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SOME PROPERTIES OF THE POSITIVE HYPER-POISSON DISTRIBUTION AND ITS APPLICATIONS

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1. INTRODUCTION

Bardwell and Crow (1964) developed a class of discrete distributions, namely the hyper-Poisson distribution (HPD), which has probability mass function (p.m.f.)

$$f(x) = \frac{\Gamma(\lambda)}{\Gamma(\lambda+x)} \frac{\theta^x}{\phi(1;\lambda;\theta)}, \tag{1}$$

for x = 0, 1, ... with $\lambda > 0$ and $\theta > 0$ in which

$$\phi(a;b;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \cdot \frac{z^k}{k!} \tag{2}$$

is the confluent hypergeometric series (also called the Kummer's series) and $(a)_k$ is the ascending factorial:

$$(a)_k = a(a+1)...(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{3}$$

for $k = 1, 2, ...$ and $(a)_0 = 1$. For further details on confluent hypergeometric series, refer to Mathai and Haubold (2008) or Abramowitz and Stegun (1965). A distribution with p.m.f. (1) hereafter is denoted as HPD(λ, θ). Clearly, when $\lambda = 1$, the HPD reduces to the Poisson distribution and when λ is a positive integer, the distribution reduces to the displaced Poisson distribution of Staff (1964). Moreover, when $\lambda < 1$, Bardwell and Crow (1964) called the distribution as sub-Poisson and when $\lambda > 1$ as super-Poisson. Note that the HPD belongs to the Kemp family of distributions studied by Kumar

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(2009). Several methods of estimation of the parameters for these distributions can be found in Bardwell and Crow (1964) and Crow and Bardwell (1965). Nisida (1962) discussed some concepts of queuing theory associated with hyper-Poisson arrivals. In addition to this, the estimation of the parameters of the hyper-Poisson distribution using the negative moments were suggested by Roohi and Ahmad (2003). Roohi and Ahmad (2003) also obtained expressions for ascending factorial moments of the hyper-Poisson distribution and formulated certain recurrence relations of its negative moments and ascending factorial moments. Kemp (2002) studied a q-analogue of the distribution and Ahmad (2007) proposed the Conway-Maxwell hyper-Poisson distribution. Further, extended versions of the hyper-Poisson distribution were studied by Kumar and Nair (2011), who discussed some of their applications.

In this paper, we develop a zero-truncated form of the hyper-Poisson distribution which we call as "the positive hyper-Poisson (PHP) distribution" and investigate some of its important properties. In Section 2 we give the definition of the PHP distribution and derive its probability generating function (p.g.f.), cumulative distribution function, expressions for factorial moments, raw moments, mean, variance, and recurrence relations for its probabilities, factorial moments, and raw moments. Also, the estimation of the parameters of the PHP distribution by the method of mixed moments and the method of maximum likelihood are discussed in Section 3 and illustrated with the help of real life data sets.

We need the following simplifying notations for $i = 0, 1, \ldots$

$$H_i = \phi(1+i;\lambda+i;\theta) - 1 \tag{4}$$

and for $k < a$

$$a^{[k]} = a \cdot (a-1) \cdot \ldots (a-k+1) = \frac{\Gamma(a+1)}{\Gamma(a-k+1)}, \tag{5}$$

where $a^{[k]}$ is the descending factorial. In addition to this, we need the following series representation in the sequel

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} B(s,r) = \sum_{r=0}^{\infty}\sum_{s=0}^{r} B(s,r-s), \tag{6}$$

as well as

$$n^r = \sum_{m=0}^{r} S(r,m)\, n^{[m]},$$

where $S(r,m)$ are the Stirling numbers of the second kind and $n^{[m]}$ is the descending factorial.

2. THE PHP DISTRIBUTION

In this section we present the definition of the PHP distribution and derive some of its main statistical properties.

DEFINITION 1. *Let X follows a HPD(λ,θ) with p.m.f $f(r)$ as given in (1). Then a discrete random variable Y is said to follow the positive hyper Poisson distribution (or in short the PHP distribution) if its p.m.f $P_r = P(Y=r)$, for $r = 1,2,...$ is given by*

$$P_r = \frac{P[X=r]}{P[X>0]}$$

$$= H_0^{-1}\frac{\Gamma(\lambda)\theta^{r}}{\Gamma(\lambda+r)}. \tag{7}$$

Clearly, when $\lambda = 1$, the p.m.f (7) reduces to the p.m.f of a zero truncated Poisson distribution (also known in the literature as the positive Poisson distribution).
Now we have the following results.

RESULT 2. *The p.g.f $G(t)$ of the PHP distribution is the following*

$$G(t) = H_0^{-1}\,[\phi(1;\lambda;\theta t) - 1]. \tag{8}$$

PROOF. By definition, the p.g.f of the PHP distribution with p.m.f (7) is given by

$$\begin{aligned}
G(t) &= \sum_{r=1}^{\infty} P_r\, t^r &(9)\\
&= \frac{(1+H_0)}{H_0}\sum_{r=1}^{\infty} t^r \frac{\Gamma(\lambda)}{\Gamma(\lambda+r)}\,\frac{\theta^r}{(1+H_0)}\\
&= \frac{(1+H_0)}{H_0}\left[\sum_{r=0}^{\infty} t^r \frac{\Gamma(\lambda)}{\Gamma(\lambda+r)}\,\frac{\theta^r}{(1+H_0)}\right] - \frac{1}{H_0},
\end{aligned}$$

which on simplification gives (8). □

RESULT 3. *The cumulative distribution function (c.d.f) of the PHP distribution is the following, for any $r \in \Re = (-\infty,\infty)$*

$$P(X \le r) = 1 - H_0^{-1}\,\theta^{r+1}\,\Gamma(\lambda)\sum_{k=0}^{\infty}\frac{\theta^k}{\Gamma(k+\lambda+r+1)}. \tag{10}$$

PROOF. By definition, the c.d.f of the PHP distribution with p.m.f (7) is

$$\begin{aligned}
P(X \leq r) &= \sum_{k=1}^{r} \frac{\Gamma(\lambda)}{\Gamma(\lambda+k)} \frac{\theta^k}{H_0} \\
&= \frac{1}{H_0}\left[\sum_{k=0}^{r} \frac{\Gamma(\lambda)\,\theta^k}{\Gamma(\lambda+k)} - 1\right] \\
&= \frac{(1+H_0)}{H_0}\left[\sum_{k=0}^{r} \frac{\Gamma(\lambda)}{\Gamma(\lambda+k)} \frac{\theta^k}{(1+H_0)} - \frac{1}{(1+H_0)}\right] \\
&= \frac{(1+H_0)}{H_0}\left[1 - \sum_{k=r+1}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\lambda+k)} \frac{\theta^k}{(1+H_0)} - \frac{1}{(1+H_0)}\right] \\
&= \frac{(1+H_0)}{H_0}\left[1 - \sum_{k=r+1}^{\infty} \frac{\Gamma(\lambda)}{\Gamma(\lambda+k)} \frac{\theta^k}{(1+H_0)} - \frac{1}{(1+H_0)}\right] \\
&= \frac{(1+H_0)}{H_0}\left[1 - \frac{\Gamma(\lambda)\,\theta^{r+1}}{(1+H_0)} \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(\lambda+k+r+1)} - \frac{1}{(1+H_0)}\right],
\end{aligned}$$

which on simplification yields to (10). □

RESULT 4. *The survival function $S(r)$ and the hazard function $h(r)$ of the PHP distribution are the following, for any $r \in \Re$*

$$S(r) = H_0^{-1}\,\theta^{r+1}\,\Gamma(\lambda) \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(k+\lambda+r+1)}$$

and

$$h(r) = \frac{1}{\theta\,\Gamma(\lambda+r)}\left[\sum_{k=0}^{\infty} \frac{\theta^k}{(k+\lambda+r+1)}\right]^{-1}.$$

PROOF. The proof is straightforward from (7) and (10), since

$$S(r) = P(X > r)$$

and

$$h(r) = \frac{P_r}{S(r)}.$$ □

RESULT 5. *Let $\mu_{[r]}(1;\lambda)$ denote the r-th factorial moment of the PHP distribution with p.g.f (8). Then, an expression for $\mu_{[r]}(1;\lambda)$ of the PHP distribution is the following, for $r \geq 1$*

$$\mu_{[r]}(1;\lambda) = \frac{r!\,\theta^r}{(\lambda)_r}\,\phi(1+r;\lambda+r;\theta). \tag{11}$$

PROOF. The factorial moment generating function $F(t)$ of the PHP distribution with p.g.f (8) is

$$\begin{aligned} F(t) &= G(1+t) \\ &= H_0^{-1}[\phi(1;\lambda;\theta(1+t))-1]. \end{aligned} \tag{12}$$

On differentiating (12) r times with respect to t and putting $t=1$, we get (11). □

RESULT 6. *Mean and variance of the PHP distribution are*

$$Mean = \frac{\theta}{\lambda}\ H_0^{-1}\,(1+H_1)$$

and

$$Variance = \frac{\theta}{\lambda}\,H_0^{-1}\,[\frac{2(1+H_2)}{(1+\lambda)} - H_1(1+H_1)].$$

RESULT 7. *Let $\mu_r(1;\lambda)$ denote the r-th raw moment of the PHP distribution with p.g.f (8). Then an expression for $\mu_r(1;\lambda)$ of the PHP distribution is the following, for $r \geq 0$*

$$\mu_r(1;\lambda) = H_0 \sum_{m=0}^{r} S(r,m)\theta^m \frac{(1)_m}{(\lambda)_m}(1+H_m), \tag{13}$$

where $S(r,m)$ are the Stirling numbers of the second kind and $n^{[m]}$ is the descending factorial. For details see (Riordan, 1968).

PROOF. The characteristic function $\Psi(t)$ of the PHP distribution with p.g.f (8) is the following, for any $t \in \Re$ and $i=\sqrt{-1}$

$$\Psi(t) = G\left(e^{it}\right) = H_0^{-1}\left[\phi\left(1;\lambda;\theta e^{it}\right)-1\right] \tag{14}$$

$$= \sum_{r=1}^{\infty} \mu_r(1;\lambda)\frac{(it)^r}{r!}. \tag{15}$$

On expanding the confluent hypergeometric series $\phi(.)$ and the exponential function e^{it} in (14) we obtain

$$\Psi(t) = H_0^{-1}\left[\sum_{n=1}^{\infty}\frac{(1)_n}{(\lambda)_n}\,\frac{\theta^n}{n!}\sum_{r=0}^{\infty}\frac{n^r\,(it)^r}{r!}\right]. \tag{16}$$

Equating the coefficients of $(r!)^{-1}(it)^r$ on right hand side expressions of (15) and (16) we get the following

$$\mu_r(1;\lambda) = H_0^{-1}\sum_{n=1}^{\infty}\frac{(1)_n\,\theta^n}{(\lambda)_n\,n!}n^r$$

$$= H_0^{-1} \sum_{n=1}^{\infty} \frac{(1)_n \theta^n}{(\lambda)_n n!} \sum_{m=0}^{r} S(r,m) n^{[m]},$$

where $S(r,m)$ are the Stirling numbers of the second kind and $n^{[m]}$ is the descending factorial. Writing $n! = n^{[m]}(n-m)!$ and rearranging the terms we get

$$\mu_r(1;\lambda) = H_0^{-1} \sum_{m=0}^{r} S(r,m)\theta^m \sum_{n=1}^{\infty} \frac{(1)_n}{(\lambda)_n} \frac{\theta^{n-m}}{(n-m)!}$$

$$= H_0^{-1} \sum_{m=0}^{r} S(r,m)\theta^m \sum_{n=1}^{\infty} \frac{(1)_{n+m}}{(\lambda)_{n+m}} \frac{\theta^n}{n!}.$$

Using $(a)_{n+m} = (a)_m (a+m)_n$ in the above expression, we obtain (13). □

RESULT 8. *For $n \geq 0$, let $P_n(1;\lambda) = P_n$. Now, a simple recurrence relation for probabilities of the PHP distribution with p.g.f. (8)*

$$P_{n+1}(1;\lambda) = \frac{\theta H_1}{\lambda(n+1)H_0} P_n(2;\lambda+1). \tag{17}$$

PROOF. On differentiating (8) and (9) with respect to t, we have

$$\frac{\partial G(t)}{\partial t} = \sum_{r=0}^{\infty} (r+1) P_{r+1}(1;\lambda) t^r$$

$$= \frac{H_0^{-1}\theta}{\lambda} \phi(2;\lambda+1;\theta t). \tag{18}$$

On replacing 1, λ by 2, λ+1 in (8) and (9) we get

$$H_1^{-1}[\phi(2;\lambda+1;\theta t) - 1] = \sum_{r=1}^{\infty} P_r(2;\lambda+1)\ t^r. \tag{19}$$

By using (19) in (18) we get

$$\sum_{r=0}^{\infty} (r+1) P_{r+1}(1;\lambda) t^r = \frac{\theta}{\lambda H_0} \left[1 + H_1 \sum_{r=1}^{\infty} P_r(2;\lambda+1) t^r \right].$$

Equating the coefficients of t^n on both sides we get (17). □

RESULT 9. *The following is a simple recurrence relation for raw moments of the PHP distribution, for $n \geq 0$ in which $\mu_0(1;\lambda) = 1$*

$$\mu_{n+1}(1;\lambda) = \frac{\theta}{\lambda H_0} \left[1 + H_1 \left\{ \sum_{s=0}^{n} \frac{n!}{s!(n-s)!} \mu_{n-s}(2;\lambda+1) - 1 \right\} \right]. \tag{20}$$

PROOF. Consider the following identity obtainable from (14) and (15) on differentiation with respect to t

$$\frac{\partial \Psi(t)}{\partial t} = \frac{ie^{it}\theta}{\lambda H_0}\phi\left(2;\lambda+1;\theta e^{it}\right)$$

$$= \sum_{r=1}^{\infty} i\mu_r(1;\lambda)\frac{(it)^{r-1}}{(r-1)!}.$$

Using (14) and (15) with 1, λ replaced by 2, λ+1, we have

$$\begin{aligned}
\sum_{r=0}^{\infty}\mu_{r+1}(1;\lambda)\tfrac{(it)^r}{r!} &= \tfrac{\theta e^{it}}{\lambda H_0}\left[1+H_1\sum_{r=1}^{\infty}\mu_r(2;\lambda+1)\tfrac{(it)^r}{r!}\right]\\
&= \tfrac{\theta}{\lambda H_0}\left[e^{it}+H_1e^{it}\left\{\sum_{r=0}^{\infty}\mu_r(2;\lambda+1)\tfrac{(it)^r}{r!}-1\right\}\right]\\
&= \tfrac{\theta}{\lambda H_0}\left[\sum_{s=0}^{\infty}\tfrac{(it)^s}{s!}+H_1\left\{\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\mu_r(2;\lambda+1)\tfrac{(it)^{r+s}}{r!s!}-\sum_{s=0}^{\infty}\tfrac{(it)^s}{s!}\right\}\right]\\
&= \tfrac{\theta}{\lambda H_0}\left[\sum_{s=0}^{\infty}\tfrac{(it)^s}{s!}+H_1\left\{\sum_{r=0}^{\infty}\sum_{s=0}^{r}\tfrac{r!}{(r-s)!s!}\mu_{r-s}(2;\lambda+1)\tfrac{(it)^r}{r!}-\sum_{s=0}^{\infty}\tfrac{(it)^s}{s!}\right\}\right],
\end{aligned}$$

in the light of (6). On equating the coefficients of $\frac{(it)^n}{n!}$, we get (20). □

RESULT 10. *The following is a simple recurrence relation for factorial moments of the PHP distribution, for* $r \geq 1$, *in which* $\mu_{[0]}(1,\lambda) = 1$

$$\mu_{[n+1]}(1;\lambda) = \frac{\theta H_1}{\lambda H_0}\mu_{[n]}(2;\lambda+1). \tag{21}$$

PROOF. The factorial moment generating function $F(t)$ of the PHP distribution with p.g.f given in (8) has the following series representation

$$\begin{aligned}
F(t) &= G(1+t) = H_0^{-1}[\phi(1;\lambda;\theta(1+t))-1]\\
&= \sum_{r=1}^{\infty}\mu_{[r]}(1;\lambda)\frac{t^r}{r!}. \qquad (22)
\end{aligned}$$

The relation (21) follows on differentiating the above equation with respect to t and equating coefficients of $(n!)^{-1}t^n$ on both sides, in the light of arguments similar to those in the proof of Result 7. □

3. MAXIMUM LIKELIHOOD ESTIMATION

Here we consider the estimation of the parameters λ and θ of the PHP distribution by the method of maximum likelihood.

Let $a(y)$ be the observed frequency of y events for any $y = 1, 2, ...$ and let z be the highest value of y observed. Then the likelihood function of the sample is

$$L(\Theta; y) = \prod_{y=1}^{z} [f(y)]^{a(y)},$$

in which $f(y)$ is the p.m.f of the PHP distribution as given in (1).

Then the log-likelihood function can be written as

$$l = \ln L(\Theta, y) = \sum_{y=1}^{z} a(y)[\ln\Gamma(\lambda) + y\ \ln\theta - \ \ln\Gamma(\lambda+y) - \ln H_0]. \tag{23}$$

Suppose that $\hat{\lambda}$ and $\hat{\theta}$ are the maximum likelihood estimators of the parameters λ and θ of the PHP distribution. On differentiating the log-likelihood function (23) with respect to the parameters λ and θ and equating to zero, we obtain the following likelihood equations.

$$\frac{\partial l}{\partial \lambda} = 0$$

implies

$$\sum_{y=1}^{z} a(y) \left[\frac{1}{H_0} \left[\sum_{k=0}^{\infty} \frac{k!\,(\lambda-1)!}{(\lambda+k-1)!}\, \psi(\lambda+k)\, \frac{\theta^k}{k!} - \psi(\lambda) \right] - \psi(\lambda+y) \right] = 0 \tag{24}$$

and

$$\frac{\partial l}{\partial \theta} = 0$$

implies

$$\sum_{y=1}^{z} a(y) \left[\frac{y}{\theta} - \frac{\Phi(2; \lambda+1; \theta)}{\lambda H_0} \right] = 0, \tag{25}$$

in which $\psi(\lambda) = \frac{\partial}{\partial\lambda} \ln\Gamma(\lambda)$.

On solving the likelihood Equations (24) and (25) with the help of some mathematical softwares such as `MATHEMATICA` or `MATHCAD` one can obtain the maximum likelihood estimators of the parameters of the distribution.

4. Testing

In order to test the significance of the parameter λ of the PHP distribution, we adopt the following generalized likelihood ratio test (GLRT) procedure. Here the null hypothesis is $H_0 : \lambda = 0$ vs. the alternative $H_1 : \lambda \neq 0$.

In case of the generalized likelihood ratio test, the test statistic is

$$-2\ln\lambda^* = 2(l_1 - l_2), \tag{26}$$

where $l_1 = \ln L(\hat{\Theta}; y)$, in which $\hat{\Theta}$ is the maximum likelihood estimator of $\Theta = (\lambda, \theta)$ with no restrictions, and $\ln L(\hat{\Theta}^*; x)$, in which $\hat{\Theta}^*$ is the maximum likelihood estimator of Θ under H_0. The test statistic given in (26) is asymptotically distributed as a chi-square with one degree of freedom.

5. Data illustration

In this section we illustrate all the procedures discussed in Sections 3 and 4 with the help of some real life data sets. The first data set is on the distribution of family of epidemics of common cold taken from Heasman and Reid (1961). The second data set is on the distribution of number of households having at least one migrant according to the number of migrants, reported by Singh and Yadava (1981) from Neyman (1939). The third data set is on the distribution of number of European red mites on apple leaves, reported by Garman (1923).

TABLE 1

Distribution of family of epidemics of common cold obtained by Heasman and Reid (1961) and the expected frequencies computed using the positive Poisson and the PHP distribution.

X	Observed frequency	PP distribution	PHP distribution
1	156	145.99	154.13
2	55	68.47	58.10
3	19	21.41	20.32
4	10	5.02	6.63
5	12	0.94	2.03
Total	242	242	242
df		2	1
Estimates		λ=1.23	λ=1.45
		θ=1.02	θ=0.58
χ^2-value		9.77	1.57
p-value		0.01	0.21
AIC		490.42	474.00
BIC		493.91	480.97

We have fitted the PHP distribution to all these data sets and considered the fitting of the positive Poisson distribution for comparison. For comparing the models we computed the values of χ^2, AIC and BIC. All these numerical results are presented in Tables 1, 2 and 3. Based on the computed values of χ^2, AIC and BIC given Tables 1, 2 and 3 it can be seen that the PHP distribution shows a better fit to all these models discussed in the paper.

TABLE 2

Distribution of number of households having at least one migrant according to the number of migrants, reported by Singh and Yadava (1981) and the expected frequencies computed using the positive Poisson and the PHP distribution.

X	Observed frequency	PP distribution	PHP distribution
1	375	354.08	377.84
2	143	167.66	138.25
3	49	52.92	48.95
4	17	12.53	6.78
5	2	2.37	5.58
6	2	0.38	1.80
7	1	0.05	0.56
8	1	0.06	0.17
Total	590	590	590
df		2	2
Estimates		λ=1.25	λ=0.63
		θ=0.4	θ=0.27
χ^2-value		8.98	0.31
p-value		0.01	0.86
AIC		1205	1184
BIC		1210	1193

TABLE 3

Distribution of number of European red mites on apple leaves, used by Jani and Shah (1979) and the expected frequencies computed using the positive Poisson and the PHP distribution.

X	Observed frequency	PP distribution	PHP distribution
1	38	28.66	34.74
2	17	25.68	21.10
3	10	15.34	11.96
4	9	6.87	6.36
5	3	2.46	3.18
6	2	0.74	1.50
7	1	0.19	0.67
Total	80	80	80
df		2	2
Estimates		λ=1.40	λ=0.30
		θ=0.40	θ=0.26
χ^2-value		10.03	2.60
p-value		0.01	0.27
AIC		247.59	236.00
BIC		249.97	240.75

We have computed the values of the test statistic given in (26) and included in Table 4. Since the critical value for the test at 5% level of significance and degree of freedom one is 3.84, the null hypothesis is rejected in all the cases. Hence we conclude that the parameter λ in the model is significant.

TABLE 4
Calculated value of the test statistic in case of generalized likelihood ratio test.

	$\ln L(\hat{\Theta}^*; y)$	$\ln L(\hat{\Theta}; y)$	Test statistic
Data set 1	-244.212	-235	18.42
Data set 2	-601.645	-590	23.29
Data set 3	-122.795	-116	13.59

6. SIMULATION

It is quite difficult to compare the theoretical performances of estimators of different parameters of the PHP obtained by the method of maximum likelihood. So, in this section, we have attempted a brief simulation study for comparing the performances of the estimators, and computed their absolute bias and standard errors.

TABLE 5
Absolute bias and standard errors (given in parenthesis) of each estimators of the PHP distribution obtained by the method of maximum likelihood in case of parameter sets 1 and 2.

		MLE	
Parameter set	Sample size	$\hat{\theta}$	$\hat{\lambda}$
	$n = 100$	0.835	0.926
(1)		(1.195)	(1.536)
	$n = 300$	0.467	0.622
		(0.73)	(1.036)
	$n = 700$	0.24	0.408
		(0.079)	(0.209)
	$n = 100$	0.727	0.889
(2)		(1.105)	(1.513)
	$n = 300$	0.201	0.433
		(0.065)	(0.335)
	$n = 700$	0.01	0.018
		(0.001)	(0.004)

The simulation results are summarized in Table 5 corresponding to the sample of sizes 100, 300 and 700 for the following two sets of parameters:

1. $\lambda = 3$, $\theta = 5$ (over-dispersion);

2. $\lambda = 9$, $\theta = 4$ (under-dispersion).

From Table 5, it can be observed that both the absolute bias and standard errors for both the parameter sets are in decreasing order as the sample size increases.

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SUMMARY

In this paper we consider a zero-truncated form of the hyper-Poisson distribution and investigate some of its crucial properties through deriving its probability generating function, cumulative distribution function, expressions for factorial moments, mean, variance and recurrence relations for probabilities, raw moments and factorial moments. Further, the estimation of the parameters of the distribution is discussed. The distribution has been fitted to certain real life data sets to test its goodness of fit. The likelihood ratio test procedure is adopted for checking the significance of the parameters and a simulation study is performed for assessing the efficiency of the maximum likelihood estimators.

Keywords: Confluent hypergeometric function; Mixed moment estimation; Maximum likelihood estimation; Stirling numbers of the second kind; GLRT; Simulation.