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QUANTILE-BASED GENERALIZED ENTROPY OF ORDER (α, β) FOR ORDER STATISTICS

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1. INTRODUCTION

Entropy, as a measure of randomness contained in a probability distribution, is a fundamental concept in information theory and cryptography. Today, information theory is considered to be a very fundamental field which intersects with physics (statistical mechanics), mathematics (probability theory), electrical engineering (communication theory) and computer science (Kolmogorov complexity) etc. The average amount of uncertainty associated with the nonnegative continuous random variable X can be measured using the differential entropy function

$$H(f) = -\int_0^\infty f(x) \log f(x) dx \,, \tag{1}$$

a continuous counterpart of the Shannon (1948) entropy in the discrete case, where $f(x)$ denotes the probability density function (pdf) of the random variable X. A huge literature devoted to the characterizations, generalizations and applications of the Shannon entropy measure is available, refer to Cover and Thomas (2006). A generalization of order (α, β) of the entropy (1) is the Varma (1966) entropy defined as

$$H^{(\alpha,\beta)}(f) = \frac{1}{(\beta-\alpha)} \log\left[\int_0^\infty f^{\alpha+\beta-1}(x) dx\right]; \; \beta \neq \alpha, \; \beta - 1 < \alpha < \beta, \; \beta \geq 1 \,. \tag{2}$$

When $\beta = 1$, $H^{(\alpha,\beta)}(f)$ reduces to $H^{\alpha}(f) = \frac{1}{(1-\alpha)} \log\left[\int_0^\infty f^{\alpha}(x) dx\right]$, the Rényi (1961) entropy, and when $\beta = 1$ and $\alpha \longrightarrow 1, H^{(\alpha,\beta)}(f) \longrightarrow H(f)$ given in (1). Varma's entropy measure is much more flexible due to the parameters α and β, enabling several

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measurements of uncertainty within a given distribution and increase the scope of application. In recent years, Varma's entropy has been used by many researchers in the context of information theory, we refer to Kayal and Vellaisamy (2011), and Kayal (2015). Suppose that $X_1, X_2, \cdots, X_n$ are n independent and identically distributed (iid) random variables with a common absolutely continuous cumulative distribution function (cdf) $F(x)$ and pdf $f(x)$. The order statistics of this sample is defined as arrangement of $X_1, X_2, \cdots, X_n$ from the smallest to the largest denoted by $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$. Then $X_{i:n}$ is called the i^{th} order statistics and its pdf is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} F(x)^{i-1}(1-F(x))^{n-i} f(x), \tag{3}$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx, a > 0, b > 0$, is the beta function with parameters a and b. Order statistics plays an important role in various applied practical problems, we refer to Arnold *et al.* (1992) and David and Nagaraja (2003) for more details. One of the important applications of order statistics is to construct median filters for image and signal processing. Various authors have worked on information properties of order statistics. Wong and Chen (1990) showed that the difference between the entropy of i^{th} order statistics and the average entropy is a constant. Ebrahimi *et al.* (2004) have explored the properties of Shannon entropy, Kullback-Leibler information, and mutual information for order statistics. Similar results on generalized residual entropy for order statistics have been derived by Abbasnejad and Arghami (2011), Zarezadeh and Asadi (2010), Thapliyal *et al.* (2015) and Baratpour and Khammar (2016).

All the theoretical investigations and applications using these information measures are based on the distribution function. A probability distribution can be specified either in terms of the distribution function or by the *quantile functions* (QF), defined by

$$Q(u) = F^{-1}(u) = \inf\{x \mid F(x) \geq u\}, \; 0 \leq u \leq 1. \tag{4}$$

Quantile functions (QFs) have several properties that are not shared by distribution functions. For example, the sum of two QFs is again a QF. Sometimes, the quantile-based approach is better in terms of tractability. In many cases, QF is more convenient as it is less influenced by extreme observations, and thus provides a straightforward analysis with a limited amount of information. It is easier to generate random numbers from the QF. In reliability analysis, a single long-term survivor can have a marked effect on mean life, especially in the case of heavy-tailed models which are commonly encountered for lifetime data. In such cases, quantile-based estimates are generally found to be more precise and robust against outliers. However, the use of QFs in the place of F provides new models, alternative methodology, easier algebraic manipulations, and methods of analysis in certain cases and some new models and characterizations that are difficult to derive by using distribution function (Gilchrist, 2000; Nair *et al.*, 2013; Parzen, 1979). There are explicit general distribution forms for the QF of order statistics. The quantile-based Shannon entropy of i^{th} order statistics studied by Sunoj *et al.* (2017). Considering the importance of generalized entropy and its order statistics, we extend the concept of generalized quantile entropy using order statistics. Motivated by these, in the present study we consider Varma's entropy for order statistics for residual and reversed residual (past) lifetime using the QFs and proved some characterization results of these.

From (4), we have $FQ(u) = u$, then pdf of the i^{th} order statistics (3) becomes

$$\begin{aligned} f_{i:n}(u) = f_{i:n}(Q(u)) &= \frac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i} f(Q(u)) \\ &= \frac{1}{B(i, n-i+1)} u^{i-1}(1-u)^{n-i} \frac{1}{q(u)} \end{aligned} \tag{5}$$

Sunoj and Sankaran (2012) have considered the quantile-based Shannon entropy and its residual form, defined as

$$\underset{\text{,}}{H} = \int_0^1 \log q(p) dp, \tag{6}$$

and

$$\underset{\text{,}}{H}(u) = \log(1-u) + (1-u)^{-1} \int_u^1 \log q(p) dp, \tag{7}$$

respectively, where $q(u) = \frac{dQ(u)}{du}$ is the quantile density function, the mean of the distribution is $E(X) = \int_0^1 Q(p)dp = \int_0^1 (1-p)q(p)dp$ and assumed to be finite. Defining the density quantile function by $fQ(u) = f(Q(u))$ and the quantile density function by $q(u)$, we have

$$q(u)f(Q(u)) = 1. \tag{8}$$

Kumar and Rani (2018) proposed the quantile version of Varma's entropy of order (α, β), which was defined as

$$\underset{\text{,}}{H}_X^{(\alpha,\beta)} = \frac{1}{(\beta-\alpha)} \log\left(\int_0^1 (q(p))^{2-\alpha-\beta} dp\right);\ \beta \neq \alpha,\ \beta - 1 < \alpha < \beta,\ \beta \geq 1, \tag{9}$$

and studied its properties. For more details we refer to Baratpour and Khammar (2018). When $\beta = 1$ and $\alpha \longrightarrow 1$, the measure (9) reduces to (6).

The paper is organized as follows. In Section 2, we introduce a quantile version of generalized entropy (2) for i^{th} order statistics, and investigate it for various type of univariate distributions as well as provides bound for it. Also we show that the generalized quantile information between $X_{i:n}$ and X is distribution free. In Section 3, we express the quantile-based generalized residual entropy (GRE) of order statistics and study its important properties. Section 4 is devoted to the characterization result for first order statistics. In Section 5, characterization result for quantile-based generalized entropy of order statistics for reversed residual lifetime has been studied for parallel systems.

2. Generalized quantile entropy for $X_{i:n}$

Analogous to (2), Thapliyal and Taneja (2012) proposed the two parameter generalized entropy for the i^{th} order statistics $X_{i:n}$ as

$$H^{(\alpha,\beta)}(f_{i:n})=\frac{1}{\beta-\alpha}\log\left(\int_0^\infty (f_{i:n}(x))^{\alpha+\beta-1}dx\right);\ \beta\neq\alpha,\ \beta-1<\alpha<\beta,\ \beta\geq 1 \quad (10)$$

and studied some properties of it. Here $f_{i:n}(x)$ is the pdf of i^{th} order statistics that is defined by (3). In the following result, we will show that the quantile-based generalized entropy of order statistics $X_{i:n}$ can be represented in terms of quantile-based generalized entropy of order statistics of standard uniform distribution.

THEOREM 1. *The quantile version of generalized entropy (2) of $X_{i:n}$ can be expressed as*

$$\underset{\sim}{H}^{(\alpha,\beta)}_{X_{i:n}}=\underset{\sim}{H}^{(\alpha,\beta)}_{U_{i:n}}+\frac{1}{\beta-\alpha}\log E_{g_i}\left(q^{2-\alpha-\beta}(Y_i)\right), \quad (11)$$

where $\underset{\sim}{H}^{(\alpha,\beta)}_{U_{i:n}}$ denotes the quantile-based generalized entropy of $X_{i:n}$ from standard uniform distribution, $E_{g_i}(X)$ denotes expectation of X over g_i and $Y_i\sim g_i$ is the beta density with parameters $(\alpha+\beta-1)(i-1)+1$ and $(\alpha+\beta-1)(n-i)+1$.

PROOF. Using (4), the quantile-based generalized entropy of order (α,β) of i^{th} order statistics is defined as

$$\underset{\sim}{H}^{(\alpha,\beta)}_{X_{i:n}}=\frac{1}{\beta-\alpha}\log\left(\int_0^1 f_{i:n}^{\alpha+\beta-1}(Q(u))d(Q(u))\right).$$

From (5), we have

$$\begin{aligned}\underset{\sim}{H}^{(\alpha,\beta)}_{X_{i:n}}&=\frac{1}{\beta-\alpha}\log\left\{\frac{1}{(B(i,n-i+1))^{\alpha+\beta-1}}\right.\\&\quad\left.\times\int_0^1 u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}(q(u))^{2-\alpha-\beta}du\right\}\\&=\frac{1}{\beta-\alpha}\log\left\{\frac{B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i)+1\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right.\\&\quad\left.\times\int_0^1\frac{u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}(q(u))^{2-\alpha-\beta}du}{B\{(\alpha+\beta-1)(i-1)+1,\ (\alpha+\beta-1)(n-i)+1\}}\right\},\end{aligned} \quad (12)$$

where the first term inside the parenthesis is the quantile-based generalized entropy for $X_{i:n}$ of standard uniform distribution, which completes the proof. □

For some specific univariate continuous distributions, the expression (11) is evaluated as given below in Table 1.

TABLE 1
Quantile-based generalized entropy $H_{X_{i:n}}^{(\alpha,\beta)}$ of i^{th} order statistics for different lifetime distributions.

Distribution	Quantile function $Q(u)$	GRQE $H_{X_{i:n}}^{(\alpha,\beta)}$
Uniform	$a+(b-a)u$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(b-a)^{(2-\alpha-\beta)}B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i)+1\}}{(B(i,n-i+1))^{(\alpha+\beta-1)}}\right)$
Exponential	$-\lambda^{-1}\log(1-u)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\lambda^{\alpha+\beta-2}B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i+1)\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Pareto-I	$b(1-u)^{-\frac{1}{a}}$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(\frac{b}{a})^{2-\alpha-\beta}B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i+1)+\frac{\alpha+\beta-2}{a}\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Log-logistic	$\frac{1}{a}\left(\frac{u}{1-u}\right)^{\frac{1}{b}}$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(ab)^{\alpha+\beta-2}B\{(\alpha+\beta-1)i+\frac{(2-\alpha-\beta)}{b},(\alpha+\beta-1)(n-i+1)+\frac{\alpha+\beta-2}{b})\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$=\frac{1}{\beta-\alpha}\log\left(\frac{B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i)+(\alpha+\beta-2)(\frac{2a+1}{a+1})+1\}}{(\frac{b}{a+1})^{\alpha+\beta-2}(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Finite range	$b(1-(1-u)^{\frac{1}{a}})$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(\frac{b}{a})^{2-\alpha-\beta}B\{(\alpha+\beta-1)(i-1)+1,(\alpha+\beta-1)(n-i+1)+\frac{2-\alpha-\beta}{a}\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Power distribution	$au^{\frac{1}{b}}$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(\frac{a}{b})^{2-\alpha-\beta}B\{(\alpha+\beta-1)i+\frac{2-\alpha-\beta}{b},(\alpha+\beta-1)(n-i)+1\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$
Govindarajulu	$a\{(b+1)u^b-bu^{b+1}\}$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\{ab(b+1)\}^{2-\alpha-\beta}B\{(\alpha+\beta-1)i+b(2-\alpha-\beta),(\alpha+\beta-1)(n-i-1)+2)\}}{(B(i,n-i+1))^{\alpha+\beta-1}}\right)$

Next we obtain the upper (lower) bound of quantile-based generalized entropy for order statistics (11) in terms of quantile entropy (9). We prove the following result.

THEOREM 2. *For any random variable X, with quantile-based generalized entropy $H_X^{(\alpha,\beta)}<\infty$, the quantile-based Varma entropy of i^{th} order statistics $X_{i:n}$, $i=1,2,\cdots,n$ is bounded above as*

$$H_{X_{i:n}}^{(\alpha,\beta)}\le C_i+H_X^{(\alpha,\beta)}, \tag{13}$$

where

$$C_i=H_{U_{i:n}}^{(\alpha,\beta)}+\frac{1}{\beta-\alpha}\log B_i,$$

and bounded below as

$$H_{X_{i:n}}^{(\alpha,\beta)}\ge H_{U_{i:n}}^{(\alpha,\beta)}+\frac{\alpha+\beta-2}{\beta-\alpha}\log M, \tag{14}$$

where $M=\frac{1}{q(m)}<\infty$, $m=sup\{u:q(u)\ge\frac{1}{M}\}$ is the mode of the distribution and $q(u)$ is quantile density function of the random variable X and B_i represents the beta distribution with parameter $(\alpha+\beta-1)(i-1)+1$ and $(\alpha+\beta-1)(n-i)+1$.

PROOF. Let g_i and m_i be the pdf and the mode of beta distribution with parameter $(\alpha+\beta-1)(i-1)+1$ and $(\alpha+\beta-1)(n-i)+1$, respectively. Since mode of the beta distribution is $m_i=\frac{i-1}{n-1}$. Thus,

$$g_i(y)\le B_i=g_i(m_i)=\frac{m_i^{(\alpha+\beta-1)(i-1)+1}(1-m_i)^{(\alpha+\beta-1)(n-i)+1}}{B\{(\alpha+\beta-1)(i-1)+1,\,(\alpha+\beta-1)(n-i)+1\}}.$$

For $\beta > 1, \beta - 1 < \alpha < \beta$, from (12)

$$\begin{aligned} H_{X_{i:n}}^{(\alpha,\beta)} - H_{U_{i:n}}^{(\alpha,\beta)} &= \frac{1}{\beta-\alpha}\log\int_0^1 g_i(u)(q(u))^{2-\alpha-\beta}du \\ &\leq \frac{1}{\beta-\alpha}\log B_i\int_0^1 (q(u))^{2-\alpha-\beta}du \\ &= \frac{1}{\beta-\alpha}\log B_i + \frac{1}{\beta-\alpha}\log\left(\int_0^1 (q(u))^{2-\alpha-\beta}du\right) \\ &= \frac{1}{\beta-\alpha}\log B_i + H_X^{(\alpha,\beta)}. \end{aligned}$$

which gives (13). From (11) we can write

$$\begin{aligned} H_{X_{i:n}}^{(\alpha,\beta)} &\geq H_{U_{i:n}}^{(\alpha,\beta)} + \left(\frac{1}{\beta-\alpha}\right)\log\int_0^1 g_i(u)M^{\alpha+\beta-2}du, \\ &= H_{U_{i:n}}^{(\alpha,\beta)} + \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log M. \end{aligned}$$

This complete the proof. □

EXAMPLE 3. *For uniform distribution over the interval* (a, b), *we have* $H_X^{(\alpha,\beta)} = \left(\frac{\alpha+\beta-2}{\alpha-\beta}\right)\log(b-a)$. *Substituting* $i = 1$ *and* $i = n$ *in Table 1 for uniform distribution, we obtain*

$$H_{U_{1:n}}^{(\alpha,\beta)} = H_{U_{n:n}}^{(\alpha,\beta)} = \frac{1}{\beta-\alpha}\{(\alpha+\beta-1)\log n - \log((\alpha+\beta-1)(n-1)+1)\}$$

and $C_1 = C_n = \left(\frac{\alpha+\beta-1}{\beta-\alpha}\right)\log n$. *Hence, using (13) we get*

$$H_{X_{1:n}}^{(\alpha,\beta)} \leq \left(\frac{\alpha+\beta-1}{\beta-\alpha}\right)\log n + \left(\frac{\alpha+\beta-2}{\alpha-\beta}\right)\log(b-a).$$

Also, for uniform distribution over the interval (a, b), $M = \frac{1}{b-a}$. *Using (14) we get*

$$\begin{aligned} H_{X_{1:n}}^{(\alpha,\beta)} \geq &\frac{1}{\beta-\alpha}\{(\alpha+\beta-1)\log n - \log((\alpha+\beta-1)(n-1)+1)\} \\ &+ \left(\frac{\alpha+\beta-2}{\alpha-\beta}\right)\log(b-a). \end{aligned}$$

Thus, for uniform distribution, we have

$$\frac{1}{\beta-\alpha}\{(\alpha+\beta-1)\log n - \log((\alpha+\beta-1)(n-1)+1)\} + \left(\frac{\alpha+\beta-2}{\alpha-\beta}\right)\log(b-a)$$

$$\leq \underset{\sim}{H}_{X_{1:n}}^{(\alpha,\beta)} \leq \left(\frac{\alpha+\beta-1}{\beta-\alpha}\right)\log n + \left(\frac{\alpha+\beta-2}{\alpha-\beta}\right)\log(b-a).$$

Similarly we can obtain the upper (lower) bound of $\underset{\sim}{H}_{X_{n:n}}^{(\alpha,\beta)}$.

2.1. Generalized quantile information (divergence) measure

Information or divergence measures play an important role in measuring the distance between two probability distribution functions. Some goodness-of- fit tests provided based on entropy and information measures, refer to Ebrahimi *et al.* (2004) and Park (2005). Let X and Y be two non-negative random variables with density functions f and g, and survival functions $\bar{F}$ and $\bar{G}$ respectively. Several divergence measures have been proposed for this purpose which the most fundamental one is Kullback and Leibler (1951). The information divergence of order (α,β) (Varma, 1966) between two distributions is defined by

$$D_{\alpha}^{\beta}(X,Y) = \frac{1}{\alpha-\beta}\log\int_0^{\infty} f(x)\left(\frac{f(x)}{g(x)}\right)^{\alpha+\beta-2} dx\ ;\ \alpha \neq \beta, \beta \geq 1, \beta-1 < \alpha < \beta. \quad (15)$$

When $\beta = 1$, $D_{\alpha}^{\beta}(X,Y)$ reduces to $D_{\alpha}(X,Y) = \frac{1}{\alpha-1}\log\int_0^{\infty} f(x)\left(\frac{f(x)}{g(x)}\right)^{\alpha-1}dx$, the Rényi divergence measure, and when $\beta = 1$ and $\alpha \longrightarrow 1$, $D_{\alpha}^{\beta}(X,Y) \longrightarrow D(f,g) = \int_0^{\infty} f(x)\log\frac{f(x)}{g(x)}dx$ is the Kullback-Leibler information between f and g.

Recently, Sankaran *et al.* (2016) and Kumar and Rani (2018) respectively introduced quantile versions of the Kullback-Leibler and generalized divergence measure of order (α,β) and studied their properties.

LEMMA 4. *The quantile-based divergence measure of order (α,β) between the distribution of i^{th} order distribution $f_{i:n}$ and the parent distribution f is given by*

$$\underset{\sim}{D}_{\alpha}^{\beta}(f_{i:n}, f) = -\underset{\sim}{H}_{U_{i:n}}^{(\alpha,\beta)}, \quad (16)$$

where $U_{i:n}$ is the standard uniform distribution for $X_{i:n}$.

PROOF. Analogous to (15), we have

$$D_{\alpha}^{\beta}(f_{i:n}, f) = \frac{1}{\alpha-\beta}\log\left(\int_0^{\infty}\left(\frac{f_{i:n}(x)}{f(x)}\right)^{\alpha+\beta-2} f_{i:n}(x)dx\right).$$

Following Kumar and Rani (2018), the quantile version of divergence measure of order

(α,β) for i^{th} order statistics is defined as

$$\begin{aligned}
\underset{\sim}{D}_{\alpha}^{\beta}(f_{i:n},f) &= \frac{1}{\alpha-\beta} \\
&\times \log\left(\int_0^1 \frac{(FQ(u))^{(\alpha+\beta-1)(i-1)}(1-FQ(u))^{(\alpha+\beta-1)(n-i)}}{(B(i,\,n-i+1))^{\alpha+\beta-1}} f(Q(u))dQ(u)\right) \\
&= \frac{1}{\alpha-\beta}\log\left(\int_0^1 \frac{u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}}{(B(i,\,n-i+1))^{\alpha+\beta-1}} f(Q(u))q(u)du\right) \\
&= \frac{1}{\alpha-\beta}\log\left(\int_0^1 \frac{u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}}{(B(i,\,n-i+1))^{\alpha+\beta-1}} du\right) \\
&= -\underset{\sim}{H}_{U_{i:n}}^{(\alpha,\beta)}. \qquad (17)
\end{aligned}$$

Hence, the quantile-based generalized information between the distribution of order statistics and the original distribution is distribution free. □

3. Generalized quantile entropy of $X_{i:n}$ for residual lifetime

In the context of reliability and life testing studies when the present age of a component needs to be incorporated, the entropy measure given in (1) is not applicable to a system which has survived for some unit of time. Ebrahimi (1996) considered the entropy of the residual lifetime $X_t=[X-t|X>t]$ as a dynamic measure of uncertainty given by

$$H(f;t)=-\int_t^\infty \frac{f(x)}{\bar{F}(t)}\log\left(\frac{f(x)}{\bar{F}(t)}\right)dx,\ t>0. \qquad (18)$$

In analogy to Ebrahimi (1996), Baig and Dar (2008) extended generalized entropy of order $(\alpha,\ \beta)$ for the residual lifetime $X_t=[X-t|X>t]$ as

$$H^{(\alpha,\beta)}(f;t)=\frac{1}{\beta-\alpha}\log\left(\int_t^\infty \frac{f^{\alpha+\beta-1}(x)}{\bar{F}^{\alpha+\beta-1}(t)}dx\right), \qquad (19)$$

and studied many properties of it. Kumar and Rani (2018) introduced a quantile-based generalized entropy of order $(\alpha,\ \beta)$ and its residual version, and studied some properties of it. Thapliyal and Taneja (2012) introduced generalized residual entropy of order $(\alpha,\ \beta)$ for the i^{th} order statistics, given by

$$H^{(\alpha,\beta)}(f_{i:n};t)=\frac{1}{\beta-\alpha}\log\left(\int_t^\infty \frac{f_{i:n}^{\alpha+\beta-1}(x)}{\bar{F}_{i:n}^{\alpha+\beta-1}(t)}dx\right), \qquad (20)$$

where

$$\bar{F}_{i:n}(x)=\frac{\bar{B}_{F(x)}(i,n-i+1)}{B(i,n-i+1)} \qquad (21)$$

is the survival function of the i^{th} order statistics and

$$\bar{B}_x(a,b) = \int_x^1 u^{a-1}(1-u)^{b-1}du, \ 0 < x < 1$$

is the incomplete beta function. For more properties and applications and recent developments of Equation (20), we refer to Kayal (2014), Kumar (2015) and Kayal (2016). It is easy to see that the measure (20) generalizes the measure (18) and (19) both.

The quantile version of *generalized residual entropy* (GRE)of i^{th} order statistics is defined as

$$\begin{aligned} H_{X_{i:n}}^{(\alpha,\beta)}(u) &= H_\alpha^\beta(X_{i:n}; Q(u)) \\ &= \frac{1}{\beta-\alpha}\log\left\{\left(\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}\right)^\alpha \int_u^1 (g_i(p))^\alpha (q(p))^{1-\alpha} dp\right\}, \end{aligned} \tag{22}$$

where $\frac{B(i,n-i+1)}{\bar{B}_u(i,n-i+1)}$ is the quantile form of survival function $\bar{F}_{i:n}(x)$. An equivalent representation of (22) is of the form of

$$(\beta-\alpha)H_{X_{i:n}}^{(\alpha,\beta)}(u) = \\ \log\left\{\frac{1}{\left(\bar{B}_u(i,n-i+1)\right)^{\alpha+\beta-1}} \int_u^1 p^{(\alpha+\beta-1)(i-1)}(1-p)^{(\alpha+\beta-1)(n-i)}(q(p))^{2-\alpha-\beta}dp\right\}, \tag{23}$$

which is rewritten as

$$e^{\{(\beta-\alpha)H_{X_{i:n}}^{(\alpha,\beta)}(u)\}}\left(\bar{B}_u(i,n-i+1)\right)^{\alpha+\beta-1} = \\ \int_u^1 p^{(\alpha+\beta-1)(i-1)}(1-p)^{(\alpha+\beta-1)(n-i)}(q(p))^{2-\alpha-\beta}dp.$$

Differentiating it with respect to u both sides, we obtain

$$\begin{aligned} &e^{\{(\beta-\alpha)H_{X_{i:n}}^{(\alpha,\beta)}(u)\}}\left\{(\beta-\alpha)(H_{X_{i:n}}^{(\alpha,\beta)}(u))'\left(\bar{B}_u(i,n-i+1)\right)^{\alpha+\beta-1}\right\} \\ &-e^{\{(\beta-\alpha)H_{X_{i:n}}^{(\alpha,\beta)}(u)\}}\left\{(\alpha+\beta-1)u^{i-1}(1-u)^{n-i}\left(\bar{B}_u(i,n-i+1)\right)^{\alpha+\beta-2}\right\} \\ &= -u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}(q(u))^{2-\alpha-\beta}. \end{aligned}$$

This gives

$$(q(u))^{2-\alpha-\beta} = e^{\{(\beta-\alpha)H_{X_{i:n}}^{(\alpha,\beta)}(u)\}}\left(\bar{B}_u(i,n-i+1)\right)^{\alpha+\beta-2} \times$$

$$\left\{(\alpha+\beta-1)u^{(i-1)(2-\alpha-\beta)}(1-u)^{(n-i)(2-\alpha-\beta)}\right\}$$
$$-\frac{(\beta-\alpha)\bar{B}_u(i,n-i+1)(\text{Ḩ}^{(\alpha,\beta)}_{X_{i:n}}(u))'}{u^{(\alpha+\beta-1)(i-1)}(1-u)^{(\alpha+\beta-1)(n-i)}}. \tag{24}$$

Equation (24) provides a direct relationship between quantile density function $q(u)$ and $\text{Ḩ}^{(\alpha,\beta)}_{X_{i:n}}(u)$, therefore $\text{Ḩ}^{(\alpha,\beta)}_{X_{i:n}}(u)$ uniquely determines the underlying distribution.

The i^{th} order statistic $X_{i:n}$ represents the lifetime of an $(n-i+1)$-out-of-n system which is a common structure of redundancy and widely used in reliability theory and survival analysis. It is to be noted that $X_{1:n}$ represents the lifetime of a series system, whereas $X_{n:n}$ that of a parallel system. The *quantile-based generalized residual entropy of order* (α,β) (QGRE (α,β)), of order statistics (22) for $i=1$ and $i=n$, that are, the lifetime of the series systems and parallel systems, respectively for several well known distributions are provided in Table 2.

TABLE 2
QGRE (α,β) of first and last order statistics for some common distributions.

Distribution	$\text{Ḩ}^{(\alpha,\beta)}_{X_{1:n}}(u)$	$\text{Ḩ}^{(\alpha,\beta)}_{X_{n:n}}(u)$
Uniform	$=\frac{1}{\beta-\alpha}\log\left(\frac{n^{\alpha+\beta-1}\{(b-a)(1-u)\}^{2-\alpha-\beta}}{n(\alpha+\beta-1)+2-\alpha-\beta}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(b-a)^{2-\alpha-\beta}(1-u)^{n(\alpha+\beta-1)+2-\alpha-\beta)}}{n^{1-\alpha-\beta}\{n(\alpha+\beta-1)+2-\alpha-\beta\}(1-u^n)^{\alpha+\beta-1}}\right)$
Exponential	$=\frac{1}{\beta-\alpha}\log\left(\frac{(n\lambda)^{\alpha+\beta-2}}{\alpha+\beta-1}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{(\alpha+\beta-1)(n-1)+1,\alpha+\beta-1\}}{n^{1-\alpha-\beta}\lambda^{2-\alpha-\beta}(1-u^n)^{\alpha+\beta-1}}\right)$
Pareto-I	$=\frac{1}{\beta-\alpha}\log\left(\frac{(na)^{\alpha+\beta-1}(1-u)^{\frac{\alpha+\beta-2}{a}}}{\{na(\alpha+\beta-1)+\alpha+\beta-2\}b^{\alpha+\beta-2}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{(\alpha+\beta-1)(n-1)+1,\alpha+\beta-1+\frac{\alpha+\beta-2}{a}\}}{n^{1-\alpha-\beta}(\frac{a}{b})^{2-\alpha-\beta}(1-u^n)^{\alpha+\beta-1}}\right)$
Log-logistic	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{\alpha+\beta-1+\frac{(2-\alpha-\beta)}{b},n(\alpha+\beta-1)+\frac{\alpha+\beta-2}{b}\}}{n^{1-\alpha-\beta}(1-u)^{n(\alpha+\beta-1)}(ab)^{2-\alpha-\beta}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{n(\alpha+\beta-1)+\frac{2-\alpha-\beta}{b},\alpha+\beta-1+\frac{(\alpha+\beta-2)}{b}\}}{n^{1-\alpha-\beta}(ab)^{2-\alpha-\beta}(1-u^n)^{\alpha+\beta-1}}\right)$
GPD	$=\frac{1}{\beta-\alpha}\log\left(\frac{n^{\alpha+\beta-1}(\frac{b}{a+1})^{2-\alpha-\beta}(1-u)^{\frac{a(\alpha+\beta-2)}{a+1}}}{\{n(\alpha+\beta-1)+\frac{a(\alpha+\beta-2)}{a+1}\}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{(\alpha+\beta-1)(n-1)+1,(\alpha+\beta-2)+\frac{2a+1}{a+1}+1\}}{n^{1-\alpha-\beta}(\frac{a+1}{b})^{2-\alpha-\beta}(1-u^n)^{\alpha+\beta-1}}\right)$
Finite range	$=\frac{1}{\beta-\alpha}\log\left(\frac{(na)^{\alpha+\beta-1}(1-u)^{\frac{2-\alpha-\beta}{a}}}{\{na(\alpha+\beta-1)+2-\alpha-\beta\}b^{\alpha+\beta-2}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{(\alpha+\beta-1)(n-1)+1,\alpha+\beta-1+\frac{2-\alpha-\beta}{a}\}}{n^{1-\alpha-\beta}(\frac{b}{a})^{\alpha+\beta-2}(1-u^n)^{\alpha+\beta-1}}\right)$
Power	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{\alpha+\beta-1+\frac{2-\alpha-\beta}{b},(n-1)(\alpha+\beta-1)+1\}}{n^{1-\alpha-\beta}(\frac{a}{b})^{\alpha+\beta-2}(1-u)^{n(\alpha+\beta-1)}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{(\frac{a}{b})^{2-\alpha-\beta}(1-u)^{n(\alpha+\beta-1)+\frac{2-\alpha-\beta}{b}}}{n^{1-\alpha-\beta}\{n(\alpha+\beta-1)+\frac{2-\alpha-\beta}{b}\}(1-u^n)^{\alpha+\beta-1}}\right)$
Govindarajulu	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{\alpha+\beta-1+b(2-\alpha-\beta),(\alpha+\beta-1)(n-2)+2\}}{n^{1-\alpha-\beta}\{ab(b+1)\}^{\alpha+\beta-2}(1-u)^{n(\alpha+\beta-1)}}\right)$	$=\frac{1}{\beta-\alpha}\log\left(\frac{\bar{B}_u\{n(\alpha+\beta-1)+b(2-\alpha-\beta),3-\alpha-\beta\}}{n^{1-\alpha-\beta}\{ab(b+1)\}^{\alpha+\beta-2}(1-u^n)^{\alpha+\beta-1}}\right)$

4. CHARACTERIZATION BASED ON SAMPLE MINIMA $X_{1:n}$

Let $X_{1:n}$ and $X_{n:n}$ be the first (minima) order statistic and last (maxima) order statistic in a random sample $\{X_1,X_2,\cdots,X_n\}$ of size n from a positive and continuous random variable X. Then the cdf and pdf of first order statistics $X_{1:n}$ are respectively given by

$$F_{1:n}(x)=1-\bar{F}^n(x)$$

and

$$f_{1:n}(x) = n\bar{F}^{n-1}(x)f(x).$$

Then QGRE (α, β) (23) of first order statistics $X_{1:n}$ is given by

$$H_{X_{1:n}}^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha}\log\left\{\frac{n^{\alpha+\beta-1}}{(1-u)^{n(\alpha+\beta-1)}}\int_u^1 (1-p)^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp\right\}. \quad (25)$$

An important quantile measure useful reliability analysis is the hazard quantile function defined by

$$K(u) = h(Q(u)) = \frac{fQ(u)}{(1-u)} = \frac{1}{(1-u)q(u)}, \quad (26)$$

where $h(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of X. By considering a relationship between the (α, β) and the *hazard quantile function* $K(u)$ of $X_{1:n}$, we characterize some specific lifetime distributions based on the quantile entropy measure (25). We give the following theorem.

THEOREM 5. *Let $X_{1:n}$ be a first order statistics with survival function $\bar{F}_{1:n}(x)$ and hazard quantile function $K_{X_{1:n}}(u)$. Then*

$$\underset{\sim}{H}_{X_{1:n}}^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha}\log c + \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log K_{X_{1:n}}(u), \quad (27)$$

where c is constant, characterize generalized Pareto distribution (GPD) with quantile function $Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right]$.

PROOF. The hazard quantile function of sample minima that is $X_{1:n}$ for GPD is given as

$$K_{X_{1:n}}(u) = \frac{f_{1:n}(Q(u))}{(1-F(Q(u)))^n} = \frac{n}{(1-u)q(u)} = \frac{n(a+1)(1-u)^{(\frac{a}{a+1})}}{b}.$$

From Table 1, the quantile-based generalized residual entropy of order (α, β) of GPD is

$$\begin{aligned}\underset{\sim}{H}_{X_{1:n}}^{(\alpha,\beta)}(u) &= \frac{1}{\beta-\alpha}\log\frac{n(a+1)}{n(a+1)(\alpha+\beta-1)+a(\alpha+\beta-2)} \\ &+ \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log\left(\frac{n(a+1)(1-u)^{\frac{a}{a+1}}}{b}\right).\end{aligned}$$

This prove the if part of the Theorem. To prove the only if part, let (27) holds. Then

$$\left\{\frac{n^{\alpha+\beta-1}}{(1-u)^{n(\alpha+\beta-1)}}\int_u^1 (1-p)^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp\right\} = c(K_{X_{1:n}}(u))^{\alpha+\beta-2}.$$

Substituting the value of $K_{X_{1:n}}(u)$ and simplifying, this gives

$$n\int_u^1 (1-p)^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}\,dp = c(1-u)^{n(\alpha+\beta-1)-(\alpha+\beta-2)}(q(u))^{2-\alpha-\beta}.$$

Differentiating both sides with respect to u, we get

$$\frac{q'(u)}{q(u)} = \left(\frac{c\{n(\alpha+\beta-1)-(\alpha+\beta-2)\}-n}{c(2-\alpha-\beta)}\right)\left(\frac{1}{1-u}\right).$$

Solving this differential equation, which gives

$$q(u) = A(1-u)^{\left[\frac{n}{c}-\{n(\alpha+\beta-1)-(\alpha+\beta-2)\}\right].\frac{1}{2-\alpha-\beta}}.$$

Substituting the value of $c = \frac{n(a+1)}{n(a+1)(\alpha+\beta-1)+a(\alpha+\beta-2)}$, we obtain

$$q(u) = A(1-u)^{-\left(\frac{2a+1}{a+1}\right)},$$

which characterizes the generalized Pareto distribution. Hence proved. □

COROLLARY 6. *Let $X_{1:n}$ be a first order statistics with hazard quantile function $K_{X_{1:n}}(u)$ and quantile-based generalized residual entropy $\underset{\sim}{H}{}^{(\alpha,\beta)}_{X_{1:n}}(u)$ given by*

$$\underset{\sim}{H}{}^{(\alpha,\beta)}_{X_{1:n}}(u) = \frac{1}{\beta-\alpha}\log c + \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log K_{X_{1:n}}(u). \tag{28}$$

If, and only if for (i) $c = \frac{1}{\alpha+\beta-1}$, X follows exponential distribution (ii) $c < \frac{1}{\alpha+\beta-1}$, X follows Pareto II distribution (iii) $c > \frac{1}{\alpha+\beta-1}$, X follows finite range distribution.

In the following, we give the characterization result of some well known distributions in terms of QGRE (α,β) for the sample minima $X_{1:n}$.

THEOREM 7. *For a non-negative random variable X, the relationship*

$$(\beta-\alpha)\frac{d}{du}(\underset{\sim}{H}{}^{(\alpha,\beta)}_{X_{1:n}}(u)) = \frac{C}{(1-u)}, \tag{29}$$

where C is constant holds, then X has
(i) a uniform distribution if and only if $C = \alpha+\beta-2$,
(ii) a exponential distribution if and only if $C = 0$,
(iii) a Pareto I distribution if and only if $C = -\frac{\alpha+\beta-2}{a}$.

PROOF. The necessary part follows from the Table 2. For the sufficiency part, let us assume that (29) holds. Equation (25) can be rewritten as

$$e^{(\beta-\alpha)\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u)}(1-u)^{n(\alpha+\beta-1)} = n^{\alpha+\beta-1}\int_u^1 (1-p)^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp.$$

Taking derivative with respect to u, after some algebraic simplification we get

$$(\beta-\alpha)\frac{d}{du}(\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u)) = \frac{n(\alpha+\beta-1)}{(1-u)} + \frac{n^{\alpha+\beta-1}(q(u))^{2-\alpha-\beta}}{(1-u)^{\alpha+\beta-1}}e^{(\alpha-\beta)\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u)}. \tag{30}$$

Using (29) in (30), we obtain

$$e^{(\alpha-\beta)\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u)} = \left(\frac{n(\alpha+\beta-1)-C}{n^{\alpha+\beta-1}}\right)\{(1-u)q(u)\}^{\alpha+\beta-2}.$$

Taking log both sides, this gives

$$(\alpha-\beta)\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u) = \log\left(\frac{n(\alpha+\beta-1)-C}{n^{\alpha+\beta-1}}\right) + (\alpha+\beta-2)\log\{(1-u)q(u)\}.$$

From (29) we obtain $(\beta-\alpha)\text{H}_{X_{1:n}}^{(\alpha,\beta)}(u) = -C\log(1-u)+\log A$. Substitute this value in the above expression we get

$$(\alpha+\beta-2)\log q(u) = \{C-(\alpha+\beta-2)\}\log(1-u) - \log A\left(\frac{n(\alpha+\beta-1)-C}{n^{\alpha+\beta-1}}\right).$$

Which leads to

$$q(u) = A_1(1-u)^{\left\{\frac{C}{(\alpha+\beta-2)}-1\right\}},$$

where $A_1 = A\left(\frac{n(\alpha+\beta-1)-C}{n^{\alpha+\beta-1}}\right)^{\frac{1}{2-\alpha-\beta}}$ is constant. Now if $C = (\alpha+\beta-2)$ and $A_1 = b-a$; then X follows uniform distribution. If $C = 0$ and $A_1 = \frac{1}{\lambda}$; $\lambda \geq 0$ then X follows exponential distribution with parameter λ ; and if $C = -\frac{\alpha+\beta-2}{a}$ and $A_1 = \frac{b}{a}$, then we have $q(u) = \frac{b}{a}(1-u)^{-(1+\frac{1}{a})}$, this implies that X follows Pareto I distribution. □

Another useful measure closely related to hazard QF is the mean residual quantile function, as given by

$$M(u) = m(Q(u)) = (1-u)^{-1}\int_u^1 (1-p)q(p)dp\,, \tag{31}$$

where $m(t)=E(X-t|X>t)=\frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)}$ is the mean residual life (MRL) function of X. Further the relationship between the quantile density function and mean residual quantile function is given by

$$q(u)=\frac{M(u)-(1-u)M'(u)}{(1-u)}.$$

For the sample minima $X_{1:n}$ the above relationship becomes

$$(1-u)q(u)=nM_{X_{1:n}}(u)-(1-u)M'_{X_{1:n}}(u). \tag{32}$$

We state a characterization result using the relationship between *quantile-based generalized residual entropy* of order (α,β) and the *mean quantile function* $M(u)$ of first order statistics $X_{1:n}$.

THEOREM 8. *Let $X_{1:n}$ be the first order statistics with mean residual quantile function $M_{X_{1:n}}(u)$. Then*

$$H_{X_{1:n}}^{(\alpha,\beta)}(u)=\frac{1}{\beta-\alpha}\log c+\left(\frac{2-\alpha-\beta}{\beta-\alpha}\right)\log M_{X_{1:n}}(u), \tag{33}$$

where c is constant, characterize generalized Pareto distribution.

PROOF. The mean residual quantile function of the first order statistics $X_{1:n}$ is given by

$$\begin{aligned} M_{X_{1:n}}(u) &= m(X_{1:n};Q(u))=\frac{\int_{Q(u)}^\infty \bar{F}_{1:n}(Q(p))d(Q(p))}{\bar{F}_{1:n}(Q(u))} \\ &= \frac{\int_{Q(u)}^\infty (\bar{F}(Q(p)))^n d(Q(p))}{(\bar{F}(Q(u)))^n}=(1-u)^{-n}\int_u^1 (1-p)^n q(p)dp. \end{aligned} \tag{34}$$

Thus mean residual quantile function of $X_{1:n}$ having a GPD is $M(X_{1:n};u)=\frac{b(1-u)^{-\frac{a}{a+1}}}{n(a+1)-a}$. From Table 1, the quantile-based generalized residual entropy of GPD is

$$H_{X_{1:n}}^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha}\log\left(\frac{(1-\frac{a}{n(a+1)})^{2-\alpha-\beta}}{(\alpha+\beta-1)+\frac{a(\alpha+\beta-2)}{n(a+1)}}\right)+\left(\frac{2-\alpha-\beta}{\beta-\alpha}\right)\log\left(\frac{b(1-u)^{-\frac{a}{a+1}}}{n(a+1)-a}\right).$$

This prove the if part of the Theorem. Assume that (33) holds. Then

$$n^{\alpha+\beta-1}\int_u^1 (1-p)^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp=c(1-u)^{n(\alpha+\beta-1)}(M_{X_{1:n}}(u))^{2-\alpha-\beta}.$$

Differentiating both sides with respect to u, we get

$$n\left(\frac{n}{(1-u)q(u)}\right)^{\alpha+\beta-2} = (M_{X_{1:n}}(u))^{2-\alpha-\beta} \times\left(nc(\alpha+\beta-1)-c(2-\alpha-\beta)(1-u)\frac{M'_{X_{1:n}}(u)}{M_{X_{1:n}}(u)}\right).$$

This can be rewritten as

$$n\left(K_{X_{1:n}}(u)M_{X_{1:n}}(u)\right)^{\alpha+\beta-2} = \left(nc(\alpha+\beta-1)-c(2-\alpha-\beta)(1-u)\frac{M'_{X_{1:n}}(u)}{M_{X_{1:n}}(u)}\right). \quad (35)$$

Using (32) in (35), we get

$$\left(K_{X_{1:n}}(u)M_{X_{1:n}}(u)\right)^{\alpha+\beta-1} + \{2c(\alpha+\beta-1)-c\}K_{X_{1:n}}(u)M_{X_{1:n}}(u) = c(2-\alpha-\beta). \quad (36)$$

Substituting $p(u) = K_{X_{1:n}}(u)M_{X_{1:n}}(u)$ in (36), this gives

$$(p(u))^{\alpha+\beta-1} + c\{2(\alpha+\beta-1)-1\}p(u) = c(2-\alpha-\beta),$$

which leads to $p(u) = \theta$, a constant. This means that $K_{X_{1:n}}(u)M_{X_{1:n}}(u) = \theta$. From (32), we have

$$\frac{n}{K_{X_{1:n}}(u)M_{X_{1:n}}(u)} = n-(1-u)\frac{M'_{X_{1:n}}(u)}{M_{X_{1:n}}(u)},$$

which gives

$$\frac{M'_{X_{1:n}}(u)}{M_{X_{1:n}}(u)} = \frac{n(1-\frac{1}{\theta})}{(1-u)}.$$

Solving this differential equation yields

$$M_{X_{1:n}}(u) = A(1-u)^{-n(1-\frac{1}{\theta})},$$

which characterizes the GPD for $\theta = \left(\frac{n(a+1)}{n(a+1)-a}\right)$. Hence, we have the required result. □

5. GENERALIZED QUANTILE ENTROPY OF $X_{i:n}$ FOR INACTIVITY TIME

In some practical situations, uncertainty is related to past life time rather than future. Based on this idea, Di Crescenzo and Longobardi (2002) have considered the entropy of the inactivity time or reversed residual lifetime ${}_tX = [t - X|X \leq t]$ given as

$$\bar{H}(f;t) = -\int_0^t \frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx\,. \quad (37)$$

Similarly, the generalized entropy of order (α,β) for inactivity time ${}_tX$ was given by

$$\bar{H}^{(\alpha,\beta)}(f;t)=\frac{1}{\beta-\alpha}\log\left(\int_0^t \frac{f^{\alpha+\beta-1}(x)}{F^{\alpha+\beta-1}(t)}dx\right);\ \beta\neq\alpha,\ \beta-1<\alpha<\beta,\ \beta\geq 1, \quad (38)$$

which is extensively studied by Nanda and Paul (2006). The past lifetime random variable ${}_tX$ is related with *the reversed hazard rate* function defined by $\bar{K}_F(x)=\frac{f(x)}{F(x)}$. The quantile versions of reversed hazard rate function is defined as

$$\bar{K}(u)=\bar{K}(Q(u))=\frac{f(Q(u))}{F(Q(u))}=(uq(u))^{-1}. \quad (39)$$

The reversed hazard rate function is quite useful in the forensic science, where exact time of failure (e.g. death in case of human beings) of a unit is of importance.
Analogous to (38), the generalized past entropy (GPE) of i^{th} order statistics $X_{i:n}$ is defined as

$$\begin{aligned}\bar{H}^{(\alpha,\beta)}(f_{i:n};t) &= \frac{1}{\beta-\alpha}\log\left(\frac{\int_0^t (f_{i:n}(x))^{\alpha+\beta-1}dx}{(F_{i:n}(t))^{\alpha+\beta-1}}\right);\ t>0,\\ &= \frac{1}{\beta-\alpha}\log\left(\frac{\int_0^t (f_{i:n}(x))^{\alpha+\beta-1}dx}{(B_{F(t)}(i,n-i+1))^{\alpha+\beta-1}}\right). \quad (40)\end{aligned}$$

Where $\beta_{F(x)}(i,n-i+1)$ is the distribution function of the i^{th} order statistics. In terms of quantile function (40) can be expressed as follows

$$\begin{aligned}\bar{\text{H}}^{(\alpha,\beta)}_{X_{i:n}}(u) &= \frac{1}{\beta-\alpha}\log\left(\frac{1}{(B_{F(Q(u))}(i,n-i+1))^{\alpha+\beta-1}}\int_0^u f_{i:n}^{\alpha+\beta-1}(Q(u))d(Q(u))\right)\\ &= \frac{1}{\beta-\alpha}\\ &\times\log\left(\frac{1}{(B_u(i,n-i+1))^{\alpha+\beta-1}}\int_0^u \frac{p^{(\alpha+\beta-1)(i-1)}(1-p)^{(\alpha+\beta-1)(n-i)}}{(B(i,n-i+1))^{\alpha+\beta-1}}(q(p))^{2-\alpha-\beta}dp\right). \quad (41)\end{aligned}$$

The measure (41) may be considered as the *quantile-based generalized past entropy of order* (α,β) *(QGPE* (α,β)*)* measure for i^{th} order statistics of inactivity time. Last order statistics ia an important case of order statistics. For $i=n$, we have $f_{n:n}(x)=nF^{n-1}(x)f(x)$ and $F_{n:n}(x)=(F(x))^n$. Thus quantile-based generalized past entropy for sample maxima $X_{n:n}$ is defined as

$$\begin{aligned}\bar{\text{H}}^{(\alpha,\beta)}_{X_{n:n}}(u) &= \frac{1}{\beta-\alpha}\log\left(\frac{1}{(B_u(n,1))^{\alpha+\beta-1}}\int_0^u \frac{p^{(\alpha+\beta-1)(n-1)}}{(B(n,1))^{\alpha+\beta-1}}(q(p))^{2-\alpha-\beta}dp\right)\\ &= \frac{1}{\beta-\alpha}\log\left(\frac{n^{2(\alpha+\beta-1)}}{u^{n(\alpha+\beta-1)}}\int_0^u p^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp\right). \quad (42)\end{aligned}$$

In the following theorem we characterize the power distribution, when $\bar{H}_{X_{n:n}}^{(\alpha,\beta)}(u)$ is expressed in terms of quantile-based reversed hazard rate function. We give the following result.

THEOREM 9. *Let $X_{n:n}$ denote the last order statistics with survival function $\bar{F}_{n:n}(x)$ and the reversed hazard quantile function $\bar{K}_{X_{n:n}}(u)$, then*

$$\bar{H}_{X_{n:n}}^{(\alpha,\beta)}(u) = \frac{1}{\beta-\alpha}\log c + \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log \bar{K}_{X_{n:n}}(u),\ \alpha+\beta>2 \tag{43}$$

if and only if X follows the power distribution.

PROOF. The reversed hazard quantile function for sample maxima $X_{n:n}$ of power distribution is $\bar{K}_{X_{n:n}}(u) = \frac{f_{n:n}(Q(u))}{F_{n:n}(Q(u))} = \frac{nf(Q(u))}{F(Q(u))} = n(uq(u))^{-1} = \frac{nb}{au^{\frac{1}{b}}}$.
Taking $c = \left(\frac{bn^{\alpha+\beta}}{nb(\alpha+\beta-1)-(\alpha+\beta-2)}\right)$ gives the if part of the theorem. To prove the only if part, consider (43) to be valid. Using (42), it gives

$$\frac{n^{2(\alpha+\beta-1)}}{u^{n(\alpha+\beta-1)}}\int_0^u p^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp = c(\bar{K}_{X_{n:n}}(u))^{\alpha+\beta-2}.$$

Substituting $\bar{K}_{X_{n:n}}(u) = \frac{n}{uq(u)}$ and simplify, gives

$$n^{\alpha+\beta}\int_0^u p^{(\alpha+\beta-1)(n-1)}(q(p))^{2-\alpha-\beta}dp = cu^{n(\alpha+\beta-1)-(\alpha+\beta-2)}(q(u))^{2-\alpha-\beta}.$$

Taking derivative with respect to u and simplifying, this reduces to

$$\frac{q'(u)}{q(u)} = \left(\frac{n^{\alpha+\beta} - c[n(\alpha+\beta-1)-(\alpha+\beta-2)]}{c(2-\alpha-\beta)}\right)\frac{1}{u},$$

which leads to

$$q(u) = Au^{\left(\frac{n^{\alpha+\beta}-c[n(\alpha+\beta-1)-(\alpha+\beta-2)]}{c(2-\alpha-\beta)}\right)} = Au^{\frac{1}{b}-1},$$

which characterizes the power distribution function. □

REMARK 10. *If $c = \left(\frac{n^{\alpha+\beta}}{n(\alpha+\beta-1)-(\alpha+\beta-2)}\right)$, then Equation (43) is a characterization of uniform distribution.*

6. Conclusion

The quantile-based generalized entropy and generalized information measure of order (α, β) for order statistics has several advantages. The computation of proposed measures is quite simple in cases where the distribution functions are not tractable while the quantile functions have simpler forms. Furthermore, there are certain properties of quantile functions that are not shared by probability distributions. Applications of these properties give some new results and better insight into the measure that are difficult to obtain in the conventional approach. The results obtained in this article are general in the sense that they reduce to some of the results for quantile based Shannon entropy and Rényi entropy for order statistics when parameters approaches to unity.

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SUMMARY

In the present paper, we propose a quantile version of generalized entropy measure for order statistics for residual and past lifetimes and study their properties. Lower and upper bound of the proposed measures are derived. It is shown that the quantile-based generalized information between i^{th} order statistics and parent random variable is distribution free. The uniform, exponential, generalized Pareto and finite range distributions, which are commonly used in the reliability modeling have been characterized in terms of the proposed entropy measure with extreme order statistics.

Keywords: Quantile function; Quantile entropy; Reliability measures; Series and parallel system.