSEs of the unknown parameter of IEGD has been obtained by minimizing the residual sum of the square, where residual is defined as the differences of theoretical CDF and empirical CDF:

$$LSE = \sum_{i=1}^{n} \left[F(x_i, \theta) - \frac{i}{n+1} \right]^2 . \tag{28}$$

Using Equation (4) in the above equation, we get

$$LSE = \sum_{i=1}^{n} \left[1 - \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i} \right) \right]^{\theta} - \left(\frac{i}{n+1} \right) \right]^2 . \tag{29}$$

The least square estimators of the parameter θ is obtained by minimizing (21) w. r. t θ and the weighted least squares estimates (WLSE) of the unknown parameter can be obtained by minimizing the following function

$$WLSE = \sum_{i=1}^{n} W_i' \left[1 - \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i} \right) \right]^{\theta} - \left(\frac{i}{n+1} \right) \right]^2 , \tag{30}$$

where W_i' is the weight function at the point i and is taken as

$$W_i' = \frac{1}{\mathrm{Var}(F(x_i))} = \frac{(n+1)^2(n+2)}{i(n-i+1)} .$$

3.4. Cramer-von-Mises estimation

Cramer-von-Mises estimation (CVME) method is proposed and used by MacDonald (1971). This method is based on minimum difference between empirical and cumulative distribution functions. See Choi and Bulgren (1968), Boos (1981) for more details about this method. The CVM estimator of the parameter is obtained by minimizing

$$CV = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)}) - \frac{2i-1}{2n} \right]^2 . \tag{31}$$

Hence, from Equation (3) and (31), we get

$$CV = \frac{1}{12n} + \sum_{i=1}^{n} \left[\left(1 - \left\{ 1 - e^{-\frac{1}{x}} \left(1 + \frac{1}{x} \right) \right\}^{\theta} \right) - \frac{2i-1}{2n} \right]^2 . \tag{32}$$

Now, differentiating Equation (32) respect to the parameter and equating to zero we get

$$\sum_{i=1}^{n} \left[W_i(\theta) - \frac{2i-1}{2n} \right] W_i'(\theta) = 0, \tag{33}$$

where $W_i'(\theta)$ is the first derivatives of CDF of IEGD. Now, again, the above equation cannot be solved analytically, hence the some numerical technique may be applied.

4. Bayesian estimation

Here, the Bayes procedure has been discussed to estimate the unknown parameter of the proposed distribution. It is to be noted that the Bayes procedure is posterior based inference and hence the posterior distribution of the parameter θ is derived under the assumption of gamma prior. Since, no conjugate prior exist for IEGD thus, gamma prior is taken under consideration. The considered prior is very flexible and can also be converted to the non-informative prior. The prior density for θ is given by

$$g(\theta) \propto \theta^{p-1} e^{-q\theta} \quad ;\theta > 0, \tag{34}$$

where p, q are the hyper-parameters of the considered prior and are assumed to be known. Now, using likelihood Equation (24) and prior (34), the posterior distribution is obtained by

$$\begin{aligned} p(\theta|\underline{x}) &= \frac{L(x|\theta)\, g(\theta)}{\int_{\theta=0}^{\infty} L(x|\theta)\, g(\theta)\, d\theta} \\ &= \frac{\theta^{n+p-1} e^{-q\theta} \prod_{i=1}^{n} \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i}\right)\right]^{\theta}}{\int_{\theta=0}^{\infty} \theta^{n+p-1} e^{-q\theta} \prod_{i=1}^{n} \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i}\right)\right]^{\theta} d\theta}. \end{aligned} \tag{35}$$

Here, we assume squared error loss function (SELF) to obtain the Bayes estimate of the parameter θ. Let $\hat{\alpha}$ is the estimate of α, then SELF is defined as

$$L_f(\alpha, \hat{\alpha}) = (\hat{\alpha} - \alpha)^2.$$

It is mentioned that under SELF, posterior mean is the Bayes estimate of the parameter

$$\hat{\theta}_b = E_\theta(\theta|\underline{x}) = \frac{\int_{\theta=0}^{\infty} \theta^{n+p}\, e^{-q\theta} \prod_{i=1}^{n} \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i}\right)\right]^{\theta} d\theta}{\int_{\theta=0}^{\infty} \theta^{n+p-1} e^{-q\theta} \prod_{i=1}^{n} \left[1 - e^{-\frac{1}{x_i}} \left(1 + \frac{1}{x_i}\right)\right]^{\theta} d\theta}, \tag{36}$$

provided the above expectation exist. The above expectation involve the ratio of two integrals; thus the explicit solution of the above equation is not possible. Thus, we used Lindley's approximation method to obtain the Bayes estimates of the parameter θ.

4.1. *Lindley's approximation method*

One of the most efficient technique to extract the Bayes estimate from the ratio of two integral was suggested by Lindley in year of 1980, see Lindley (1980). Thus, by applying this approximation technique, the Bayes estimator of θ is given by

$$\hat{\theta}_{bl} = \hat{\theta}_{ml} + \hat{\tau}_\theta \hat{\sigma}_{\theta\theta} + \frac{1}{2} \hat{\sigma}^2_{\theta\theta} \hat{L}_{\theta\theta\theta}, \tag{37}$$

where

$$\frac{\partial^2 L}{\partial \theta^2} = L_{\theta\theta} = -\frac{n}{\theta^2}, \quad \frac{\partial^3 L}{\partial \theta^3} = L_{\theta\theta\theta} = \frac{2n}{\theta^3}$$

$$\sigma_{\theta\theta} = -\left(\frac{1}{L_{\theta\theta}}\right), \quad \tau_\theta = \frac{a-1}{\theta} - b.$$

All the above derivatives are evaluated at the point $\hat{\theta}_{ml}$. If n is sufficiently large then $\hat{\alpha}_{bl} \to \hat{\alpha}_{ml}$.

5. SIMULATION STUDY

In this section, the performances of the estimators obtained by the different methods of estimation are investigated using Monte Carlo simulation study. The simulation is performed for different values of sample size, in particular it is taken as $n = 15, 30, 45, 60, 90, 120, \& 150$ when $\theta = 0.75, 2$. In simulation-based result $\hat{\theta}_{ml}, \hat{\theta}_{mp}, \hat{\theta}_{ls}, \hat{\theta}_{wls}, \hat{\theta}_{cv}$ represents the estimates obtained by method of MLE, method of MPSE, method of LSE, method of WLSE, method of CVME. $\hat{\theta}_{bl1}, \hat{\theta}_{bl2}$ denote the Bayes estimate under informative prior and non- informative prior respectively. Since, the classical estimators are not explicitly obtained, thus `optim()` function has been used solve them. The Bayes estimate is computed by using Lindley's approximation method. The values of hyper parameters are chosen such that the prior mean accurately matches with true beliefs and variance is very less. The average estimates of the parameter and corresponding mean square errors (MSEs) are reported based on 3000 replications in Table 1. From this extensive simulation study, it has been remarked that the Bayes estimator under informative prior is more precise as compared to all estimators however under non-informative priors the Bayes estimator is almost same as the MLE and also ensure the consistency property of the estimators, i.e., the MSEs of each estimator are decreasing when n increases. In case of large sample MLE, MPS and Bayes estimators are more or less the same. Further, among the classical estimation methods (MLE, MPSE, LSE, WLSE & CVME), the maximum product spacing estimation method provides the more efficient result as compared to the rest. The graph of the MSEs of each estimators under $\theta = 0.75$, $\theta = 2$ are displayed in Figure 5.

TABLE 1
Average estimate and corresponding mean square error (second row).

θ	n	$\hat{\theta}_{ml}$	$\hat{\theta}_{mp}$	$\hat{\theta}_{ls}$	$\hat{\theta}_{wls}$	$\hat{\theta}_{cv}$	$\hat{\theta}_{bl1}$	$\hat{\theta}_{bl2}$
0.75	15	0.80370	0.79911	0.79269	0.78819	0.80147	0.73286	0.80370
		0.05150	0.04579	0.06756	0.06283	0.06906	0.04103	0.05150
	30	0.77315	0.77217	0.76480	0.76386	0.76937	0.73018	0.77315
		0.02210	0.02111	0.02802	0.02572	0.02842	0.01968	0.02210
	45	0.76724	0.76680	0.76228	0.76134	0.76536	0.73525	0.76724
		0.01392	0.01351	0.01807	0.01631	0.01826	0.01278	0.01392
	60	0.76120	0.76097	0.75590	0.75670	0.75820	0.73547	0.76120
		0.01029	0.01007	0.01316	0.01070	0.01325	0.00970	0.01029
	90	0.75790	0.75779	0.75538	0.75169	0.75691	0.73904	0.75790
		0.00645	0.00636	0.00832	0.00783	0.00836	0.00621	0.00645
	120	0.75535	0.75530	0.75303	0.75466	0.75418	0.74026	0.75535
		0.00496	0.00491	0.00650	0.00569	0.00653	0.00482	0.00496
	150	0.75394	0.75390	0.75307	0.75495	0.75400	0.74134	0.75393
		0.00380	0.00377	0.00504	0.00440	0.00506	0.00373	0.00380
2	15	2.14454	1.95683	2.09878	2.08512	2.12157	2.05925	2.14453
		0.35057	0.27753	0.46629	0.42728	0.47600	0.13167	0.35057
	30	2.08556	1.97094	2.05404	2.04752	2.06601	2.06359	2.08556
		0.15851	0.13505	0.21161	0.18967	0.21447	0.10993	0.15851
	45	2.05516	1.97043	2.03625	2.02928	2.04431	2.04606	2.05516
		0.09431	0.08417	0.12482	0.11230	0.12618	0.07576	0.09431
	60	2.03848	1.97103	2.01560	2.01083	2.02166	2.03359	2.03848
		0.06994	0.06464	0.09131	0.08163	0.09192	0.05984	0.06994
	90	2.03243	1.98354	2.01777	2.01207	2.02182	2.03000	2.03242
		0.04433	0.04132	0.05998	0.05279	0.06030	0.04007	0.04433
	120	2.03322	1.99442	2.01468	2.01113	2.01773	2.03149	2.03321
		0.03687	0.03423	0.04927	0.04334	0.04947	0.03424	0.03687
	150	2.02109	1.98905	2.00732	2.00139	2.00974	2.02018	2.02109
		0.02625	0.02484	0.03735	0.03245	0.03745	0.02480	0.02625

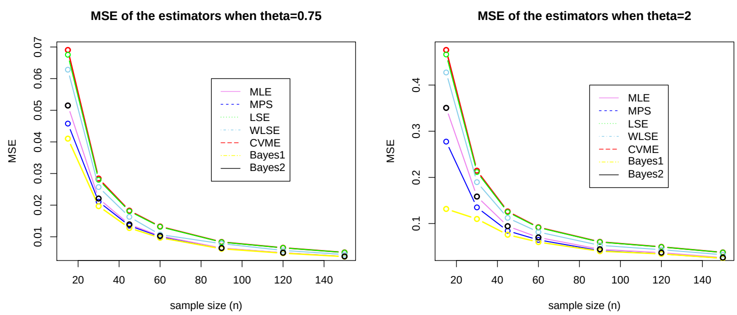

Figure 5 – Mean square error of the estimators obtained by simulation.

6. REAL DATA ILLUSTRATION

In this section, we consider a real data to show the suitability of IEGD. The considered real data set represents the 46 repair times (in hours) for an airborne communication transceiver, taken from Chhikara and Folks (1977). The data set is given below:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1, 1, 1, 1, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2, 2, 2.2, 2.5, 2.7, 3, 3, 3.3, 3.3, 4, 4, 4.5, 4.7, 5, 5.4, 5.4, 7, 7.5, 8.8, 9, 10.3, 22, 24.5.

The average repair times for an airborne communication transceiver is 3.607 hours with standard deviation 4.9444. Also, the coefficient of skewness and kurtosis are 2.8883 and 11.8025 respectively. The values of the skewness is positive with kurtosis greater than 3, which indicates that the considered data set is positively skewed with leptokurtic shape.

The fitting of the data to the proposed model has been checked by using one sample Kolmogorov-Smirnov (KS) goodness of fit test based on the following null hypothesis

$$H_0 : \text{Data comes from IEGD}$$

The value of the KS statistic and corresponding p-value based on MLE of unknown parameter are obtained. Following the idea of Ristić *et al.* (2018), we also applied the parametric bootstrap technique and simulated $B = 1000$ samples of size $n = 46$ from the IEGD and then again computed KS value and p-value using the parametric bootstrap MLE. The same is reported in Table 2, and it is observed that p-value for usual MLE and bootstrap MLE is greater the $\alpha = 0.05$ (level of significance), which indicates that the strong evidence to accept the null hypothesis. Further, the empirical cumulative distribution function (ECDF) plot is also plotted for usual MLE and bootstrap MLE, in which the empirical and theoretical CDFs are very close. Hence, we may conclude that the proposed model is well fitted to the considered data set, see Figure 9.

Further, the summary of the data set has been presented through box plot (see Figure 6) and it is observed that the some outlier values are presents in the considered data set, which may indicate the value due to heavy tailed property of the phenomenon from which the data came. The nature of failure rate for the considered real data sets has been examined by considering total time on test (TTT) plot and it is noticed that data set possess UBT type hazard rate (see Figure 7). The estimated density and hazard plots are also plotted for the MLE obtained for the considered data and MLE for the parametric bootstrap samples, see Figure 8. The classical and Bayes estimates of the unknown parameter are also computed for the considered real data set, see Table 3.

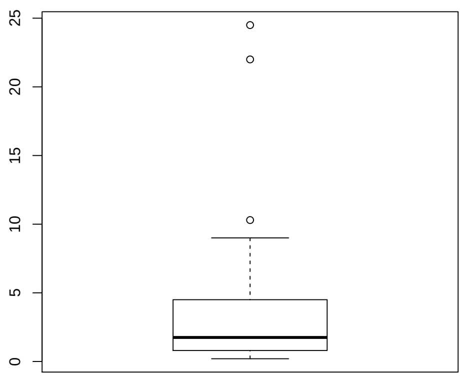

Figure 6 – Box plot of the considered data set.

TABLE 2
The value of the KS test and p-value.

Model	Estimate	KS value	p-value
$IEGD$	0.3925	0.1415	0.3160
$IEGD_B$	0.3725	0.1328	0.3910

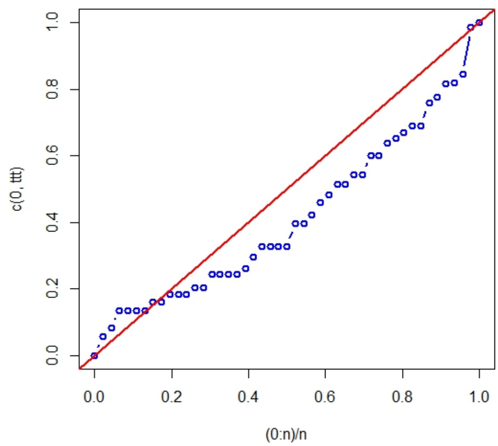

Figure 7 – TTT plot for data set.

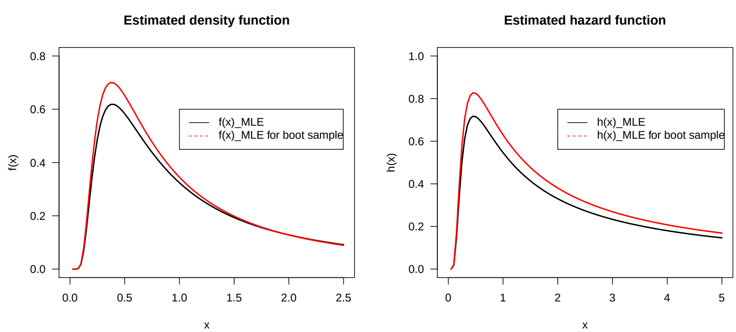

Figure 8 – Estimated plots for the considered data set.

TABLE 3
Estimate for the real dataset obtained by different methods of estimation.

θ	n	$\hat{\theta}_{ml}$	$\hat{\theta}_{mp}$	$\hat{\theta}_{ls}$	$\hat{\theta}_{wls}$	$\hat{\theta}_{cv}$	$\hat{\theta}_{bl2}$
	46	0.39259	0.38919	0.33517	0.35346	0.33647	0.39259

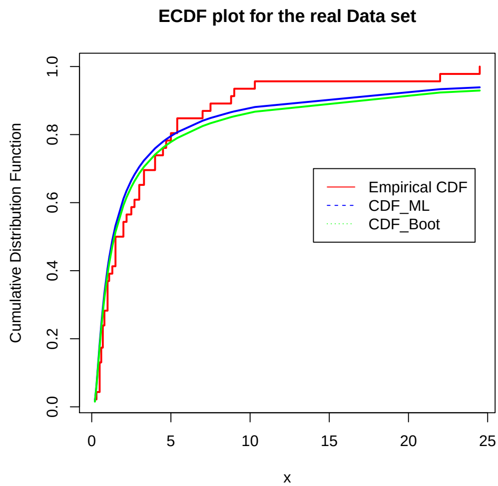

Figure 9 – Empirical CDF plot for the considered data set.

7. Conclusions

In this article, the inverted version of the exponentiated gamma distribution, named as IEGD has been proposed as a heavy-tailed UBT shaped probability model. The related distributional properties are derived and presented in respective sections. The shape of the distribution has been traced graphically as well as mathematically and it is observed that the proposed model accommodate the nature of the upside down bathtub hazard rate which is very useful to analyze the medical and engineering data. Secondly, the estimation of the unknown parameter of the IEGD model has been addressed by different classical (MLE, MPSE, LSE, WLSE & CVME) methods of estimation and Bayes estimation method. The Bayes estimates of the parameter are obtained with the gamma prior under SELF by applying Lindley's approximation method. The Monte Carlo simulation study has been performed to investigate the performances of the obtained estimators under both setups, and it is remarked that the Bayes estimator under informative prior is superior to that of other classical estimators while under non-informative prior the Bayes, MLE, and MPS estimators are same. Among all traditional estimation methods MPSE method outperforms than others. Lastly, a real data set has been considered to demonstrate the applicability of the proposed model and estimation procedures.

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SUMMARY

In this article, we proposed and studied the inverted exponentiated gamma distribution (IEGD). IEGD is obtained by considering the inverse transformation of exponentiated gamma variate. This distribution has been motivated by the extensive use of the exponentiated gamma model in many applied areas and also due to the fact that this new generalization provides more flexibility to analyze real data with upside down bathtub (UBT) hazard rate. The shape of the distribution has been traced mathematically and found that the proposed model is compatible with UBT hazard rate models. The tail area property is also presented based on the idea of Marshall and Olkin (2007) and it is concluded that the new model belongs to the family of heavy-tailed distributions. Some other characteristics such as reliability, hazard, the quantile function, skewness and kurtosis, stochastic ordering, stress-strength reliability and order statistics have been explicitly derived. The classical and Bayesian estimation procedures have been discussed to estimate the unknown parameter of IEGD. The performances of classical and Bayes estimators are studied in terms of average mean square error (MSE) by conducting Monte Carlo simulations. Finally, a real data set with UBT type hazard rate is analyzed for the illustrative purpose of the study.

Keywords: Inverted exponentiated gamma distribution; Stochastic ordering; Order statistics; Different method of classical estimation; Bayes estimation.