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ON A GENERALIZATION OF THE POSITIVE EXPONENTIAL FAMILY OF DISTRIBUTIONS AND THE ESTIMATION OF RELIABILITY CHARACTERISTICS

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1. Introduction

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if a random variable (rv) X denotes the lifetime of an item or a system, then $R(t)=P(X>t)$. Another measure of reliability under stress-strength setup is the probability $P=P(X>Y)$, which represents the reliability of an item or a system of random strength X subject to random stress Y. A lot of work has been done in the literature for the point estimation and testing of $R(t)$ and P. For a brief review, one may refer to Bartholomew (1957, 1963), Pugh (1963), Basu (1964), Tong (1974, 1975), Kelly *et al.* (1976), Sathe and Shah (1981), Chao (1982), Awad and Gharraf (1986), Tyagi and Bhattacharya (1989), Chaturvedi and Rani (1997, 1998), Chaturvedi and Surinder (1999), Chaturvedi and Tomer (2002, 2003), Kotz and Pensky (2003), Chaturvedi and Singh (2006, 2008), Saracoglu and Kaya (2007), Krishnamoorthy *et al.* (2007, 2009), Baklizi (2008a,b), Eryilmaz (2008a,b, 2010, 2011), Kundu and Raqab (2009), Krishnamoorthy and Lin (2010), Rezaei *et al.* (2010), Chaturvedi and Pathak (2012, 2013), Chaturvedi and Kumari (2016), Chaturvedi and Malhotra (2017), Chaturvedi and Vyas (2017), Chaturvedi *et al.* (2018) and others.

Constantine *et al.* (1986) have derived the UMVUE and MLE of P when X and Y follow gamma distributions with shape parameters to be integer-valued. Huang and Wang (2012) have generalized these results for the case when shape parameters are positive-valued. Liang (2008) proposed a family of lifetime distributions, which he named as a positive exponential family of distributions. He showed that three distributions, exponential, Weibull and gamma to be the particular cases of this family. Recently, Chaturvedi and Malhotra (2018) developed estimation procedures for the reliability characteristics

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of the positive exponential family of distributions.

The purpose of present paper is manifold. We propose a generalization of the positive exponential family of distributions which covers as many as ten distributions to be particular cases. We derive uniformly minimum variance unbiased estimators (UMVUEs), maximum likelihood estimators (MLEs) and method of moment estimators (MMEs) of the reliability characterstics. In Section 2, we propose the generalized exponential family of distributions and study its properties. In Section 3, we derive UMVUEs, MLEs and MMEs. In Section 4, we derive MLES and MMES when all the parameters are unknown. In Section 5, we present simulation studies and in Section 6, we present two examples of real data. Finally in Section 7, we give the concluding remarks.

2. THE GENERALIZED POSITIVE EXPONENTIAL FAMILY OF DISTRIBUTIONS AND ITS PROPERTIES

A random variable X is said to follow generalized positive exponential family of distributions if its probability density function is given by

$$f(x;\alpha,\beta,\nu,\theta)=\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma(\nu)}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right);x>0,\alpha,\beta,\nu,\theta>0. \tag{1}$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x)=\frac{\gamma\left(\nu,\frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}. \tag{2}$$

where $\gamma(x,a)=\int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.

We note that this family covers the following distributions as special cases.

1. For $\alpha=\nu=\beta=1$, we get one parameter exponential distribution (see Johnson and Kotz, 1970, pp. 197).

2. For $\alpha=\beta=1$, it gives a gamma distribution. Further, for integral values of α, it gives an Erlang distribution (see Johnson and Kotz, 1970, pp. 197).

3. For $\beta=1$, it leads to generalized gamma distribution (see Johnson and Kotz, 1970, pp. 197).

4. For $\beta=\nu=1$, it turns out to be a Weibull distribution (see Johnson and Kotz, 1970, pp. 250).

5. For $\nu=\frac{1}{2},\beta=1,\alpha=2$, it is known as half normal distribution (see Davis, 1952).

6. For $\nu=\frac{m}{2}$, $\alpha=2$, $\beta=\frac{1}{2},m>0$ we get a chi distribution (see Patel *et al.*, 1976, pp. 173) and for $m=3$ we get a Maxwell distribution (see Tyagi and Bhattacharya, 1989).

7. For $\alpha=2$, $\nu=1$, $\beta=1$, we get a Rayleigh distribution (see Sinha, 1986, pp. 200).

8. For $\alpha=2$, $\beta=1$, $\nu=k+1$; $k\geq 0$ we get a generalized Rayleigh distribution of (see Voda, 1978).

9. For $\nu=\beta$ and $\alpha=2$, $\nu>o$, $\beta>0$ we get the Nakagami (1960) distribution.

10. For $\beta=1$, we get the positive exponential family of distribution (see Liang, 2008; Chaturvedi and Malhotra, 2018).

2.1. Distribution properties

Here we discuss some important distribution properties of the generalized positive exponential family of distributions.

1. The r^{th} raw moment is given by

$$\mu_r^{'}=\frac{(\theta/\beta)^{(r/\alpha)}}{\Gamma(\nu)}\Gamma\left(\frac{r}{\alpha}+\nu\right);\ r=1,2,...,$$

so that

$$\text{Mean}=\frac{(\theta/\beta)^{(1/\alpha)}}{\Gamma(\nu)}\Gamma\left(\frac{1}{\alpha}+\nu\right).$$

2. Mean square error is given by

$$\text{Mean square}=\frac{(\theta/\beta)^{(2/\alpha)}}{\Gamma(\nu)}\left[\Gamma\left(\frac{2}{\alpha}+\nu\right)-\frac{1}{\Gamma(\nu)}\left(\Gamma\left(\frac{1}{\alpha}+\nu\right)\right)^2\right].$$

3. Mode is the value of x for which $f(x)$ is maximum. The mode of the distribution is given by

$$\text{Mode}=\left(\frac{\theta}{\beta}\frac{(\alpha\nu-1)}{\alpha}\right)^{\left(\frac{1}{\alpha}\right)}.$$

4. Median is the solution of the following equation:

$$F(Md)=0.5,$$

$$\Rightarrow\frac{\gamma\left(\nu,\frac{\beta(Md)^\alpha}{\theta}\right)}{\Gamma(\nu)}-0.5=0.$$

2.2. Reliability characteristics

Here we discuss some reliability characterstics of this family of distributions.

1. Mean time to system failure for this family of distributions is given by

$$MTSF = \frac{\left(\frac{\theta}{\beta}\right)^{\left(\frac{1}{\alpha}\right)}}{\Gamma(\nu)}\Gamma\left(\frac{1}{\alpha}+\nu\right).$$

2. The relibility function is given by

$$R(x) = P[X > x] = 1 - \frac{1}{\Gamma(\nu)}\gamma\left(\nu, \frac{\beta x^{\alpha}}{\theta}\right).$$

3. The mean residual life is given by

$$\mu(x) = \frac{\left[\int_x^{\infty} 1 - \frac{\gamma\left(\nu, \frac{\beta u^{\alpha}}{\theta}\right)}{\Gamma(\nu)}\right]}{\left[1 - \frac{\gamma\left(\nu, \frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}\right]}.$$

4. The failure rate function of this family of distributions is given by

$$\frac{\alpha\left(\frac{\beta}{\theta}\right)^{\nu}\frac{1}{\Gamma(\nu)}x^{\alpha\nu-1}\exp\left(\frac{-\beta x^{\alpha}}{\theta}\right)}{1 - \frac{\gamma\left(\nu, \frac{\beta x^{\alpha}}{\theta}\right)}{\Gamma(\nu)}}.$$

3. UMVUEs, MLEs and MMEs of $\theta^q, R(t)$ & P

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the distribution given in (1). Then, assuming ν, β and α to be known, the likelihood function of the parameter θ given the sample observations $\underline{x} = (x_1, x_2, \ldots, x_n)$ is:

$$L(\theta \mid \underline{x}) = \left(\frac{\alpha}{\Gamma(\nu)}\right)^n \left(\frac{\beta}{\theta}\right)^{n\nu} e^{-\frac{\beta}{\theta}\sum_{i=1}^{n} x_i^{\alpha}} \prod_{i=1}^{n} x_i^{\alpha\nu-1}. \tag{3}$$

The following theorem provides UMVUEs of powers of θ.

THEOREM 1. *For $q \in (-\infty, \infty)$, the UMVUE of θ^q is given by:*

$$\widetilde{\theta}^q = \begin{cases} \left\{\frac{\Gamma(n\nu)}{\Gamma(n\nu+q)}\right\} S^q; & n\nu + q > 0 \\ 0; & otherwise \end{cases}$$

where $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and $S = \beta\sum_{i=1}^{n} X_i^{\alpha}$.

PROOF. It follows from (3) and factorization theorem (see Rohatgi and Saleh, 2012, pp. 367) that S is sufficient statistic for θ and the pdf of S is

$$f_s(s \mid \theta) = \frac{s^{n\nu-1}}{\Gamma(n\nu)\theta^{n\nu}} \exp\left(-\frac{s}{\theta}\right); \quad \nu > 0,\ \theta > 0,\ s \geq 0. \tag{4}$$

From (3), since the distribution of S belongs to exponential family, it is also complete. Now it follows from (4) that

$$E[S^q] = \left\{\frac{\Gamma(n\nu + q)}{\Gamma(n\nu)}\right\} \theta^q, \tag{5}$$

and the theorem follows. □

In the following theorem, we obtain UMVUE of the sampled pdf at a specified point.

THEOREM 2. *The UMVUE of the sampled pdf at a specified point x is:*

$$\tilde{f}(x;\theta) = \begin{cases} \frac{\alpha}{\beta(\nu,(n-1)\nu)} \left(\frac{\beta}{S}\right)^{\nu} x^{\alpha\nu-1} \left[1 - \frac{\beta x^{\alpha}}{S}\right]^{(n-1)\nu-1}; & \beta x^{\alpha} < S \\ 0; & \textit{otherwise} \end{cases}$$

where $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ *is the beta function.*

PROOF. We can write

$$f(x;\theta) = \alpha \left(\frac{\beta}{\theta}\right)^{\nu} \frac{1}{\Gamma(\nu)} x^{\alpha\nu-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\beta x^{\alpha}}{\theta}\right)^i.$$

Applying Theorem 1

$$\begin{aligned} \tilde{f}(x;\theta) &= \frac{\alpha\beta^{\nu} x^{\alpha\nu-1}}{\Gamma(\nu)} \sum_{i=0}^{\infty} \frac{(-1)^i (\beta x^{\alpha})^i}{i!} (\tilde{\theta})^{(-(\nu+i))}, \\ &= \frac{\alpha\left(\frac{\beta}{S}\right)^{\nu} x^{(\alpha\nu-1)}}{\beta(\nu,(n-1)\nu)} \sum_{i=0}^{(n-1)\nu-1} (-1)^i \binom{(n-1)\nu-1}{i} \left(\frac{\beta x^{\alpha}}{S}\right)^i, \end{aligned}$$

and the result follows. □

The following theorem provides UMVUE of the reliability function $R(t)$.

THEOREM 3. *The UMVUE of* $R(t)$ *is:*

$$\tilde{R}(t) = \begin{cases} 1 - I_{\frac{\beta t^{\alpha}}{S}}(\nu,(n-1)\nu); & \beta t^{\alpha} < S \\ 0; & \textit{otherwise} \end{cases}$$

where $I_x(p,q) = \frac{1}{\beta(p,q)} \int_0^x y^{p-1}(1-y)^{q-1} dy;\ 0 \leq y \leq 1,\ x < 1,\ p,q > 0$ *is the incomplete beta function.*

PROOF. We note that the expectation of $\int_t^\infty \widetilde{f}(x;\theta)dx$ with respect to S is $R(t)$. Thus, applying Theorem 2,

$$\widetilde{R}(t) = \int_t^\infty \widetilde{f}(x;\theta)dx,$$
$$= \frac{\alpha}{\beta(\nu,(n-1)\nu)}\left(\frac{\beta}{S}\right)^{\nu}\int_t^\infty x^{\alpha\nu-1}\left[1-\frac{\beta x^\alpha}{S}\right]^{(n-1)\nu-1}dx,$$

and the result follows by substituting $\frac{\beta x^\alpha}{S} = z$. □

Let X and Y be two independent random variables with respective *pdf*:

$$f(x;\theta_1) = \alpha_1\left(\frac{\beta_1}{\theta_1}\right)^{\nu_1}\frac{1}{\Gamma(\nu_1)}x^{\alpha_1\nu_1-1}\exp\left(\frac{-\beta_1 x^{\alpha_1}}{\theta_1}\right),$$

and

$$f(y;\theta_2) = \alpha_2\left(\frac{\beta_2}{\theta_2}\right)^{\nu_2}\frac{1}{\Gamma(\nu_2)}y^{\alpha_2\nu_2-1}\exp\left(\frac{-\beta_2 y^{\alpha_2}}{\theta_2}\right).$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $f(x;\theta_1)$ and $Y_1, Y_2, \ldots, Y_m$ be a random sample of size m from $f(y;\theta_2)$. Define, $S = \sum_{i=1}^{n}\beta_1 X_i^{\alpha_1}$ and $T = \sum_{i=1}^{m}\beta_2 Y_i^{\alpha_2}$. Now the UMVUE of P is given in the following theorem.

THEOREM 4. *The UMVUE of P is*

$$\tilde{P} = \begin{cases} \int_{z=0}^{1}\frac{1}{\beta\{\nu_1,(n-1)\nu_1\}}z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}I_{\left\{\frac{\beta_2\left(\frac{Sz}{\beta_1}\right)^{\left(\frac{\alpha_2}{\alpha_1}\right)}}{T}\right\}}(\nu_2,(m-1)\nu_2) \\ \text{where}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} \le \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}. \\ 1-\frac{1}{\beta\{\nu_2,(m-1)\nu_2\}\beta\{\nu_1,(n-1)\nu_1\}}\int_{z=0}^{1}z^{\nu_2-1}(1-z)^{(m-1)\nu_2-1} \\ \times\int_{w=0}^{(zT/\beta_2)^{\alpha_1/\alpha_2}}w^{\nu_1-1}(1-w)^{(n-1)\nu_1-1}dw, \\ \text{where}\left(\frac{S}{\beta_1}\right)^{1/\alpha_1} > \left(\frac{T}{\beta_2}\right)^{1/\alpha_2}. \end{cases}$$

PROOF. It follows from Theorem 3 that

$$\widetilde{f}(x;\theta_1) = \begin{cases} \frac{\alpha_1}{\beta(\nu_1,(n-1)\nu_1)}\left(\frac{\beta_1}{S}\right)^{\nu_1}x^{\alpha_1\nu_1-1}\left[1-\frac{\beta_1 x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1} & ;\beta_1 x^{\alpha_1} < S \\ 0 & ;\text{otherwise} \end{cases}$$

and

$$\tilde{f}(y;\theta_2)=\begin{cases}\frac{\alpha_2}{\beta(\nu_2,(n-1)\nu_2)}\left(\frac{\beta_2}{T}\right)^{\nu_2} y^{\alpha_2\nu_2-1}\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1} & ;\beta_2 y^{\alpha_2}<T\\ 0 & ;\text{otherwise}\end{cases}$$

From the arguments similar to those used in Theorem 2

$$\begin{aligned}\tilde{P} &= \int_{x=0}^{\infty}\int_{y=0}^{x}\tilde{f}(x;\theta_1)\tilde{f}(y;\theta_2)dx\,dy,\\ &= \int_{x=0}^{\min\left(\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}},\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\right)}\frac{\alpha_1 x^{\alpha_1\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}\left(\frac{\beta_1}{S}\right)^{\nu_1}\left[1-\frac{x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1},\\ &\times I_{\frac{\beta_2 x^{\alpha_2}}{T}}(\nu_2,(m-1)\nu_2)dx.\end{aligned} \tag{6}$$

When $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} \leq \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, we substitute $\beta_1\frac{x^{\alpha_1}}{S}=z$ and the first assertion follows. When $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}} > \left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, we first replace $\int_{x=y}^{\infty}\tilde{f}(x;\theta_1)dx$ by $\tilde{R}(y)$ and then substitute $\beta_2\frac{y^{\alpha_2}}{T}=z$ and the second assertion follows. □

It is interesting to note that on putting $\alpha_1=\alpha_2=1$ and $\beta_1=\beta_2=1$ we get the UMVUE of $P(X>Y)$ obtained by Huang and Wang (2012). Hence we were able to obtain a generalized expression of UMVUE of $P(X>Y)$ by a different yet simpler approach. Assuming both shape parameters ν_1 and ν_2 to be integers, Constantine *et al.* (1986) showed that UMVUE of $P(X>Y)$ can be expressed in terms of an incomplete beta function and hypergeometric series. Following which, in Corollary 5, we derive a generalized expression of UMVUE of $P(X>Y)$ when both the shape parameters ν_1 and ν_2 are integers.

COROLLARY 5. *The UMVUE of P when the shape parameters ν_1 and ν_2 are integers is:*

$$\widetilde{P}=\begin{cases}\dfrac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)}\displaystyle\sum_{i=0}^{(m-1)\nu_2-1}\frac{(-1)^i}{\nu_2+i}\binom{(m-1)\nu_2-1}{i}\\ \cdot\displaystyle\int_0^1 z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}\left[\frac{\beta_2\left(\frac{zS}{\beta_1}\right)^{\frac{\alpha_2}{\alpha_1}}}{T}\right]^{\nu_2+i}dz;\ \left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}\leq\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\\ \\ 1-\dfrac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)}\displaystyle\sum_{i=0}^{(n-1)\nu_1-1}\frac{(-1)^i}{\nu_1+i}\binom{(n-1)\nu_1-1}{i}\\ \cdot\displaystyle\int_0^1 z^{\nu_2-1}(1-z)^{(m-1)\nu_2-1}\left[\frac{\beta_1\left(\frac{zT}{\beta_2}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right]^{\nu_1+i}dz;\ \left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}>\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}\end{cases}$$

PROOF. From Theorem 4, for $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}\leq\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$,

$$\widetilde{P}=\int_{z=0}^{1}\frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)},$$
$$\cdot\int_{w=0}^{\frac{\beta_2}{T}\left(\frac{zS}{\beta_1}\right)^{\frac{\alpha_2}{\alpha_1}}} w^{\nu_2-1}(1-w)^{(m-1)\nu_2-1}dw\,dz,$$

and the first assertion follows by binomial expansion of $(1-w)^{(m-1)\nu_2-1}$. For $\left(\frac{S}{\beta_1}\right)^{\frac{1}{\alpha_1}}>\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}$, we consider

$$\begin{aligned}\widetilde{P}&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\,dy,\\ &=\int_{y=0}^{\left(\frac{T}{\beta_2}\right)^{\frac{1}{\alpha_2}}}\frac{\alpha_2 y^{\alpha_2\nu_2-1}}{\beta(\nu_2,(m-1)\nu_2)}\left(\frac{\beta_2}{T}\right)^{\nu_2}\left[1-\frac{\beta_2 y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1}\\ &\times[1-I_{\frac{\beta_1 y^{\alpha_1}}{S}}(\nu_1,(n-1)\nu_1)]dy.\end{aligned}$$

and the second assertion follows on substituting $\frac{\beta_2 y^{\alpha_2}}{T}=z$. □

It is interesting to note that on putting $\alpha_1=\alpha_2=1$ and $\beta_1=\beta_2=1$, we get the UMVUE of $P(X>Y)$ obtained by Constantine *et al.* (1986). Hence we were able to obtain another generalized expression of UMVUE of $P(X > Y)$ by a different yet simpler approach when the shape parameters ν_1 and ν_2 are assumed to be integers.

Now we provide MLE of $R(t)$ in the following theorem.

THEOREM 6. *The MLE of $R(t)$ is given by:*

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\nu, \frac{n\nu\beta t^{\alpha}}{S}\right)}{\Gamma(\nu)},$$

where $\gamma(a, r) = \int_0^r y^{a-1} e^{-y} dy$ is the lower incomplete gamma function.

PROOF. It can be easily seen from (3) that the MLE of θ is $\widehat{\theta} = \frac{S}{n\nu}$, where $S = \beta \Sigma x_i{}^{\alpha}$. Now from the invariance property of MLE, the MLE of sampled *pdf* is:

$$\widehat{f}(x;\theta) = \frac{\alpha x^{\alpha\nu-1}}{\Gamma(\nu)} \left(\frac{n\nu\beta}{S}\right)^{\nu} \exp\left\{\frac{-n\nu\beta x^{\alpha}}{S}\right\}.$$

Thus, $\widehat{R}(t) = \int_t^{\infty} \widehat{f}(x;\theta) dx$ and the theorem follows. □

The MLE of P is given in the following theorem:

THEOREM 7. *The MLE of P is:*

$$\widehat{P} = 1 - \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_{z=0}^{\infty} z^{\nu_2 - 1} e^{-z} \gamma\left(\nu_1, \frac{n\nu_1\beta_1 \left(\frac{zT}{m\nu_2\beta_2}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right) dz.$$

PROOF. We have,

$$\begin{aligned}
\widehat{P} &= \int_{y=0}^{\infty} \int_{x=y}^{\infty} \widehat{f}(x;\theta_1)\widehat{f}(y;\theta_2) dx\, dy, \\
&= \int_{y=0}^{\infty} \widehat{R}_X(y)\widehat{f}(y;\theta_2) dy, \\
&= \int_{y=0}^{\infty} \left[1 - \frac{\gamma\left(\nu_1, \frac{n\nu_1\beta_1 y^{\alpha_1}}{S}\right)}{\Gamma(\nu_1)}\right] \frac{\alpha_2 y^{\alpha_2\nu_2 - 1}}{\Gamma(\nu_2)} \left(\frac{m\nu_2\beta_2}{T}\right)^{\nu_2} \exp\left\{\frac{-m\nu_2\beta_2 y^{\alpha_2}}{T}\right\} dy,
\end{aligned}$$

and the theorem follows on substituting $\frac{m\nu_2\beta_2 y^{\alpha_2}}{T} = z$. □

4. MLEs AND MMEs WHEN ALL THE PARAMETERS ARE UNKNOWN

Now we discuss the case when all the four parameters α,β ν and θ are unknown. For MLES, the log-likelihood function of the parameters α, ν, β and θ given the sample observations $\underline{x}$ is:

$$l(\alpha,\nu,\beta,\theta \mid \underline{x}) = n\log(\alpha) - n\log(\Gamma(\nu)) + n\nu\log(\beta) - n\nu\log(\theta) - \frac{\beta}{\theta}\sum_{i=1}^{n} x_i^{\alpha} + (\alpha\nu - 1)\sum_{i=1}^{n}\log(x_i).$$

The MLE's of α, ν, β and θ are given by the simultaneous solution of the following three equations:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \frac{\beta}{\theta}\sum_{i=1}^{n} x_i^{\alpha}\log(x_i) + \nu\sum_{i=1}^{n}\log(x_i) = 0, \tag{7}$$

$$\frac{\partial l}{\partial \nu} = \frac{-n}{\Gamma(\nu)}\frac{d\Gamma(\nu)}{d\nu} - n\log(\theta) + n\log(\beta) + \alpha\sum_{i=1}^{n}\log(x_i) = 0, \tag{8}$$

$$\frac{\partial l}{\partial \beta} = \frac{n\nu}{\beta} - \frac{\sum_{i=1}^{n} x_i^{\alpha}}{\theta} = 0, \tag{9}$$

$$\frac{\partial l}{\partial \theta} = \frac{-n\nu}{\theta} + \frac{\beta\sum_{i=1}^{n} x_i^{\alpha}}{\theta^2} = 0. \tag{10}$$

Since these non-linear equations don't have a closed form solution, therefore we apply Newton Raphson algorithm to compute MLEs of α, β and ν. These values of MLEs of α, β and ν so obtained can be substituted in equation (10) to obtain MLE of θ. From (10), the MLE of θ is

$$\widehat{\theta} = \frac{\widehat{\beta}\sum_{i=1}^{n} x_i^{\widehat{\alpha}}}{n\widehat{\nu}},$$

where $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\nu}$ are the MLEs of α, β and ν respectively. It is to be noted that from Theorem 6, Theorem 7, and invariance property of MLE, the MLE of $R(t)$ is given as:

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\widehat{\nu}, \frac{n\widehat{\nu}\widehat{\beta}t^{\widehat{\alpha}}}{S}\right)}{\Gamma(\widehat{\nu})},$$

where $S = \widehat{\beta}\sum_{i=1}^{n} X_i^{\widehat{\alpha}}$ and the MLE of P is given as:

$$\widehat{P} = 1 - \frac{1}{\Gamma(\widehat{\nu}_1)\Gamma(\widehat{\nu}_2)}\int_{z=0}^{\infty} z^{(\widehat{\nu}_2-1)}e^{-z}\gamma\left(\widehat{\nu}_1, \frac{n\widehat{\nu}_1\widehat{\beta}_1\left(\frac{zT}{m\widehat{\nu}_2}\widehat{\beta}_2\right)^{\frac{\widehat{\alpha}_1}{\widehat{\alpha}_2}}}{S}\right)dz,$$

where $S = \widehat{\beta}_1 \sum_{i=1}^{n} X_i^{\widehat{\alpha}_1}$, $T = \widehat{\beta}_2 \sum_{i=1}^{m} Y_i^{\widehat{\alpha}_2}$.

Next we derive the moment estimators of the parameters α, ν, β and θ of this family of distributions. From equation (1), we obtain the rth moment as:

$$\begin{aligned} E(X^r) &= \int_0^\infty \alpha \left(\frac{\beta}{\theta}\right)^\nu \frac{1}{\Gamma\nu} x^{r+\alpha\nu-1} \exp\left(\frac{-\beta x^\alpha}{\theta}\right) dx, \\ &= \frac{\left(\frac{\theta}{\beta}\right)^{\frac{r}{\alpha}}}{\Gamma(\nu)} a_r, \end{aligned}$$

on substituting $\frac{\beta x^\alpha}{\theta} = y$, where $a_r = \Gamma\left(\frac{r}{\alpha} + \nu\right)$.

For $r = 1, 2, 3$ and denoting $E(X^r)$ by $\overline{X}^r$, we obtain the following equations:

$$\Gamma(\nu)\overline{X} - a_1 \left(\frac{\theta}{\beta}\right)^{\frac{1}{\alpha}} = 0, \tag{11}$$

$$\Gamma(\nu)\overline{X^2} - a_2 \left(\frac{\theta}{\beta}\right)^{\frac{2}{\alpha}} = 0, \tag{12}$$

$$\Gamma(\nu)\overline{X^3} - a_3 \left(\frac{\theta}{\beta}\right)^{\frac{3}{\alpha}} = 0, \tag{13}$$

$$\Gamma(\nu)\overline{X^4} - a_4 \left(\frac{\theta}{\beta}\right)^{\frac{4}{\alpha}} = 0. \tag{14}$$

These equations can be simultaneously solved using the `uniroot` function in the `R` software to obtain MMEs $\widehat{\alpha_M}$, $\widehat{\nu_M}$, $\widehat{\beta_M}$ and $\widehat{\theta_M}$ of the parameters α, ν, β and θ.

For α, β, ν known, the moment estimator of θ is given by

$$\widehat{\theta_M^{\frac{1}{\alpha}}} = \frac{\Gamma(\nu)}{\Gamma\left(\frac{1}{\alpha} + \nu\right)} \overline{X} \beta^{\frac{1}{\alpha}}.$$

5. SIMULATION STUDY

Firstly, we conduct Monte Carlo simulation studies to compare the performance of $\widetilde{\theta^q}$, $\widehat{\theta_M}^q$ and $\widehat{\theta^q}$ for different sample sizes and powers of parameter θ. For $\alpha = 3$ and $\beta = \nu = 2$, we generate 10,000 samples each of size n from generalization of positive exponential family of distributions and repeat this procedure for several values of θ.

Figure 1 shows the mean square error (MSE) of the UMVUE, MMSE and MLE of θ^q. From these figures we note that for smaller sample sizes and for q=2, the MLE performs the best and the MME performs the worst. The performance of UMVUE is in between the two. As the sample size increases the three curves come close to each other.

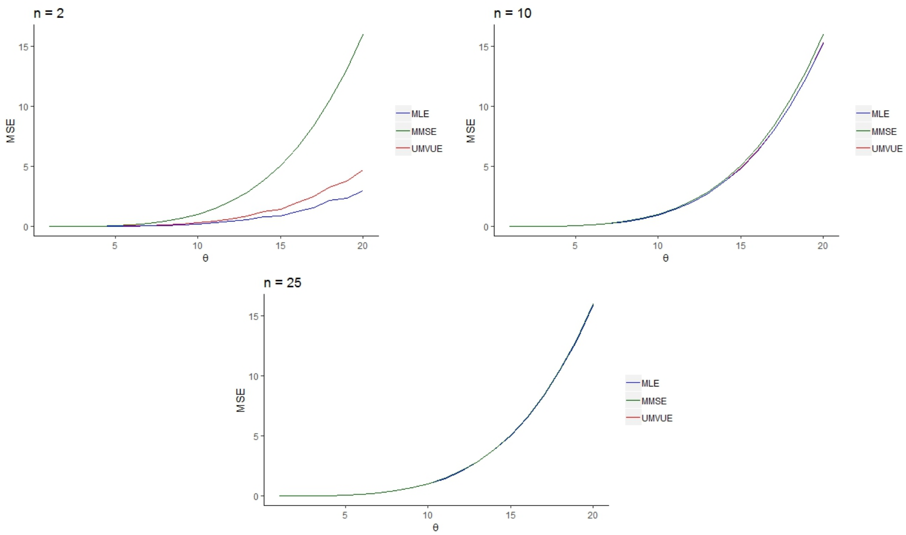

Figure 1 – MSE of the UMVUE, MLE and MMSE of θ^q for different sample sizes.

On similar lines, we perform simulation studies to compare the performance of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different sample sizes. For $t = 7$ and $\alpha = \beta = \nu = 2$, we generate 10,000 samples each of size n from the generalization of positive exponential family of distributions and repeat this procedure for several values of $R(t)$. Figure 2 shows the MSE of the UMVUE and MLE of $R(t)$. From these figures we note that the MSE of the UMVUE of $R(t)$ is always greater than that of the MLE, however for large sample sizes these estimators of $R(t)$ are better and almost equally efficient.

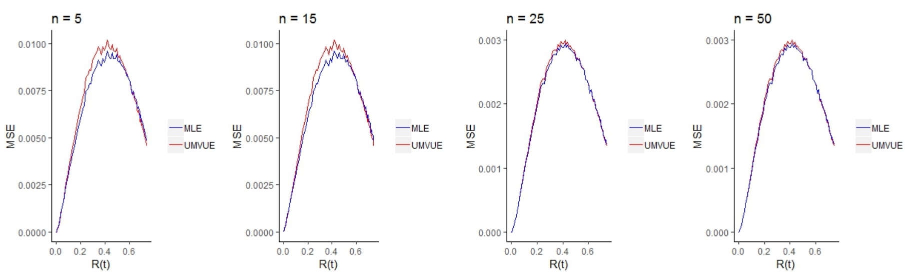

Figure 2 – MSE of the UMVUE and MLE of $R(t)$ for different sample sizes.

Now, we compare the performance of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. By Monte Carlo simulation, for $\alpha_1 = 1$ and $\beta_1 = \nu_1 = 2$ and $\alpha_2 = 2$ and $\beta_2 = \nu_2 = 3$, we generate 10,000 samples each of size n and m from generalization of positive exponential family of distributions and repeat this procedure for several values of P. Figure 3 shows the MSE of the UMVUE and MLE of P. From these figures we note that the MSE of the UMVUE of P is always greater than that of the MLE, however for large sample sizes these estimators of P are better and almost equally efficient.

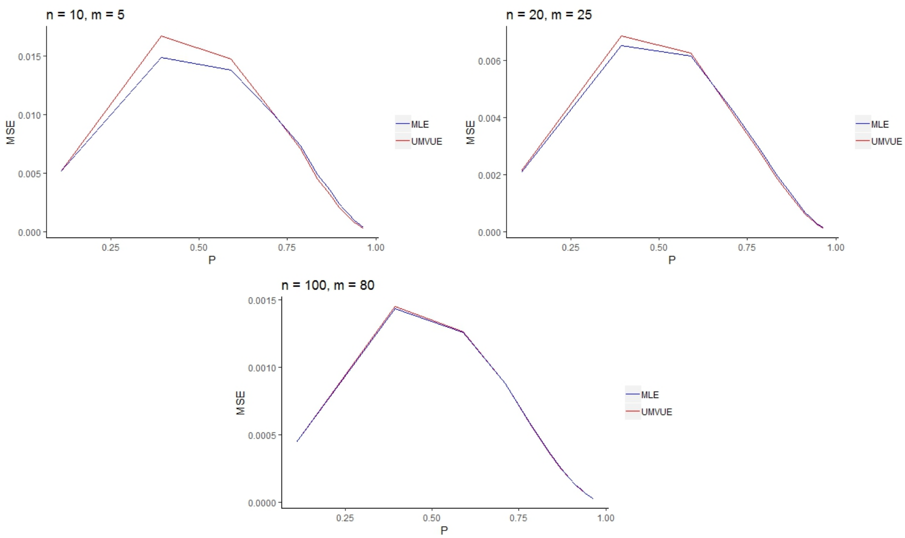

Figure 3 – MSE of the UMVUE and MLE of P for different sample sizes.

Figure 4 shows estimation of pdf in Equation (1) based on MLE and UMVUE.

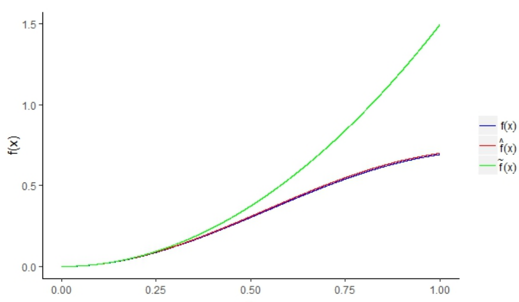

Figure 4 – MLE and UMVUE of sampled pdf.

6. Real life data examples

This section deals with examples of real data to illustrate the proposed estimation methods.

Example 1. This data set was originally reported in Schafft *et al.* (1987) (see also Kumar *et al.*, 2017), represents hours to failure of 59 conductors of 400-micrometer length. All specimens ran to failure at a certain high temperature and current density. The 59 specimens were all tested under the same temperature and current density.We observe that Nakagami distribution, which is the special case of the generalization of the positive exponential family of distributions, fits well to this data as shown in Figure 5. Let us assign the random variable $X \sim f(x;\beta,\theta)$ to Data set I that has been reproduced in the Table 1.

TABLE 1
Data set I, example 1.

6.545	9.289	7.543	6.956	6.492	5.459	8.120	4.706	8.687	2.997
8.591	6.129	11.038	5.381	6.958	4.288	6.522	4.137	7.459	7.495
6.573	6.538	5.589	6.087	5.807	6.725	8.532	9.663	6.369	7.024
8.336	9.218	7.945	6.869	6.352	4.700	6.948	9.254	5.009	7.489
7.398	6.033	10.092	7.496	4.531	7.974	8.799	7.683	7.224	7.365
6.923	5.640	5.434	7.937	6.515	6.476	6.071	10.491	5.923	

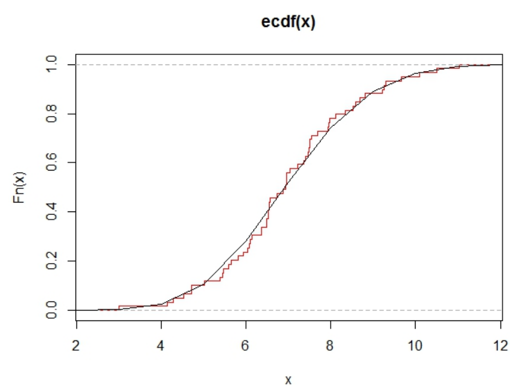

Figure 5 – The empirical and theoretical cdf of Nakagami(β,θ) model.

Now for the above data set we obtain the various estimators of β, θ and $R(t)$ and the results are presented in Table 2.

TABLE 2
The MLE, UMVUE and MME of parameters β and θ of Nakagami model and its corresponding reliability function $R_X(t)$ for time $t = 2$ based on Data set I.

$\widehat{\beta}$	$\widehat{\theta}$	$\widetilde{\theta}$	$\widehat{\beta}_M$	$\widehat{\theta}_M$	$\widehat{R}_X(t)$	$\widetilde{R}_X(t)$
4.834	51.282	51.282	4.872	51.282	0.999	0.999

Example 2. The data on breaking strength of jute fibers were proposed by Xia *et al.* (2009) (see also Chaturvedi *et al.*, 2018). The Jute fibers were tested under tension at gauge lengths of 5, 10, 15, and 20 mm. In our study, we consider data on breaking strength of jute fibers under gauge lengths 15 mm and 20 mm. These data are reported in Tables 3 and 4, respectively.

TABLE 3
Data set I, example 2.

594.40	202.75	168.37	574.86	225.65	76.38
156.67	127.81	813.87	562.39	468.47	135.09
72.24	497.94	355.56	569.07	640.48	200.76
550.42	748.75	489.66	678.06	457.71	106.73
716.30	42.66	80.40	339.22	70.09	193.42

TABLE 4
Data set II, example 2.

771.46	419.02	284.64	585.57	456.60	113.85
187.85	688.16	662.66	45.58	578.62	756.70
594.29	166.49	99.72	707.36	765.14	187.13
145.96	350.70	547.44	116.99	375.81	581.60
119.86	48.01	200.16	36.75	244.53	83.55

First of all these two data sets are used to fit the exponential distribution, separately (see Figures 6 and 7).

Let us assign the random variable $X \sim f(x;\theta_1)$ to Data set I that has been reproduced in Table 3 and let us assign the random variable $Y \sim f(y;\theta_2)$ to Data set II that has been reproduced in Table 4. Now, for the above two data sets, we obtain estimators of $P = P(X > Y)$ and the results are presented in Table 5.

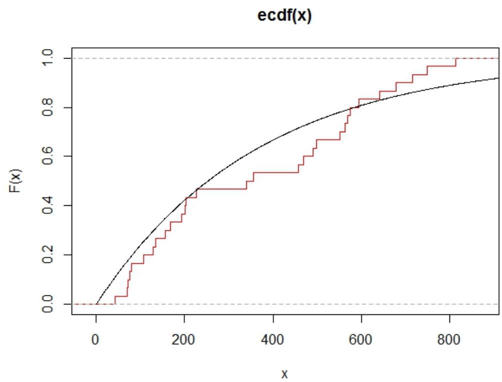

Figure 6 – The empirical and theoretical cdf of exponential(θ_1) model.

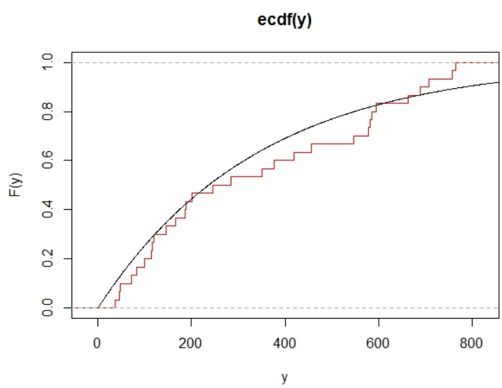

Figure 7 – The empirical and theoretical cdf of exponential(θ_2) model.

TABLE 5
The MLE and UMVUE of $P(X > Y)$.

$\widehat{P}$	$\widetilde{P}$
0.516	0.517

7. Conclusions

In the present paper, we have generalized the results of Chaturvedi and Malhotra (2018) to a family of distributions which we name as generalized exponential family of distributions. This family of distribution covers as many as ten distributions as particular cases. UMVUEs, MLEs and MMEs are developed for the powers of parameters, $R(t) = P(X > t)$ and $P = P(X > Y)$. Efficiency comparison of the three methods of estimation is done.

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SUMMARY

A generalization of positive exponential family of distributions developed by Liang (2008) is taken into consideration. Its properties are studied. Two measures of reliability are discussed. Uniformly minimum variance unbiased estimators (UMVUES), maximum likelihood estimators (MLES) and method of moment estimators (MMES) are developed for the reliability functions. The performances of three types of estimators are compared through Monte Carlo simulation. Real life data sets are also analyzed.

Keywords: Generalized positive exponential family; MLE; MME; Reliability; UMVUE.