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A NEW GENERALIZATION OF THE FRÉCHET DISTRIBUTION: PROPERTIES AND APPLICATION

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1. Introduction

Maurice Fréchet (1878-1973), a French mathematician, introduced the Fréchet distribution. This distribution is also known as the type II extreme value distribution and is a family of continuous probability distributions developed within the general extreme value theory, which deals with the stochastic behavior of the maximum and minimum of independent and identically distributed (i.i.d.) random variables (Kotz and Nadarajah, 2000). Fréchet distribution has found to be a better model for describing stochastic phenomena, like the lifetime of components and analyzing the extreme events such as floods, earthquakes, rainfall, wind speed, sea currents, etc.

The random variable X is said to follow Fréchet distribution with the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$, if the cumulative distribution function (cdf) is given by

$$G(x) = e^{-(\frac{\beta}{x})^{\alpha}}, x > 0. \quad (1)$$

This distribution is equivalent to taking the reciprocal of values from a standard Weibull distribution. Extreme value theories shows that the Weibull distribution is a better model for the minimum of large number of independent positive random variables, where as, the Fréchet distribution is a better model for the maximum of a large number of random variables from a certain class of distributions, see Abbas and Tang (2015).

There have been growing interest in developing new classes of distributions for modelling a variety of data sets from various fields (see Jayakumar and Pillai, 1993; Pillai and Jayakumar, 1995; Jayakumar, 2003; Nadarajah *et al.*, 2013; Jayakumar and Babu, 2018).

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Gumbel (1965) studied the parameter estimation of Fréchet distribution. Nadarajah and Kotz (2008) discussed the various sociological models based on Fréchet random variables. Several extensions of the Fréchet distribution are discussed in the literature. Some of them are as follows: the exponentiated Fréchet (EF) distribution in Nadarajah and Kotz (2003), the beta Fréchet (BF) distribution in Barreto-Souza *et al.* (2011), the Marshall-Olkin Fréchet (MOF) distribution in Krishna *et al.* (2013), the transmuted Fréchet (TF) distribution in Mahmoud and Mandouh (2013), the gamma extended Fréchet distribution in da Silva *et al.* (2013), the transmuted exponentiated Fréchet (TEF) distribution in Elbatal *et al.* (2014), the Kumaraswamy Fréchet distribution in Mead and Abd-Eltawab (2014), the transmuted Marshall-Olkin Fréchet distribution in Afify *et al.* (2015), the Kumaraswamy transmuted Marshall-Olkin Fréchet (KTMOF) distribution in Yousof *et al.* (2016), the Weibull Fréchet distribution in Afify *et al.* (2016), the beta exponential Fréchet distribution in Mead *et al.* (2017), the Burr-X exponentiated Fréchet (BXEF) distribution in Zayed and Butt (2017) and the generalized transmuted Fréchet (GTFr) distribution in Nofal and Ahsanullah (2019). In the present paper, we study the properties and application of a new generalization of Fréchet distribution, namely, exponential transmuted Fréchet (ETF) distribution.

This paper is organized as follows. In Section 2, we introduce the ETF distribution and provide its sub models. In Section 3, some structural properties of ETF distribution, including the quantile function, moments, moment generating function and order statistics are studied. The method of maximum likelihood is used to estimate the unknown parameters of the model and a simulation study is carried out to check the performance of the MLEs of the model parameters. These results are presented in Section 4. In Section 5, we study empirically the flexibility of ETF distribution by using a real data set. Finally, conclusions are presented in Section 6.

2. A NEW GENERALIZATION OF THE FRÉCHET DISTRIBUTION

Even though there are many generalizations of Fréchet distribution are available in the literature, the complexity in modelling extreme values demands more flexible distributions. To generate one such flexible generalization of Fréchet distribution, we consider the T-transmuted X family of Jayakumar and Babu (2017), which is defined as

$$F(x) = R\left\{-\ln\left[1 - G(x)\left(1 + \lambda\bar{G}(x)\right)\right]\right\}, \tag{2}$$

where $R\{t\}$ is the cdf of the random variable T with probability density function (pdf) $r(t)$, $G(x)$ is the base distribution function, $\bar{G}(x) = 1 - G(x)$ and $|\lambda| \leq 1$.
We consider a member of this family, where $R(t)$ follows exponential distribution with cdf $R(t) = 1 - e^{-\theta t}, \theta > 0, t > 0$, and $G(x)$ is the Fréchet distribution with cdf, $G(x) =$

$e^{-(\frac{\beta}{x})^\alpha}, \alpha > 0, \beta > 0, x > 0$. Then

$$F(x) = 1 - \left[1 - e^{-(\frac{\beta}{x})^\alpha}\left[1 + \lambda(1 - e^{-(\frac{\beta}{x})^\alpha})\right]\right]^\theta, x > 0. \tag{3}$$

We call the distribution (3) as exponential transmuted Fréchet(ETF) distribution with parameters $\alpha > 0, \beta > 0, \theta > 0$ and $|\lambda| \leq 1$. The pdf, survival function and hazard rate function(hrf) of ETF distribution are respectively

$$f(x) = \theta\alpha\beta^\alpha x^{-(\alpha+1)} \frac{e^{-(\frac{\beta}{x})^\alpha}(1 + \lambda - 2\lambda e^{-(\frac{\beta}{x})^\alpha})}{\left[1 - e^{-(\frac{\beta}{x})^\alpha}\left[1 + \lambda(1 - e^{-(\frac{\beta}{x})^\alpha})\right]\right]^{1-\theta}}, \tag{4}$$

$$S(x) = 1 - F(x) = \left[1 - e^{-(\frac{\beta}{x})^\alpha}\left[1 + \lambda(1 - e^{-(\frac{\beta}{x})^\alpha})\right]\right]^\theta, \tag{5}$$

and

$$h(x) = \frac{f(x)}{1 - F(x)} = \theta\alpha\beta^\alpha x^{-(\alpha+1)} \frac{e^{-(\frac{\beta}{x})^\alpha}(1 + \lambda - 2\lambda e^{-(\frac{\beta}{x})^\alpha})}{1 - e^{-(\frac{\beta}{x})^\alpha}\left[1 + \lambda(1 - e^{-(\frac{\beta}{x})^\alpha})\right]}. \tag{6}$$

2.1. *Sub models*

The following are the sub models of the ETF distribution given in (3).

1. When $\lambda = 0$, exponentiated Fréchet distribution studied in Nadarajah and Kotz (2003).
2. When $\theta = 1$, transmuted exponentiated Fréchet distribution studied in Elbatal *et al.* (2014).
3. When $\theta = 1$ and $\beta = 1$, transmuted Fréchet distribution studied in Mahmoud and Mandouh (2013).
4. When $\theta = 1, \beta = 1$ and $\lambda = 0$, Fréchet distribution.
5. When $\theta = 1, \beta = 1$ and $\alpha = 1$, transmuted inverse exponential distribution studied in Oguntunde and Adejumo (2015).
6. When $\theta = 1, \beta = 1, \alpha = 1$ and $\lambda = 0$, inverse exponential distribution studied in Keller and Kamath (1982).
7. When $\theta = 1, \beta = 1$ and $\alpha = 2$, transmuted inverse Rayleigh distribution studied in Ahmad *et al.* (2014).
8. When $\theta = 1, \beta = 1, \alpha = 2$ and $\lambda = 0$, inverse Rayleigh distribution studied in Voda (1972).

3. Structural properties

The shape of the pdf of ETF distribution can be described analytically by examining the roots of the equation $\frac{\partial \ln(f(x))}{\partial x} = 0$. It can be easily seen that $\lim_{x\to\infty} f(x) = 0$. The following result shows $\lim_{x\to 0} f(x) = 0$.

Proposition 1. $\lim_{x\to 0} f(x) = 0$.

Proof. We have

$$\begin{aligned}\lim_{x\to 0} f(x) &= \theta\alpha\beta^{\alpha} \lim_{x\to 0}\left(x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\right)\\ &\quad \lim_{x\to 0}\left(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}}\right)\\ &\quad \lim_{x\to 0}\left(\left(1-e^{-(\frac{\beta}{x})^{\alpha}}\left[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\right]\right)^{\theta-1}\right).\end{aligned} \tag{7}$$

Since $x > 0, \alpha > 0$ and $\beta > 0$, we have $0 \leq e^{-(\frac{\beta}{x})^{\alpha}} \leq 1$.
Next, we can show that $\lim_{x\to 0}\left(x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\right) = 0$.
We know that for all $n \in \mathbf{N}$, $\lim_{x\to\infty} x^n e^{-x} = 0$.
Letting $u = (\frac{\beta}{x})^{\alpha}$, we have, $x = \frac{\beta}{u^{\frac{1}{\alpha}}}$. Thus $x \to 0$ if and only if $u \to \infty$ and therefore,

$$\lim_{x\to 0}\left(x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\right) = \lim_{u\to\infty} \frac{u^{\frac{\alpha+1}{\alpha}} e^{-u}}{\beta^{\alpha+1}}. \tag{8}$$

Now, let $n \in \mathbf{N}$ such that $\frac{\alpha+1}{\alpha} \leq n$.
Then for $u \geq 1$, we have $u^{\frac{\alpha+1}{\alpha}} \leq u^n$ and thus

$$0 \leq \lim_{u\to\infty} \frac{u^{\frac{\alpha+1}{\alpha}} e^{-u}}{\beta^{\alpha+1}} \leq \lim_{u\to\infty} \frac{u^n e^{-u}}{\beta^{\alpha+1}} = 0. \tag{9}$$

Thus

$$\lim_{x\to 0}\left(x^{-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\right) = 0. \tag{10}$$

Using (10) in (7), we obtain $\lim_{x\to 0} f(x) = 0$. □

Since $\lim_{x\to\infty} f(x) = \lim_{x\to 0} f(x) = 0$, the pdf of ETF distribution must have at least one mode.

We have

$$\begin{aligned}\frac{\partial \ln(f(x))}{\partial x} &= \frac{-(\alpha+1)}{x}+\frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}-\frac{2\lambda\alpha\beta^{\alpha}e^{-(\frac{\beta}{x})^{\alpha}}}{x^{\alpha+1}(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}})}\\ &\quad -(1-\theta)\frac{\frac{\alpha\beta^{\alpha}}{x^{\alpha+1}}\big[\lambda(2e^{-(\frac{\beta}{x})^{\alpha}}-1)-1\big]}{1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big]}=0. \qquad (11)\end{aligned}$$

Here the Equation (11) may have more than one root. If $x = x_0$ is a root, then it corresponds to a local maximum if $\frac{\partial^2 \ln(f(x))}{\partial x^2} < 0$, a local minimum if $\frac{\partial^2 \ln(f(x))}{\partial x^2} > 0$, and a point of inflection if $\frac{\partial^2 \ln(f(x))}{\partial x^2} = 0$.
From (11) we have

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{-(\alpha+1)+\alpha(\frac{\beta}{x})^{\alpha}}{x}-\frac{2\lambda\alpha(\frac{\beta}{x})^{\alpha}e^{-(\frac{\beta}{x})^{\alpha}}}{x(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}})}\\ &\quad -\frac{(1-\theta)(\frac{\beta}{x})^{\alpha}(\frac{\alpha}{x})e^{-(\frac{\beta}{x})^{\alpha}}(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}})}{1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big]}\\ &= \frac{s(x)}{x(1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}})(1-e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\big])}, \qquad (12)\end{aligned}$$

where

$$\begin{aligned}s(x) &= \big[\alpha(\frac{\beta}{x})^{\alpha}-\alpha-1\big]\big[1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}}\big]\\ &\quad \big[1-e^{-(\frac{\beta}{x})^{\alpha}}[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})]\big]\\ &\quad -2\lambda\alpha(\frac{\beta}{x})^{\alpha}e^{-(\frac{\beta}{x})^{\alpha}}\big[1-e^{-(\frac{\beta}{x})^{\alpha}}[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})]\big]+\\ &\quad \alpha(1-\theta)(\frac{\beta}{x})^{\alpha}e^{-(\frac{\beta}{x})^{\alpha}}\big[1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}}\big]^2. \qquad (13)\end{aligned}$$

Now, put $y = (\frac{\beta}{x})^{\alpha}$. Since $x > 0$, we have $y > 0$. Then

$$\begin{aligned}s(y) &= \big[1-e^{-y}(1+\lambda)+\lambda e^{-2y})\big]\\ &\quad \big[(1+\lambda)(y(\alpha-1)-1)-2\lambda e^{-y}(y(2\alpha-1)-1)\big]\\ &\quad +\alpha(1-\theta)ye^{-y}\big[1+\lambda-2\lambda e^{-y}\big]^2\\ &= u(y)v(y)+w(y), \qquad (14)\end{aligned}$$

where, $u(y) = 1-e^{-y}(1+\lambda)+\lambda e^{-2y})$, $v(y) = (1+\lambda)(y(\alpha-1)-1)-2\lambda e^{-y}(y(2\alpha-1)-1)$ and $w(y) = \alpha(1-\theta)ye^{-y}\big[1+\lambda-2\lambda e^{-y}\big]^2$. The function $u(y)$ is positive for $y > 0$ and

$|\lambda| \leq 1$. Also note that $s(y) < 0$ for $0 < \alpha \leq 1, \theta > 1$ and $|\lambda| \leq 1$. Hence, $s(x) < 0$ and $f(x)$ is a decreasing function. In all other cases, $f(x)$ is a unimodal function and the mode is obtained by solving the non linear Equation (11). Some possible shapes of the pdf and hrf for selected parameter values for the ETF distribution are presented in Figure 1 and Figure 2 respectively. These figures shows the flexibility of the ETF distribution. The hrf is initially increasing and then decreasing.

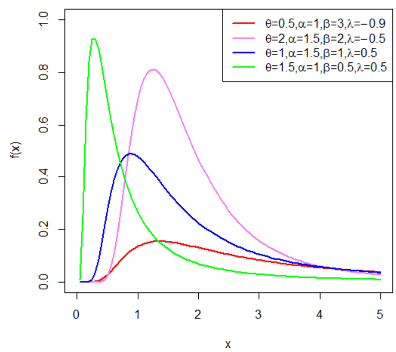

Figure 1 – Plot of the pdf of ETF distribution for given parameter values.

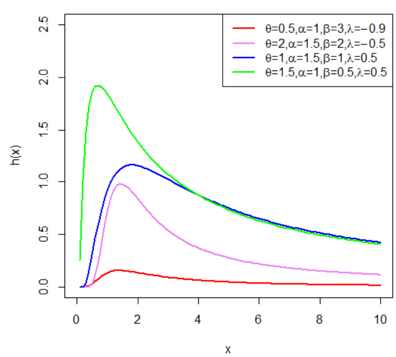

Figure 2 – Plot of the hrf of ETF distribution for given parameter values.

3.1. Quantile function

The following theorem gives the quantile function of ETF distribution.

THEOREM 2. *The u^{th} quantile $\phi(u)$ of ETF distribution is given by*

$$\phi(u)=\begin{cases}\beta\left[-\ln\left(\frac{(1+\lambda)-[(1-\lambda)^2+4\lambda(1-u)^{\frac{1}{\theta}}]^{\frac{1}{2}}}{2\lambda}\right)\right]^{-\frac{1}{\alpha}}, & \text{if } |\lambda|\le 1 \text{ and } \lambda\neq 0,\\ \beta\left[-\ln(1-(1-u)^{\frac{1}{\theta}})\right]^{-\frac{1}{\alpha}}, & \text{if } \lambda=0.\end{cases} \tag{15}$$

PROOF. Using (3), we have

$$\begin{aligned} & 1-\left[1-e^{-(\frac{\beta}{x_u})^\alpha}\left[1+\lambda(1-e^{-(\frac{\beta}{x_u})^\alpha})\right]\right]^\theta=u\\ \Rightarrow\quad & \lambda e^{-2(\frac{\beta}{x_u})^\alpha}-(1+\lambda)e^{-(\frac{\beta}{x_u})^\alpha}+\left[1-(1-u)^{\frac{1}{\theta}}\right]=0. \end{aligned} \tag{16}$$

Put $k=e^{-(\frac{\beta}{x_u})^\alpha}$, and this implies $x_u=\beta[-\ln(k)]^{-\frac{1}{\alpha}}$.
Now from (16),

$$\lambda k^2-(1+\lambda)k+\left[1-(1-u)^{\frac{1}{\theta}}\right]=0. \tag{17}$$

Here (17) is a quadratic equation in k and the possible root is

$$k=\frac{(1+\lambda)-\left[(1-\lambda)^2+4\lambda(1-u)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}}{2\lambda}, \tag{18}$$

where $\lambda\neq 0$. That is

$$\phi(u)=x_u=\beta\left[-\ln\left(\frac{(1+\lambda)-\left[(1-\lambda)^2+4\lambda(1-u)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}}{2\lambda}\right)\right]^{-\frac{1}{\alpha}}.$$

Now if $\lambda=0$, then we have

$$F(x)=1-[1-e^{-(\frac{\beta}{x})^\alpha}]^\theta,$$

which implies

$$\phi(u)=\beta\left[-\ln(1-(1-u)^{\frac{1}{\theta}})\right]^{-\frac{1}{\alpha}}.$$

This completes the proof. □

Using (15) we can generate random numbers from ETF distribution.
In particular when $u=0.5$, the median is given by

$$\text{Median}=\phi(0.5)=\beta\left[-\ln\left(\frac{(1+\lambda)-\left[(1-\lambda)^2+\lambda 2^{2-\frac{1}{\theta}}\right]^{\frac{1}{2}}}{2\lambda}\right)\right]^{-\frac{1}{\alpha}}. \tag{19}$$

The skewness and kurtosis can be defined based on the quantile function. The Galton's skewness S and the Moors kurtosis K are, respectively

$$
\begin{aligned}
S &= \frac{Q(\frac{6}{8})-2Q(\frac{4}{8})+Q(\frac{2}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}, \\
K &= \frac{Q(\frac{7}{8})-Q(\frac{5}{8})+Q(\frac{3}{8})-Q(\frac{1}{8})}{Q(\frac{6}{8})-Q(\frac{2}{8})}.
\end{aligned}
$$

For $S = 0$, the distribution is symmetric, when $S > 0$ (or $S < 0$), the distribution is right (or left) skewed. As the value of kurtosis increases, the tail of the distribution becomes heavier. Table 1, shows the changes of skewness and kurtosis for various parameter values of the ETF distribution. Here we can observe that the skewness and kurtosis of the ETF distribution, (i) decreases as α increases and β, θ, λ are fixed , (ii) decreases as θ increases and α, β, λ are fixed , (iii) remains constant as β increases and α, θ, λ are fixed, and (iv) increasing when $-1 \leq \lambda < 0$ and decreasing when $0 \leq \lambda \leq 1$, for fixed α, β and θ.

TABLE 1
Measures of skewness and kurtosis of ETF distribution for given parameter values.

α	β	θ	λ	Skewness (S)	Kurtosis (K)
0.5	0.5	0.5	-1.0	0.910	16.730
			-0.5	0.913	16.815
			0.0	0.911	16.730
			0.5	0.885	14.962
			1.0	0.729	4.529
1.0	1.0	1.0	1.0	0.330	1.599
		1.5		0.260	1.460
		2.0		0.230	1.397
		5.0		0.144	1.292
1.0	0.5	1.0	0.9	0.351	1.681
	1.0			0.351	1.681
	2.0			0.351	1.681
	5.0			0.351	1.681
0.5	0.5	0.5	0.9	0.778	6.857
1.0				0.527	2.680
2.0				0.350	1.788
5.0				0.231	1.479

3.2. Moments

The following theorem gives the r^{th} raw moment of the ETF distribution.

THEOREM 3. *If X has the ETF distribution with* $|\lambda| \leq 1$, *then the* r^{th} *raw moment is given by*

$$\begin{aligned} \mu_r^{'}(x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}\theta\beta^r(1+\lambda)\binom{\theta-1}{j}\binom{j}{k}\binom{k}{l}\frac{\Gamma(1-\frac{r}{\alpha})}{(1+j+l)^{1-\frac{r}{\alpha}}} \\ &- \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}2(-1)^{j+l}\theta\beta^r\lambda^{k+1}\binom{\theta-1}{j}\binom{j}{k}\binom{k}{l}\frac{\Gamma(1-\frac{r}{\alpha})}{(2+j+l)^{1-\frac{r}{\alpha}}}. \end{aligned} \tag{20}$$

PROOF. We have

$$\begin{aligned} \mu_r^{'}(x) = E(X^r) &= \int_0^{\infty} x^r f(x) dx \\ &= \int_0^{\infty} \frac{\theta\alpha\beta^{\alpha}x^{r-(\alpha+1)}e^{-(\frac{\beta}{x})^{\alpha}}\left[1+\lambda-2\lambda e^{-(\frac{\beta}{x})^{\alpha}}\right]}{\left[1-e^{-(\frac{\beta}{x})^{\alpha}}\left[1+\lambda(1-e^{-(\frac{\beta}{x})^{\alpha}})\right]\right]^{1-\theta}}\, dx. \end{aligned} \tag{21}$$

Let $t = (\frac{\beta}{x})^{\alpha}$, then $x = \beta t^{-(\frac{1}{\alpha})}$. Therefore (21) becomes

$$\mu_r^{'}(x) = \int_0^{\infty} \frac{\theta\beta^r t^{-(\frac{r}{\alpha})}e^{-t}\left[1+\lambda-2\lambda e^{-t}\right]}{\left[1-e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^{1-\theta}}\, dt. \tag{22}$$

Using the expansion $(1-z)^{\theta-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\theta-1}{j}z^j$, we have

$$\left[1-e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^{\theta-1} = \sum_{j=0}^{\infty}(-1)^j\binom{\theta-1}{j}\left[e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^j.$$

Thus

$$\begin{aligned}
\mu_r'(x) &= \int_0^\infty \theta\beta^r t^{-(\frac{r}{\alpha})} e^{-t}\left[1+\lambda-2\lambda e^{-t}\right] \\
&\quad \sum_{j=0}^{\infty}(-1)^j \binom{\theta-1}{j}\left[e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^j dt \\
&= \int_0^\infty \theta\beta^r t^{-(\frac{r}{\alpha})} e^{-t}(1+\lambda) \\
&\quad \sum_{j=0}^{\infty}(-1)^j \binom{\theta-1}{j}\left[e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^j dt \\
&\quad -\int_0^\infty 2\theta\lambda\beta^r t^{-(\frac{r}{\alpha})} e^{-2t} \\
&\quad \sum_{j=0}^{\infty}(-1)^j \binom{\theta-1}{j}\left[e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^j dt \\
&= I_1 - I_2,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
I_1 &= \int_0^\infty \theta\beta^r t^{-(\frac{r}{\alpha})} e^{-t}(1+\lambda) \\
&\quad \sum_{j=0}^{\infty}(-1)^j \binom{\theta-1}{j}\left[e^{-t}\left[1+\lambda(1-e^{-t})\right]\right]^j dt \\
&= \int_0^\infty \theta(1+\lambda)\beta^r t^{-(\frac{r}{\alpha})} \\
&\quad \sum_{j=0}^{\infty}(-1)^j \binom{\theta-1}{j} e^{-(1+j)t} \sum_{k=0}^{\infty}\binom{j}{k}\left[\lambda(1-e^{-t})\right]^k dt \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\theta(1+\lambda)\lambda^k\beta^r(-1)^{j+l}\binom{\theta-1}{j}\binom{j}{k}\binom{k}{l} \\
&\quad \int_0^\infty t^{(\frac{r}{\alpha})} e^{-(1+j+l)t} dt \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}(-1)^{j+l}\theta(1+\lambda)\beta^r\binom{\theta-1}{j}\binom{j}{k}\binom{k}{l}\frac{\Gamma(1-\frac{r}{\alpha})}{(1+j+l)^{1-\frac{r}{\alpha}}}.
\end{aligned} \tag{24}$$

Similarly

$$
\begin{aligned}
I_2 &= \int_0^\infty 2\theta\lambda\beta^r t^{-(\frac{r}{\alpha})} e^{-2t} \sum_{j=0}^{\infty} (-1)^j \binom{\theta-1}{j} \Big[e^{-t}\big[1+\lambda(1-e^{-t})\big]\Big]^j \, dt \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2(-1)^{j+l}\theta\beta^r\lambda^{k+1} \binom{\theta-1}{j}\binom{j}{k}\binom{k}{l} \frac{\Gamma(1-\frac{r}{\alpha})}{(2+j+l)^{1-\frac{r}{\alpha}}}. \qquad (25)
\end{aligned}
$$

Substituting (24) and (25) in (23) we get the result (20).
This completes the proof. □

3.3. *Moment generating function*

The moment generating function (mgf) of the ETF distribution is given in the following theorem.

THEOREM 4. *If X has the ETF distribution with* $|\lambda| \leq 1$, *then the mgf is given by*

$$
M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r^{'}(x), \qquad (26)
$$

where

$$
\begin{aligned}
\mu_r^{'}(x) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{j+l}\theta\beta^r(1+\lambda) \binom{\theta-1}{j}\binom{j}{k}\binom{k}{l} \frac{\Gamma(1-\frac{r}{\alpha})}{(1+j+l)^{1-\frac{r}{\alpha}}} \\
&- \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} 2(-1)^{j+l}\theta\beta^r\lambda^{k+1} \binom{\theta-1}{j}\binom{j}{k}\binom{k}{l} \frac{\Gamma(1-\frac{r}{\alpha})}{(2+j+l)^{1-\frac{r}{\alpha}}}.
\end{aligned}
$$

Proof follows easily.

3.4. *Order statistics*

Let $X_1, X_2, ..., X_n$ be a random sample of size n taken from the ETF$(\alpha, \beta, \theta, \lambda)$ distribution and let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denotes the corresponding order statistics. Then for the

k^{th} order statistic, say $Z = X_{(k)}$, the pdf and cdf are respectively given by

$$\begin{aligned} f_Z(z) &= \frac{n!}{(k-1)!(n-k)!} F^{k-1}(z)[1-F(z)]^{n-k} f(z) \\ &= \frac{n!}{(k-1)!(n-k)!} \Big[\theta\alpha\beta^\alpha z^{-(\alpha+1)} e^{-(\frac{\beta}{z})^\alpha}(1+\lambda-2\lambda e^{-(\frac{\beta}{z})^\alpha})\Big] \\ &\quad \Big[1-\big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\big]^\theta\Big]^{k-1} \\ &\quad \Big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\Big]^{\theta(n-k+1)-1}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} F_Z(z) &= \sum_{j=k}^{n} \binom{n}{j} F^j(z)[1-F(z)]^{n-j} \\ &= \sum_{j=k}^{n} \binom{n}{j} \Big[1-\big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\big]^\theta\Big]^j \\ &\quad \Big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\Big]^{\theta(n-j)}. \end{aligned} \tag{28}$$

The pdf of minimum is

$$\begin{aligned} f_{X_{(1)}}(z) &= n\theta\alpha\beta^\alpha z^{-(\alpha+1)} e^{-(\frac{\beta}{z})^\alpha}(1+\lambda-2\lambda e^{-(\frac{\beta}{z})^\alpha}) \\ &\quad \Big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\Big]^{n\theta-1}, \end{aligned} \tag{29}$$

and the pdf of the maximum is

$$\begin{aligned} f_{X_{(n)}}(z) &= n\Big[\theta\alpha\beta^\alpha z^{-(\alpha+1)} e^{-(\frac{\beta}{z})^\alpha}(1+\lambda-2\lambda e^{-(\frac{\beta}{z})^\alpha})\Big] \\ &\quad \Big[1-\big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\big]^\theta\Big]^{n-1} \\ &\quad \Big[1-e^{-(\frac{\beta}{z})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{z})^\alpha})\big]\Big]^{\theta-1}. \end{aligned} \tag{30}$$

4. Maximum likelihood estimates of the parameters

In this section, we obtain the maximum likelihood estimates (MLEs) of the parameters of ETF$(\alpha, \beta, \theta, \lambda)$ distribution. Let $x_1, x_2, ..., x_n$ be a random sample of size n from

ETF$(\alpha,\beta,\theta,\lambda)$ distribution. The likelihood function L is given by

$$L=\theta^n\alpha^n\beta^{n\alpha}\frac{e^{-\sum_{i=1}^n(\frac{\beta}{x_i})^\alpha}\prod_{i=1}^n x_i^{-(\alpha+1)}\prod_{i=1}^n(1+\lambda-2\lambda e^{-(\frac{\beta}{x_i})^\alpha})}{\prod_{i=1}^n\Big[1-e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{x_i})^\alpha})\big]\Big]^{1-\theta}}. \tag{31}$$

The log-likelihood function can be written as

$$\begin{aligned}\log(L) &= n\log(\theta)+n\log(\alpha)+n\alpha\log(\beta)-\beta^\alpha\sum_{i=1}^n x_i^{-\alpha}\\ &\quad-(\alpha+1)\sum_{i=1}^n\log(x_i)+\sum_{i=1}^n\log(1+\lambda-2\lambda e^{-(\frac{\beta}{x_i})^\alpha})\\ &\quad+(\theta-1)\sum_{i=1}^n\log\Big[1-e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{x_i})^\alpha})\big]\Big].\end{aligned} \tag{32}$$

The MLE, $\hat{\mathbf{e}}=(\hat\alpha,\hat\beta,\hat\theta,\hat\lambda)^T$ of $\mathbf{e}=(\alpha,\beta,\theta,\lambda)^T$ is obtained by maximizing the log-likelihood function. We have

$$\begin{aligned}\frac{\partial\log(L)}{\partial\alpha} &= \frac{n}{\alpha}+n\log(\beta)-\beta^\alpha\log(\beta)\sum_{i=1}^n x_i^{-\alpha}+\beta^\alpha\sum_{i=1}^n x_i^{-\alpha}\log(x_i)\\ &\quad-\sum_{i=1}^n\log(x_i)+2\lambda\sum_{i=1}^n\frac{(\frac{\beta}{x_i})^\alpha\log(\frac{\beta}{x_i})e^{-(\frac{\beta}{x})^\alpha}}{1+\lambda-2\lambda e^{-(\frac{\beta}{x})^\alpha}}\\ &\quad+(\theta-1)\sum_{i=1}^n\frac{(\frac{\beta}{x_i})^\alpha\log(\frac{\beta}{x_i})e^{-(\frac{\beta}{x})^\alpha}\big[1+\lambda-2\lambda e^{-(\frac{\beta}{x})^\alpha}\big]}{1-e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{x_i})^\alpha})\big]}=0,\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial\log(L)}{\partial\beta} &= \frac{n\alpha}{\beta}-\frac{\alpha}{\beta}\sum_{i=1}^n(\frac{\beta}{x_i})^\alpha+\frac{2\lambda\alpha}{\beta}\sum_{i=1}^n\frac{(\frac{\beta}{x_i})^\alpha e^{-(\frac{\beta}{x_i})^\alpha}}{1+\lambda-2\lambda e^{-(\frac{\beta}{x})^\alpha}}\\ &\quad+(\theta-1)\sum_{i=1}^n\frac{\frac{\alpha}{\beta}(\frac{\beta}{x_i})^\alpha e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda-2\lambda e^{-(\frac{\beta}{x})^\alpha}\big]}{1-e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{x_i})^\alpha})\big]}=0,\end{aligned} \tag{34}$$

$$\frac{\partial\log(L)}{\partial\theta} = \frac{n}{\theta}+\sum_{i=1}^n\log\Big[1-e^{-(\frac{\beta}{x_i})^\alpha}\big[1+\lambda(1-e^{-(\frac{\beta}{x_i})^\alpha})\big]\Big]=0, \tag{35}$$

and

$$\frac{\partial \log(L)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2e^{-(\frac{\beta}{x_i})^{\alpha}}}{1+\lambda-2\lambda e^{-(\frac{\beta}{x_i})^{\alpha}}} - \sum_{i=1}^{n} \frac{(\theta-1)e^{-(\frac{\beta}{x_i})^{\alpha}}(1-e^{-(\frac{\beta}{x_i})^{\alpha}})}{1-e^{-(\frac{\beta}{x_i})^{\alpha}}\left[1+\lambda(1-e^{-(\frac{\beta}{x_i})^{\alpha}})\right]} = 0. \tag{36}$$

These equations cannot be solved analytically and the R software can be used to solve them numerically. The normal approximation of the MLE of $\mathbf{e}$ can be used for constructing the approximate confidence limits and for testing hypothesis on the parameters α, β, θ and λ. Under the conditions that are fulfilled for parameters in the interior of the parameter space, we have $\sqrt{n}(\hat{\mathbf{e}}-\mathbf{e}) \sim \mathbf{N}(0, \mathbf{K}_e^{-1})$, where $\sim$ means approximately distributed and $\mathbf{K}_e$ is the unit expected information matrix. The asymptotic behavior is valid if $\mathbf{K}_e = \lim_{n\to\infty} n^{-1}\mathbf{I}_n(\mathbf{e})$, where $\mathbf{I}_n(\mathbf{e})$ is the observed information matrix. The Fisher's information matrix is given by

$$I_X(\mathbf{e}) = \begin{bmatrix} -E(\frac{\partial^2 ln(L)}{\partial \alpha^2}) & -E(\frac{\partial^2 ln(L)}{\partial \alpha \partial \beta}) & -E(\frac{\partial^2 ln(L)}{\partial \alpha \partial \theta}) & -E(\frac{\partial^2 ln(L)}{\partial \alpha \partial \lambda}) \\ -E(\frac{\partial^2 ln(L)}{\partial \beta \partial \alpha}) & -E(\frac{\partial^2 ln(L)}{\partial \beta^2}) & -E(\frac{\partial^2 ln(L)}{\partial \beta \partial \theta}) & -E(\frac{\partial^2 ln(L)}{\partial \beta \partial \lambda}) \\ -E(\frac{\partial^2 ln(L)}{\partial \theta \partial \alpha}) & -E(\frac{\partial^2 ln(L)}{\partial \theta \partial \beta}) & -E(\frac{\partial^2 ln(L)}{\partial^2 \theta}) & -E(\frac{\partial^2 ln(L)}{\partial \theta \partial \lambda}) \\ -E(\frac{\partial^2 ln(L)}{\partial \lambda \partial \alpha}) & -E(\frac{\partial^2 ln(L)}{\partial \lambda \partial \beta}) & -E(\frac{\partial^2 ln(L)}{\partial \lambda \partial \theta}) & -E(\frac{\partial^2 ln(L)}{\partial^2 \lambda}) \end{bmatrix}. \tag{37}$$

Here, the ETF$(\alpha, \beta, \theta, \lambda)$ distribution satisfies the regularity conditions which are full filled for the parameters in the interior of the parameter space, but not on the boundary. Hence, the vector $\hat{\mathbf{e}}$ is consistent and asymptotically normal. That is, $\sqrt{I_X(\mathbf{e})}[\hat{\mathbf{e}}-\mathbf{e}]$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix. The Fisher's information matrix can be computed using the approximation,

$$I_X(\hat{\mathbf{e}}) \approx \begin{bmatrix} -\frac{\partial^2 ln(L)}{\partial \alpha^2}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \alpha \partial \beta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \alpha \partial \theta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \alpha \partial \lambda}|_{\hat{\mathbf{e}}} \\ -\frac{\partial^2 ln(L)}{\partial \beta \partial \alpha}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial^2 \beta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \beta \partial \theta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \beta \partial \lambda}|_{\hat{\mathbf{e}}} \\ -\frac{\partial^2 ln(L)}{\partial \theta \partial \alpha}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \theta \partial \beta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial^2 \theta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \theta \partial \lambda}|_{\hat{\mathbf{e}}} \\ -\frac{\partial^2 ln(L)}{\partial \lambda \partial \alpha}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \lambda \partial \beta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial \lambda \partial \theta}|_{\hat{\mathbf{e}}} & -\frac{\partial^2 ln(L)}{\partial^2 \lambda}|_{\hat{\mathbf{e}}} \end{bmatrix}. \tag{38}$$

We compute the maximized unrestricted and restricted log-likelihood ratio (LR) test statistic for testing on some ETF sub models. We can use the LR test statistic to check whether the ETF distribution for a given data set is statistically superior to the sub models. For example, $H_0 : \theta = 1$ versus $H_1 : \theta \neq 1$ is equivalent to compare the ETF distribution and transmuted exponentiated Fréchet (TGF) distribution and the LR test statistic reduce to $\omega = 2(l(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda}) - l(\hat{\alpha}', \hat{\beta}', 1, \hat{\lambda}'))$, where $(\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$ and $(\hat{\alpha}', \hat{\beta}', \hat{\lambda}')$ are the MLEs under H_1 and H_0, respectively. The test statistic ω is asymptotically (as $n \to \infty$) distributed as $\chi^2_{(k)}$, where k is the length of the parameter vector of interest. The LR test rejects H_0 if $\omega > \chi^2_{(k,\alpha)}$ where $\chi^2_{(k,\alpha)}$ denotes the upper $100(1-\alpha)\%$ quantile of the $\chi^2_{(k)}$ distribution.

4.1. Simulation study

This section explains the performance of the MLEs of the model parameters of the ETF distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. The algorithm for the simulation study is given below.

Step 1. Input the value of replication (N).

Step 2. Specify the sample size n and the values of the parameters α, β, θ and λ.

Step 3. Generate $u_i \sim$ Uniform $(0, 1)$, $i = 1, 2, ..., n$.

Step 4. Obtain the random observations from the ETF distribution using (15).

Step 5. Compute the MLEs of the four parameters.

Step 6. Repeat steps 3 to 5, N times.

Step 7. Compute the parameter estimate, standard error of estimate, average bias, mean square error (MSE) and coverage probability (CP) for each parameter.
Here the expected value of the estimator is $E(\hat{\mathbf{e}}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\mathbf{e}}_i$, with
$E(SE(\hat{\mathbf{e}})) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(-\frac{\partial^2 \log(L)}{\partial \mathbf{e}_i^2}\right)}$,
Average Bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{e}}_i - \mathbf{e})$, MSE$(\hat{\mathbf{e}}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\mathbf{e}}_i - \mathbf{e})^2$ and
Coverage Probability = Probability of $\mathbf{e}_i \in \left(\hat{\mathbf{e}}_i \pm 1.96\sqrt{-\frac{\partial^2 \log(L)}{\partial \mathbf{e}_i^2}}\right)$.

We take random samples of size n=50, 100, 200 and 500 respectively. The MLEs of the parameter vector $\mathbf{e} = (\alpha, \beta, \theta, \lambda)$ are determined by maximizing the log-likelihood function in (32) by using the *optim* package of R software based on each generated samples. This simulation is repeated 1000 times and the average estimate and its standard error, average bias, MSE and CP are computed and presented in Table 2. From Table 2, it can be seen that, as sample size increases the estimates of bias and MSE are decreasing. Also note that the CP values are quite close to the 95% nominal level.

TABLE 2
The parameter estimate, standard error, average bias, MSE and CP for given parameter values.

Parameter($\mathbf{e}$)	Samples(n)	$E(\hat{\mathbf{e}})(E(SE(\hat{\mathbf{e}})))$	Average bias	MSE	CP
$\alpha=0.5$	50	0.538(0.010)	0.027	0.002	85.2
	100	0.490(0.008)	-0.018	0.002	89.4
	200	0.499(0.005)	-0.003	0.001	90.8
	500	0.505(0.003)	0.005	0.001	94.3
$\beta=1.5$	50	1.715(0.060)	0.204	0.051	82.3
	100	1.638(0.051)	0.146	0.020	85.3
	200	1.476(0.044)	-0.034	0.002	89.1
	500	1.494(0.035)	-0.006	0.001	93.1
$\theta=3$	50	3.161(0.054)	0.146	0.036	87.5
	100	3.136(0.067)	0.125	0.017	90.9
	200	3.083(0.036)	0.068	0.007	92.2
	500	3.053(0.077)	0.043	0.003	94.4
$\lambda=-0.9$	50	-0.602(0.011)	0.318	0.079	83.8
	100	-0.650(0.010)	0.280	0.063	87.3
	200	-0.707(0.009)	0.179	0.037	89.8
	500	-0.761(0.008)	0.122	0.019	91.2
$\alpha=1.5$	50	1.942(0.072)	0.442	0.037	86.7
	100	1.781(0.059)	0.137	0.029	88.9
	200	1.713(0.052)	0.112	0.011	90.2
	500	1.668(0.033)	0.109	0.007	92.8
$\beta=0.5$	50	0.327(0.011)	-0.162	0.017	89.3
	100	0.399(0.008)	-0.148	0.011	90.1
	200	0.429(0.008)	-0.093	0.009	91.7
	500	0.473(0.007)	-0.041	0.004	93.4
$\theta=1$	50	1.319(0.218)	0.297	0.091	91.6
	100	1.212(0.173)	0.201	0.062	92.5
	200	1.114(0.109)	0.183	0.044	93.6
	500	1.105(0.098)	0.105	0.039	93.9
$\lambda=0.5$	50	0.869(0.028)	0.342	0.018	84.8
	100	0.814(0.022)	0.299	0.017	86.1
	200	0.722(0.019)	0.241	0.014	86.9
	500	0.631(0.017)	0.183	0.011	88.7

5. Data application

In this section, in order to show the flexibility of the ETF distribution to model real-life data, we use the data set represents the remission times (in months) of 128 bladder cancer patients (Lee and Wang, 2003). The data are as follows:

0.080 0.200 0.400 0.500 0.510 0.810 0.900 1.050 1.190 1.260 1.350 1.400 1.460 1.760 2.020 2.020 2.070 2.090 2.230 2.260 2.460 2.540 2.620 2.640 2.690 2.690 2.750 2.830 2.870 3.020 3.250 3.310 3.360 3.360 3.480 3.520 3.570 3.640 3.700 3.820 3.880 4.180 4.230 4.260 4.330 4.340 4.400 4.500 4.510 4.870 4.980 5.060 5.090 5.170 5.320 5.320 5.340 5.410 5.410 5.490 5.620 5.710 5.850 6.250 6.540 6.760 6.930 6.940 6.970 7.090 7.260 7.280 7.320 7.390 7.590 7.620 7.630 7.660 7.870 7.930 8.260 8.370 8.530 8.650 8.660 9.020 9.220 9.470 9.740 10.06 10.34 10.66 10.75 11.25 11.64 11.79 11.98 12.02 12.03 12.07 12.63 13.11 13.29 13.80 14.24 14.76 14.77 14.83 15.96 16.62 17.12 17.14 17.36 18.10 19.13 20.28 21.73 22.69 23.63 25.74 25.82 26.31 32.15 34.26 36.66 43.01 46.12 79.05.

The fit of the data set is compared with the sub models of the ETF distribution and the competitive models namely, Kumaraswamy Fréchet(KF) distribution, transmuted Marshall-Olkin Fréchet(TMOF) distribution and Weibull Fréchet(WF) distribution. The pdfs of these distributions are, respectively:

KF: $f(x) = ab\beta\theta^{\beta}x^{-\beta-a}e^{-a(\frac{\theta}{x})^{\beta}}(1-e^{-a(\frac{\theta}{x})^{\beta}})^{b-1}, x > 0;$

TMOF: $f(x) = \frac{\alpha\beta\sigma^{\beta}x^{-\beta-1}e^{-(\frac{\sigma}{x})^{\beta}}}{(\alpha+(1-\alpha)e^{-a(\frac{\theta}{x})^{\beta}})^2}\left[1+\lambda-\frac{2\lambda e^{-a(\frac{\theta}{x})^{\beta}}}{\alpha+(1-\alpha)e^{-a(\frac{\theta}{x})^{\beta}}}\right], x > 0;$

WF: $f(x) = ab\beta\alpha^{\beta}x^{-\beta-1}e^{-b(\frac{\alpha}{x})^{\beta}}(1-e^{-b(\frac{\alpha}{x})^{\beta}})^{-b-1}e^{-a(e^{-b(\frac{\alpha}{x})^{\beta}}-1)^{-b}}, x > 0.$

Descriptive statistics of the data set are given in Table 3.

TABLE 3
Descriptive statistics of the remission times (in months) of 128 bladder cancer patients.

Descriptive Statistics	
Sample size(n)	128.000
Mean	9.366
SD	10.508
Minimum	0.080
Maximum	79.050
Skewness	3.326
Kurtosis	16.154

The total time on test (TTT) curve of the given data set is plotted to obtain the empirical behavior of the hazard rate function. Figure 3, show that the hazard rate function of the data set is an upside-down bathtub shape. Also, the data set is highly positively skewed and leptokurtic and hence we choose to fit the ETF distribution for the given data set.

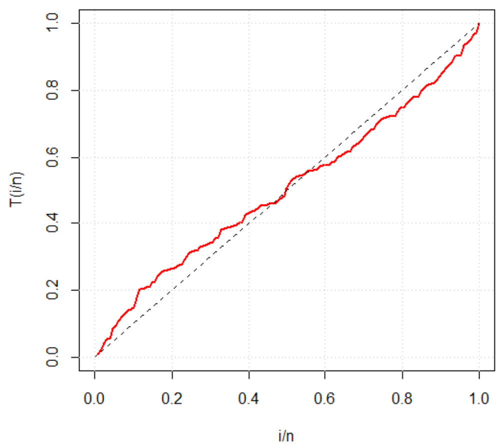

Figure 3 – Empirical TTT plot of the data set.

The estimates of the unknown parameters are obtained by the maximum-likelihood estimation method. To compare the distributions we consider the criteria like, Kolmogorov -Smirnov (K-S) statistic (the distance between the empirical cdf's and the fitted cdf's), Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion (CAIC), Cramér - von Mises criterion (W*) and Anderson - Darling criterion (A*). The best distribution corresponds to lower -logL, AIC, BIC, CAIC, K-S distance, A*, W* statistics and higher p value. Here, AIC=$-2\log L + 2k$, CAIC=$-2\log L + (\frac{2kn}{n-k-1})$ and BIC=$-2\log L + k\log n$ where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size. The K-S distance, $D_n = \sup_x |F(x) - F_n(x)|$, where, $F_n(x)$ is the empirical distribution function. Let $F(x;\mathbf{e})$ be the cdf and the form of F is known but $\mathbf{e}$ is unknown. Then the statistics W* and A* are computed as follows: (i) compute $\xi_i = F(x_i;\hat{\mathbf{e}})$ where the x_i's are in ascending order; (ii) compute $x_i = \phi^{-1}(\xi_i)$, where $\phi(.)$ is the normal cdf and $\phi^{-1}(.)$ is its inverse; (iii) compute $y_i = \phi((x_i - \bar{x})/s_x)$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$; (iv) calculate

$$W^2 = \sum_{i=1}^{n}\left(y_i - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}$$

and

$$A = -n - \frac{1}{n}\sum_{1=1}^{n}\Big[(2i-1)\log(y_i) + (2n+1-2i)\log(1-y_i)\Big];$$

(v) $W^* = W^2(1+\frac{0.5}{n})$ and $A^* = A^2(1+\frac{0.75}{n}+\frac{2.25}{n^2})$, see Chen and Balakrishnan (1995).

TABLE 4
The parameter estimates with standard error (SE) and -log(likelihood).

Model	ML estimates(SE)	-log L
F	$\hat{\alpha}=0.752(0.040)$, $\hat{\beta}=3.256(0.410)$	444.001
TMOF	$\hat{\alpha}=2.711(0.630)$, $\hat{\beta}=0.795(0.090)$, $\hat{\sigma}=0.445(0.150)$, $\hat{\lambda}=-0.999(0.030)$	438.799
TGF	$\hat{\alpha}=0.836(0.050)$, $\hat{\beta}=1.707(0.220)$, $\hat{\lambda}=-0.856(0.090)$	436.678
KF	$\hat{a}=1.969(0.210)$, $\hat{b}=54.159(19.650)$, $\hat{\beta}=0.241(0.030)$, $\hat{\theta}=168.832(16.430)$	412.473
WF	$\hat{\alpha}=118.595(34.540)$, $\hat{\beta}=0.209(0.010)$, $\hat{a}=36.738(7.850)$, $\hat{b}=2.377(0.100)$	411.511
ETF	$\hat{\alpha}=0.323(0.040)$, $\hat{\beta}=53.030(37.330)$, $\hat{\theta}=31.519(17.930)$, $\hat{\lambda}=-0.966(0.030)$	410.833

TABLE 5
Goodness of fit statistics.

Model	AIC	CAIC	BIC	A*	W*	K-S	p-value
F	892.002	892.098	897.706	6.121	0.980	0.427	$2.2\text{x}10^{-16}$
TMOF	885.599	885.924	897.007	6.859	1.146	0.155	0.004
TGF	879.356	879.549	887.912	4.588	0.698	0.124	0.039
KF	832.946	833.271	844.354	0.591	0.085	0.053	0.865
WF	831.023	831.348	842.431	0.411	0.063	0.055	0.839
ETF	829.666	829.991	841.074	0.236	0.030	0.039	0.989

The values in Table 4 and Table 5 show that the ETF distribution leads to better fit compared to the other five models. Figure 4, show the fitted cdfs with the empirical distribution of the data set.

The LR test statistic to test the hypothesis $H_0:\theta=1$ versus $H_1:\theta\neq 1$ for the given data set is $\omega=51.69>3.841=\chi^2_{(1,0.05)}$ with p value $6.499X10^{-13}$. So we reject the null hypothesis.

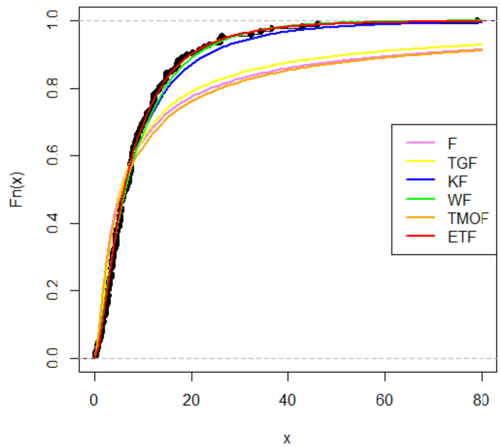

Figure 4 – Fitted cdf plots and the empirical distribution of the data set.

6. Conclusion

In this paper, we propose a new four-parameter model named as the exponential transmuted Fréchet (ETF) distribution, which extends the Fréchet distribution. We study some of its mathematical and statistical properties. The expressions for the quantile function, moments, moment generating function and order statistics are derived. The model parameters are estimated using maximum likelihood estimation method and a simulation study to illustrate the performance of the method is presented. The new distribution was applied to a real data set to show its flexibility for data modelling.

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SUMMARY

A new generalization of the Fréchet distribution is introduced and studied. Its structural properties including the quantile function, random number generation, moments, moment generating function and order statistics are investigated. The unknown parameters of the model are estimated using maximum likelihood estimation method and a simulation study is carried out to check the performance of the method. The new model is applied to a real data set to prove empirically its flexibility.

Keywords: Fréchet distribution; Hazard rate function; Maximum likelihood estimation; Moments; T-X family of distributions.