ESTIMATION IN INVERSE WEIBULL DISTRIBUTION BASED ON RANDOMLY CENSORED DATA

Kapil Kumar [1]
Department of Statistics, Central University of Haryana, Mahendergarh-123031, India
Indrajeet Kumar
Department of Statistics, Central University of Haryana, Mahendergarh-123031, India

1. Introduction

The Weibull distribution is the most popular and widely used lifetime model in reliability and life testing experiments because of the flexibility of its probability density and failure rate functions. The Weibull distribution can have increasing, decreasing or constant failure rate depending upon the values of its shape parameter. However, it has been found that the Weibull distribution does not provide satisfactory parametric fit for lifetime data depicting non-monotone failure rate pattern. This encourages authors to look for other plausible lifetime models. The failure rate function of inverse Weibull (IW) distribution is unimodal or decreasing depending on its shape parameter. There are various real life examples where data show the non-monotone, unimodel failure rate, like remission time of cancer patients, wind speed data, rainfall data, etc. Therefore, if the empirical study suggests that the failure rate function of the underlying distribution is of unimodal shape then the IW distribution may be used to analyze such data sets. Kundu and Howlader (2010) studied Bayesian inferences and prediction of the IW distribution for type II censored data, Sultan *et al.* (2014) discussed Bayesian and maximum likelihood estimation methods of the IW distribution parameters under progressive type II censoring, Akgül *et al.* (2016) used IW distribution for the wind speed data, Akgül and Şenoğlu (2018) compared different estimation methods for rainfall data fitted on IW distribution, Krishna *et al.* (2019) studied stress-strength reliability of IW distribution under progressive first failure censoring.

The probability density function (pdf), cumulative distribution function (cdf) and failure rate function of the IW distribution are, respectively, given by

$$f(x;\alpha,\beta)=\alpha\beta x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}\ ;\ x>0,\alpha>0,\beta>0, \tag{1}$$

[1] Corresponding Author. E-mail: kapilstats@gmail.com

$$F(x;\alpha,\beta)=e^{-\beta x^{-\alpha}}\ ;\ x>0,\alpha>0,\beta>0, \tag{2}$$

and

$$h(x;\alpha,\beta)=\frac{\alpha\beta x^{-(\alpha+1)}}{(e^{\beta x^{-\alpha}}-1)}\ ;\ x>0,\alpha>0,\beta>0, \tag{3}$$

where, α and β are the shape and scale parameters, respectively. The plot of the failure rate function for various values of the shape parameter α and scale parameter β is shown in Figure 1 (see Appendix A).

In life testing experiments, the data are frequently censored. Censoring arises in a life testing experiment, when exact lifetimes are known only for a portion of test items and remainder of the lifetimes are known only to exceed certain values under a life test. There are several types of censoring schemes which are used in life testing experiments. In literature, the two most popular censoring schemes are conventional Type I and Type II censoring schemes. These censoring schemes do not allow removal of units from the test at points other than the final termination point. Such intermittent removals are studied in progressive censoring, see, Balakrishnan and Aggarwala (2000), Kumar *et al.* (2017), Chaturvedi *et al.* (2018). Another type of censoring called random censoring occurs when the item under study is lost or removed from the experiment before its failure.

Random censoring is an important censoring in which the time of censoring is not fixed but taken as random. The random censoring arise in the situation when item or subject under study is lost or removed randomly from the experiment before its failure i.e. some items or subjects in the study have not experienced the event of interest at the end of the study. For example, in a clinical trial or a medical study, some patients may still be untreated and leave the course of treatment before its completion. In a social study, some subjects are lost for the follow-up in the middle of the survey. In reliability engineering, an electrical or electronic device such as bulb on test may break before its failure. In such cases, the exact survival time (or time to event of interest) of the subjects is unknown; therefore they are called randomly censored observations. The random censoring was first used in literature by Gilbert (1962) in his Ph.D. thesis. According to David and Moeschberger (1978), type I censoring is a particular case of random censoring in which censoring takes place at some fixed time point. Some early works on random censoring can also be found in Breslow and Crowley (1974), Koziol and Green (1976). Some recent studies on random censoring can be found in Ghitany and Al-Awadhi (2002), Kumar and Garg (2014), Krishna *et al.* (2015), Garg *et al.* (2016), Kumar (2018), Krishna and Goel (2018), etc.

Mathematically, random censoring can be described as follows: In random censoring, n patients are under study in identical conditions with their lifetimes as $X_1, X_2, \ldots, X_n$ which are independent and identically distributed (iid) random variables with pdf $f_X(x)$ and cdf $F_X(x)$. Also, their random censoring times are $T_1, T_2, \ldots, T_n$ with respective pdf and cdf $f_T(t)$ and $F_T(t)$. Let $X_i's$ and $T_i's$ be mutually independent. Note that, between $X_i's$ and $T_i's$, only one will be observed. Further, let the actual observed time be $Y_i = \min(X_i, T_i);\ i = 1, 2, \ldots, n$. Also, define the indicator variable D_i as $D_i = 1\ if\ X_i \leq T_i$

otherwise 0. Note that D_i is a random variable with Bernoulli probability mass function given by $p[D_i = j] = p^j(1-p)^{1-j}$; $j = 0,1$ and $p = [X_i \leq T_i]$. Since X_i and T_i are independent, will be Y_i and D_i, $\forall\, i = 1,2,\ldots,n$. Now, it is simple to show that the joint pdf of Y and D is

$$f_{Y,D}(y,d) = \{f_X(y)\bar{F}_T(y)\}^d\{f_T(y)\bar{F}_X(y)\}^{1-d} \ ; \ y > 0, d = 0,1$$

where, $\bar{F}_T(y) = 1 - F_T(y)$ and $\bar{F}_X(y) = 1 - F_X(y)$. Also, the probability of observing failure is given by

$$p = P[X \leq T] = \int_0^\infty F_X(t)dF_T(t) = \int_0^\infty \bar{F}_T(x)dF_X(x).$$

Now, let the lifetime X and censoring time T follow $IW(\alpha,\beta)$ and $IW(\alpha,\lambda)$, respectively. Then the joint pdf of randomly censored variables (Y,D) can be written as

$$f_{Y,D}(y,d,\alpha,\beta,\lambda) = \alpha\beta^d\lambda^{1-d}y^{-(\alpha+1)}e^{-y^{\alpha(\beta d+\lambda(1-d))}}\left(1-e^{-\lambda y^{-\alpha}}\right)^d\left(1-e^{-\beta y^{-\alpha}}\right)^{1-d} \tag{4}$$

and the probability of failure is obtained as

$$p = \frac{\lambda}{\beta+\lambda} \tag{5}$$

In this article, our main objective is to develop the classical and Bayesian estimation procedures in IW distribution based on the randomly censored data. Rest of the paper is organized as follows. In Section 2, we derive maximum likelihood estimators of the parameters and reliability characteristics like survival and failure rate functions. Also, asymptotic confidence intervals and coverage probabilities of the unknown parameters are constructed based on expected Fisher information matrix. Section 3 deals with expected test time of the experiment based on randomly censored data from IW distribution. In Section 4, Bayes estimators of the parameters and reliability characteristics under squared error loss function (SELF) with gamma informative and non-informative priors using Tierney-Kadane's approximation method and Markov chain Monte Carlo (MCMC) techniques are obtained. Highest posterior density (HPD) credible intervals for the parameters based on MCMC techniques are also developed. Section 5 deals with a simulation study to compare the performance of the estimators developed. In Section 6 findings are illustrated by a randomly censored real data set. Finally, article is concluded in Section 7.

2. Maximum likelihood estimation

Let $(\mathbf{y},\mathbf{d})=((y_1,d_1),(y_2,d_2),\ldots,(y_n,d_n))$ be a randomly censored sample from model in (4). The likelihood function is given by

$$L(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d})=\alpha^n\beta^m\lambda^{(n-m)}\prod_{i=1}^{n}y_i^{-(\alpha+1)}e^{-\left(\beta\sum_{i=1}^{n}d_iy_i^{-\alpha}+\lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)}$$
$$\prod_{i=1}^{n}\left(1-e^{-\lambda y_i^{-\alpha}}\right)^{d_i}\prod_{i=1}^{n}\left(1-e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{6}$$

where $m=\sum_{i=1}^{n}d_i$ denotes the number of failures.
Thus, the log-likelihood function becomes

$$l(\alpha,\beta,\lambda)=n\ln\alpha+m\ln\beta+(n-m)\ln\lambda-(\alpha+1)S-\beta\sum_{i=1}^{n}d_iy_i^{-\alpha}-\lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}$$
$$+\sum_{i=1}^{n}d_i\ln\left(1-e^{-\lambda y_i^{-\alpha}}\right)+\sum_{i=1}^{n}(1-d_i)\ln\left(1-e^{-\beta y_i^{-\alpha}}\right), \tag{7}$$

where $S=\sum_{i=1}^{n}\ln y_i$ denotes the log total time on test.
The corresponding normal equations are obtained as

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\alpha}=\frac{m}{\beta}-S+\beta\sum_{i=1}^{n}d_iy_i^{-\alpha}\ln y_i+\lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i$$
$$-\sum_{i=1}^{n}\frac{d_i\lambda e^{-\lambda y_i^{-\alpha}}y_i^{-\alpha}\ln y_i}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)}-\sum_{i=1}^{n}\frac{(1-d_i)\beta e^{-\beta y_i^{-\alpha}}\ln y_i}{\left(1-e^{-\beta y_i^{-\alpha}}\right)}=0 \tag{8}$$

$$\frac{\partial l(\alpha,\beta,\lambda)}{d\beta}=\frac{m}{\beta}-\sum_{i=1}^{n}d_iy_i^{-\alpha}+\sum_{i=1}^{n}\frac{(1-d_i)e^{-\beta y_i^{-\alpha}}y_i^{-\alpha}}{\left(1-e^{-\beta y_i^{-\alpha}}\right)}=0 \tag{9}$$

$$\frac{l(\alpha,\beta,\lambda)}{d\beta}=\frac{n-m}{\lambda}-\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}+\sum_{i=1}^{n}\frac{d_ie^{-\lambda y_i^{-\alpha}}y_i^{-\alpha}}{(1-e^{-\lambda y_i^{-\alpha}})}=0 \tag{10}$$

The MLEs of α, β and λ say $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively, are the solutions of the normal equations (8), (9) and (10). For the solution of the system of these normal equations, some suitable iterative procedure like Newton-Raphson method can be used. It is important to note that for the computation purpose, we have used a Broyden-Fletcher-Goldfarb-Shanno (BFGS) quasi-Newton method in this study. Once we get desired

MLEs using the invariance property of MLEs, see, Zehna (1966), the MLEs of the survival and failure rate functions, respectively, are obtained as

$$\hat{S}(t) = 1 - e^{-\hat{\beta} t^{-\hat{\alpha}}} \; ; \; t > 0 \quad \text{and} \quad \hat{h}(t) = \frac{\hat{\alpha}\hat{\beta} t^{-(\hat{\alpha}+1)}}{e^{\hat{\beta} x^{-\hat{\alpha}}} - 1} \; ; \; t > 0.$$

2.1. *Fisher information matrix*

Here, we derive Fisher information matrix for construction of asymptotic confidence intervals of the parameters. Zheng and Gastwirth (2001) suggested the expected Fisher information in randomly censored data using the failure rate functions. The Fisher information about parameters $\theta=(\alpha, \beta, \lambda)$ contained in randomly censored sample $(\mathbf{y}, \mathbf{d})$ of size n from model in (4) is given by

$$I^{Y,D}(\theta) = n \times \begin{bmatrix} I_{11}(\theta) & I_{12}(\theta) & I_{13}(\theta) \\ & I_{22}(\theta) & I_{23}(\theta) \\ & & I_{33}(\theta) \end{bmatrix}$$

where

$$\begin{aligned}
I_{11}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx + \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx \\
&= \alpha\beta \int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\beta x^{-\alpha} \ln x}{(1-e^{-\beta x^{-\alpha}})}\right)^2 x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} (1-e^{-\lambda x^{-\alpha}}) dx \\
&\quad + \alpha\lambda \int_0^\infty \left(\frac{1}{\alpha} - \ln x - \frac{\lambda x^{-\alpha} \ln x}{(1-e^{-\lambda x^{-\alpha}})}\right)^2 x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}} (1-e^{-\beta x^{-\alpha}}) dx, \\
I_{12}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_X(x)\right)\left(\frac{\partial}{\partial \beta} \ln h_X(x)\right) f_X \bar{F}_T(x)dx \\
&\quad + \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_T(x)\right)\left(\frac{\partial}{\partial \beta} \ln h_T(x)\right) f_T(x)\bar{F}_X(x)dx \\
&= \alpha\beta \int_0^\infty \left(\frac{1}{\alpha} - \ln x + \frac{\beta x^{-\alpha} \ln x}{(1-e^{-\beta x^{-\alpha}})}\right)\left(\frac{1}{\beta} + \frac{x^{-\alpha}}{(1-e^{-\beta x^{-\alpha}})}\right) \\
&\quad \times x^{-(\alpha+1)} e^{-\beta x^{-\alpha}} \left(1-e^{-\lambda x^{-\alpha}}\right) dx, \\
I_{13}(\theta) &= \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_X(x)\right)\left(\frac{\partial}{\partial \lambda} \ln h_X(x)\right) f_X(x)\bar{F}_T(x)dx \\
&\quad + \int_0^\infty \left(\frac{\partial}{\partial \alpha} \ln h_T(x)\right)\left(\frac{\partial}{\partial \lambda} \ln h_T(x)\right) f_T \bar{F}_X(x)dx \\
&= \alpha\lambda \int_0^\infty \left(\frac{1}{\alpha} - \ln x + \frac{\lambda x^{-\alpha} \ln x}{(1-e^{-\lambda x^{-\alpha}})}\right)\left(\frac{1}{\lambda} - \frac{x^{-\alpha}}{(1-e^{-\lambda x^{-\alpha}})}\right) \\
&\quad \times x^{-(\alpha+1)} e^{-\lambda x^{-\alpha}} \left(1-e^{-\beta x^{-\alpha}}\right) dx,
\end{aligned}$$

$$I_{22}(\theta)=\int_0^\infty\left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)^2 f_X(x)\bar{F}_T(x)dx+\int_0^\infty\left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)^2 f_T(x)\bar{F}_X(x)dx$$
$$=\alpha\beta\int_0^\infty\left(\frac{1}{\beta}-\frac{x^{-\alpha}}{(1-e^{-\beta x^{-\alpha}})}\right)^2 x^{-(\alpha+1)}e^{-\beta x^{-\alpha}}(1-e^{-\lambda x^{-\alpha}})dx,$$

$$I_{23}(\theta)=\int_0^\infty\left(\frac{\partial}{\partial\beta}\ln h_X(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_X(x)\right)f_X(x)\bar{F}_T(x)dx$$
$$+\int_0^\infty\left(\frac{\partial}{\partial\beta}\ln h_T(x)\right)\left(\frac{\partial}{\partial\lambda}\ln h_T(x)\right)f_T(x)\bar{F}_X(x)dx=0,$$

$$I_{33}(\theta)=\int_0^\infty(\frac{\partial}{\partial\lambda}\ln h_X(x))^2 f_X(x)\bar{F}_T(x)dx+\int_0^\infty(\frac{\partial}{\partial\lambda}\ln h_T(x))^2 f_T(x)\bar{F}_X(x)dx$$
$$=\alpha\lambda\int_0^\infty\left(\frac{1}{\lambda}-\frac{x^{-\alpha}}{(1-e^{-\lambda x^{-\alpha}})}\right)^2 x^{-(\alpha+1)}e^{-\lambda x^{-\alpha}}(1-e^{-\beta x^{-\alpha}})dx.$$

Here, h_X and h_T are the failure rate functions of $IW(\alpha,\beta)$ and $IW(\alpha,\lambda)$, respectively. The elements of the expected Fisher information matrix $I^{Y,D}(\theta)$ need to be computed numerically.

Under some mild regularity conditions, $\hat{\theta}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$ follows approximately trivariate normal distribution with mean $(\alpha,\ \beta,\ \lambda)$ and covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$. In practice, covariance matrix $\left[I^{Y,D}(\theta)\right]^{-1}$ is estimated by observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ to obtain the required asymptotic confidence intervals, see, Lawless (2003). Therefore, two sided equal tail 100(1-ξ)% asymptotic confidence intervals for the parameters α, β and λ are, respectively, given by

$(\hat{\alpha}\pm z_{\xi/2}\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\alpha})})$, $(\hat{\beta}\pm z_{\xi/2}\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\beta})})$ and $(\hat{\lambda}\pm z_{\xi/2}\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\lambda})})$.

Here, $\hat{\mathrm{V}}\mathrm{ar}(\hat{\alpha})$, $\hat{\mathrm{V}}\mathrm{ar}(\hat{\beta})$ and $\hat{\mathrm{V}}\mathrm{ar}(\hat{\lambda})$ are diagonal elements of the observed covariance matrix $\left[I^{Y,D}(\hat{\theta})\right]^{-1}$ and $z_{\xi/2}$ is the upper $(\xi/2)^{th}$ percentile of the standard normal distribution. Also. the coverage probabilities for the parameters α, β and λ are, respectively, given by

$CP_\alpha=\left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\alpha})}}\right|\le z_{\xi/2}\right]$, $CP_\beta=\left[\left|\frac{\hat{\beta}-\beta}{\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\beta})}}\right|\le z_{\xi/2}\right]$, and $CP_\lambda=\left[\left|\frac{\hat{\lambda}-\alpha}{\sqrt{\hat{\mathrm{V}}\mathrm{ar}(\hat{\lambda})}}\right|\le z_{\xi/2}\right]$.

3. Expected time on test

In this section, we study the expected time on test (ETT) of a randomly censored life testing experiment. In real life applications, ETT is useful to have an idea about the number of items to be put on test, the expected duration and cost of the life testing experiment. The following result is required for ETT.

THEOREM 1. *In randomly censored sampling plan, the expectation of the largest order statistic* $Z = \max(Y_1, Y_2, \ldots, Y_n)$ *is given by*

$$E[Z] = \int_0^\infty \left[1 - (1 - \bar{F}_X(z)\bar{F}_T(z))^n\right] dz.$$

PROOF. Since, $Y_i, i = 1, 2, \ldots, n$ are iid, the cdf of Z is given by

$$F_Z(z) = P[Z \le z] = P[\max(Y_1, Y_2, \ldots, Y_n) \le z] = \{P[Y_i \le z]\}^n \;; z > 0.$$

Note that

$$\begin{aligned} F_Y(z) &= P[Y_i \le z] = P\left[\min(X_i, T_i) \le z\right] = 1 - P[\min(X_i, T_i) > z] \\ &= 1 - P[X_i > z]P[T_i > z] = 1 - \bar{F}_X(z)\bar{F}_T(z). \end{aligned}$$

Therefore

$$E[Z] = \int_0^\infty (1 - F_Z(z)) dz = \int_0^\infty \left[1 - (1 - \bar{F}_X(z)\bar{F}_T(z))^n\right] dz.$$

□

Now, if the failure time X follows $IW(\alpha, \beta)$ and censoring time T follows $IW(\alpha, \lambda)$, the ETT for randomly censored experiment is given by

$$ETT = \int_0^\infty \left[1 - \{1 - (1 - e^{-\beta z^{-\alpha}})(1 - e^{-\lambda z^{-\alpha}})\}^n\right] dz. \tag{11}$$

ETT obtained in Equation (11) can be computed numerically for the given values of the parameters and the sample size n. Also, the observed time on the test (OBTT) is given by $OBTT = \max(y_1, y_2, \ldots, y_n)$. We compute, the average absolute bias (AB) and mean squared error (MSE) for OBTT based on 1,000 randomly censored simulated samples from the model in (4). The values of ETT and AB, MSE for OBTT under randomly censored IW distribution for different values of the parameters and sample size n are reported in Table 1 (see Appendix B). Table 1 shows that the OBTT estimates the ETT quite closely and efficiently.

4. Bayesian estimation

Here, we assume the following piecewise independent gamma priors for the parameters α, β and λ as

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} e^{-b_1\alpha} \ ; \alpha, a_1, b_1 > 0,$$
$$g_2(\beta) = \frac{b_2^{a_2}}{\Gamma(a_2)} \beta^{a_2-1} e^{-b_2\beta} \ ; \beta, a_2, b_2 > 0,$$
$$g_3(\lambda) = \frac{b_3^{a_3}}{\Gamma(a_3)} \lambda^{a_3-1} e^{-b_3\lambda} \ ; \lambda, a_3, b_3 > 0, \text{ respectively.}$$

Thus, the joint prior distribution of α, β and λ can be written as

$$g(\alpha,\beta,\lambda) \propto \alpha^{a_1-1}\beta^{a_2-1}\lambda^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\lambda)}. \tag{12}$$

The assumption of the piecewise independent gamma priors is quite reasonable. Many authors have used these priors on the shape and scale parameters of the IW distribution, see Singh *et al.* (2013); Krishna *et al.* (2019). It is noted that the non-informative priors are the special cases of independent gamma priors when hyper-parameters $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$ in (12).

Based on the observed randomly censored data $(\mathbf{y}, \mathbf{d})$, likelihood function in (6) and joint prior distribution of (α,β,λ) in (12), the joint posterior distribution of α, β and λ is given by

$$\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) = \frac{L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}$$

$$\pi(\alpha,\beta,\lambda|\mathbf{y},\mathbf{d}) \propto \alpha^{n+a_1-1}e^{-\alpha\left(b_1+\sum_{i=1}^{n}\ln y_i\right)}\beta^{m+a_2-1}e^{-\beta\left(b_2+\sum_{i=1}^{n}d_i y_i^{-\alpha}\right)}\lambda^{n-m+a_3-1}$$
$$e^{-\lambda\left(b_2+\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right)}\prod_{i=1}^{n}(1-e^{-\lambda y_i^{-\alpha}})^{d_i}\prod_{i=1}^{n}(1-e^{-\beta y_i^{-\alpha}})^{1-d_i}. \tag{13}$$

Therefore, the Bayes estimator of any function of α, β and λ, say, $\phi(\alpha,\beta,\lambda)$ under SELF is the posterior expectation of $\phi(\alpha,\beta,\lambda)$ and is given by

$$E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \phi(\alpha,\beta,\lambda)L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}{\int_0^\infty\int_0^\infty\int_0^\infty L(\mathbf{y},\mathbf{d}|\alpha,\beta,\lambda)g(\alpha,\beta,\lambda)d\alpha d\beta d\lambda}. \tag{14}$$

From the above Equation (14) we observe that the Bayes estimator is in the form of ratio of two integrals for which closed form solution is not available. The above ratio of integrals can be solved numerically. Here, we use Tierney-Kadane's (T-K) approximation method proposed by Tierney and Kadane (1986) and MCMC techniques like Gibbs sampling method and Metropolis-Hastings algorithm to derive Bayes estimates.

4.1. T-K approximation method

According to T-K approximation method, the approximate Bayes estimator of $\phi(\alpha,\beta,\lambda)$ under SELF is given by

$$\hat{\phi}_{TK} = E[\phi(\alpha,\beta,\lambda)|\mathbf{y},\mathbf{d}] = \frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{n\delta^*_\phi(\alpha,\beta,\lambda)} d\alpha d\beta d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty e^{n\delta(\alpha,\beta,\lambda)} d\alpha d\beta d\lambda}, \tag{15}$$

where $\delta(\alpha,\beta,\lambda) = \frac{1}{n}[l(\alpha,\beta,\lambda) + \rho(\alpha,\beta,\lambda)]$ and $\delta^*(\alpha,\beta,\lambda) = \left[\delta(\alpha,\beta,\lambda) + \frac{1}{n} ln\phi(\alpha,\beta,\lambda)\right]$, here, $l(\alpha,\beta,\lambda)$ is the log-likelihood function and $\rho(\alpha,\beta,\lambda) = \ln g(\alpha,\beta,\lambda)$.
The expression (15) is approximated by the T-K method as

$$\hat{\phi}(\alpha,\beta,\lambda) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}} e^{n[\delta^*_\phi(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)]}, \tag{16}$$

where $|\Sigma^*|$ and $|\Sigma|$ are the determinants of inverse of negative Hessian of $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$ at $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)$, respectively. Also, $(\hat{\alpha}_{\delta^*},\hat{\beta}_{\delta^*},\hat{\lambda}_{\delta^*})$ and $(\hat{\alpha}_\delta,\hat{\beta}_\delta,\hat{\lambda}_\delta)$ maximize $\delta^*(\alpha,\beta,\lambda)$ and $\delta(\alpha,\beta,\lambda)$, respectively. Next, we observe that

$$\begin{aligned}
\delta(\alpha,\beta,\lambda) = \frac{1}{n}\Big[&(n+a_1-1)\ln\alpha + (m+a_2-1)\ln\beta + (n-m+a_3-1)\ln\lambda - (\alpha+1)S \\
&-\beta\big(b_2 + \sum_{i=1}^n d_i y_i^{-\alpha}\big) - \lambda\big(b_3 + \sum_{i=1}^n (1-d_i) y_i^{-\alpha}\big) - b_1\alpha \\
&+\sum_{i=1}^n d_i \ln\big(1-e^{-\lambda y_i^{-\alpha}}\big) + \sum_{i=1}^n (1-d_i)\ln\big(1-e^{-\beta y_i^{-\alpha}}\big)\Big].
\end{aligned}$$

Then, $(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)$ are computed by solving the following non-linear equations

$$\frac{\partial \delta}{\partial \alpha} = \frac{n+a_1-1}{\alpha} - S + \beta\sum_{i=1}^{n} d_i y_i^{-\alpha}\ln y_i + \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i - b_1$$
$$-\lambda\sum_{i=1}^{n}\frac{d_i e^{-\lambda y^{-\alpha}} y_i^{-\alpha}\ln y_i}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)} - \beta\sum_{i=1}^{n}\frac{(1-d_i)e^{-\beta y_i^{-\alpha}} y_i^{-\alpha}\ln y_i}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0$$
$$\frac{\partial \delta}{\partial \beta} = \frac{m+a_2-1}{\beta} - \left(b_2 + \sum_{i=1}^{n} d_i y_i^{-\alpha}\right) + \sum_{i=1}^{n}\frac{(1-d_i)e^{-\beta y_i^{-\alpha}} y_i^{-\alpha}}{\left(1-e^{-\beta y_i^{-\alpha}}\right)} = 0$$
$$\frac{\partial \delta}{\partial \lambda} = \frac{n-m+a_3-1}{\lambda} - \left(b_3 + \sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\right) + \sum_{i=1}^{n}\frac{d_i e^{-\lambda y_i^{-\alpha}} y_i^{-\alpha}}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)} = 0.$$

Now, obtain $|\Sigma|$ from

$$\Sigma^{-1} = \frac{1}{n}\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix},$$

where

$$\delta_{11} = -\frac{\partial^2 \delta}{\partial \alpha^2} = \frac{n+a_1-1}{\alpha^2} + \beta\sum_{i=1}^{n} d_i y_i^{-\alpha}(\ln y_i)^2 + \lambda\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}(\ln y_i)^2$$
$$+\lambda\sum_{i=1}^{n}\frac{d_i y_i^{-\alpha}(\ln y_i)^2 e^{-\lambda y_i^{-\alpha}}(e^{-\lambda y_i^{-\alpha}} + \lambda y_i^{-\alpha} - 1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2}$$
$$+\beta\sum_{i=1}^{n}\frac{(1-d_i)y_i^{-\alpha}(\ln y_i)^2 e^{-\beta y_i^{-\alpha}}(e^{-\beta y_i^{-\alpha}} + \beta y_i^{-\alpha} - 1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2}$$
$$\delta_{12} = \delta_{21} = -\frac{\partial^2 \delta}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} d_i y_i^{-\alpha}\ln y_i - \sum_{i=1}^{n}\frac{(1-d_i)y_i^{-\alpha}\ln y_i e^{-\beta y_i^{-\alpha}}(e^{-\beta y_i^{-\alpha}} + \beta y_i^{-\alpha} - 1)}{\left(1-e^{-\beta y_i^{-\alpha}}\right)^2}$$
$$\delta_{13} = \delta_{31} = -\frac{\partial^2 \delta}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{n}(1-d_i)y_i^{-\alpha}\ln y_i - \sum_{i=1}^{n}\frac{d_i y_i^{-\alpha}\ln y_i e^{-\lambda y_i^{-\alpha}}(e^{-\lambda y_i^{-\alpha}} + \lambda y_i^{-\alpha} - 1)}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2}$$
$$\delta_{22} = -\frac{\partial^2 \delta}{\partial \beta^2} = \frac{m+a_2-1}{\beta^2} + \sum_{i=1}^{n}\frac{(1-d_i)y_i^{-2\alpha}e^{-\beta y_i^{-\alpha}}}{(1-e^{-\beta y_i^{-\alpha}})^2}, \quad \delta_{23} = \delta_{32} = -\frac{\partial^2 \delta}{\partial \beta \partial \lambda} = 0,$$
$$\delta_{33} = \frac{\partial^2 \delta}{\partial \lambda^2} = \frac{n-m+a_3-1}{\lambda^2} + \sum_{i=1}^{n}\frac{d_i y_i^{-2\alpha}e^{-\lambda y_i^{-\alpha}}}{\left(1-e^{-\lambda y_i^{-\alpha}}\right)^2}$$

In order to compute the Bayes estimator of α we take $\phi(\alpha,\beta,\lambda) = \alpha$ and accordingly function $\delta^*(\alpha,\beta,\lambda)$ becomes

$$\delta^*_\alpha(\alpha,\beta,\lambda) = \delta(\alpha,\beta,\lambda) + \frac{1}{n}\ln\alpha$$

, and then $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*})$ are obtained as solution of the following non-linear equations

$$\frac{\partial \delta^*_\alpha}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} + \frac{1}{\alpha} = 0, \quad \frac{\partial \delta^*_\alpha}{\partial \beta} = \frac{\partial \delta}{\partial \beta} = 0, \quad \frac{\partial \delta^*_\alpha}{\partial \lambda} = \frac{\partial \delta}{\partial \lambda} = 0$$

and obtain $|\Sigma^*|$ from

$$\Sigma^{*-1}_\alpha = \frac{1}{n} \begin{bmatrix} \delta^*_{11} & \delta^*_{12} & \delta^*_{13} \\ \delta^*_{21} & \delta^*_{22} & \delta^*_{23} \\ \delta^*_{31} & \delta^*_{32} & \delta^*_{33} \end{bmatrix},$$

where

$\delta^*_{11} = -\frac{\partial^2 \delta^*_\alpha}{\partial \alpha^2} = -\frac{\partial^2 \delta}{\partial \alpha^2} + \frac{1}{\alpha^2}$, $\delta^*_{12} = \delta_{12}$, $\delta^*_{13} = \delta_{13}$, $\delta^*_{21} = \delta_{21}$, $\delta^*_{22} = \delta_{22}$, $\delta^*_{23} = \delta_{23}$, $\delta^*_{31} = \delta_{31}$, $\delta^*_{32} = \delta_{32}$, $\delta^*_{33} = \delta_{33}$.

Thus, the approximate Bayes estimator of α under SELF is given by

$$\hat{\alpha}_{TK} = \sqrt{\frac{|\Sigma^*_\alpha|}{|\Sigma|}} e^{n[\delta^*_\beta(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)]}.$$

Similarly, we can derive the approximate Bayes estimator of β and λ as

$$\hat{\beta}_{TK} = \sqrt{\frac{|\Sigma^*_\beta|}{|\Sigma|}} e^{n[\delta^*_\beta(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)]}$$

$$\hat{\lambda}_{TK} = \sqrt{\frac{|\Sigma^*_\lambda|}{|\Sigma|}} e^{n[\delta^*_\lambda(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_\delta, \hat{\beta}_\delta, \hat{\lambda}_\delta)]},$$

respectively. Next, we compute the Bayes estimator of survival function S(t). In this case, $\phi(\alpha, \beta, \lambda) = 1 - e^{-\beta t^{-\alpha}}$, then

$$\delta^*_\lambda(\alpha, \beta, \lambda) = \delta(\alpha, \beta, \lambda) + \frac{1}{n} \ln\left(1 - e^{-\beta t^{-\alpha}}\right).$$

Now compute $(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*})$ by solving the following non-linear equations:

$$\frac{\partial \delta^*_{S(t)}}{\partial \alpha} = \frac{\partial \delta}{\partial \alpha} - \frac{\beta e^{-\beta t^{-\alpha}} t^{-\alpha} \ln t}{(1 - e^{-\beta t^{-\alpha}})} = 0,$$

$$\frac{\partial \delta^*_{S(t)}}{\partial \beta} = \frac{\partial \delta}{\partial \beta} + \frac{e^{-\beta t^{-\alpha}} t^{-\alpha}}{(1 - e^{-\beta t^{-\alpha}})} = 0,$$

$$\frac{\partial \delta^*_{S(t)}}{\partial \lambda} = \frac{\partial \delta}{\partial \lambda} = 0.$$

Now we find $\Sigma^*_{S(t)}$ from

$$\Sigma^{*\,-1}_{S(t)} = \frac{1}{n}\begin{bmatrix} \delta^*_{S(t)11} & \delta^*_{S(t)12} & \delta^*_{S(t)13} \\ \delta^*_{S(t)21} & \delta^*_{S(t)22} & \delta^*_{S(t)23} \\ \delta^*_{S(t)31} & \delta^*_{S(t)32} & \delta^*_{S(t)33} \end{bmatrix},$$

where

$$\delta^*_{S(t)11} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha^2} = -\frac{\partial^2 \delta}{\partial \alpha^2} - \frac{\beta e^{-\beta t^{-\alpha}} t^{-\alpha}(\ln t)^2 (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)12} = \delta^*_{S(t)21} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \beta} = -\frac{\partial^2 \delta}{\partial \alpha \partial \beta} - \frac{e^{-\beta t^{-\alpha}} t^{-\alpha} \ln t (e^{-\beta t^{-\alpha}} + \beta t^{-\alpha} - 1)}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)13} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \alpha \partial \lambda} = -\frac{\partial^2 \delta}{\partial \alpha \partial \lambda}, \; \delta^*_{S(t)22} = -\frac{\partial^2 \delta^*_{S(t)}}{\partial \beta^2} = -\frac{\partial^2 \delta}{\partial \beta^2} + \frac{t^{-2\alpha} e^{-\beta t^{-\alpha}}}{(1 - e^{-\beta t^{-\alpha}})^2},$$

$$\delta^*_{S(t)23} = \delta_{23}, \delta^*_{S(t)31} = \delta_{31}, \; \delta^*_{S(t)33} = \delta_{33}.$$

Thus, the Bayes estimator of $S(t)$ is given by

$$\hat{S}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{S(t)}|}{|\Sigma|}}\, e^{n[\delta^*_{S(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_{\delta}, \hat{\beta}_{\delta}, \hat{\lambda}_{\delta})]}.$$

Similarly, the Bayes estimator of failure rate function is given by

$$\hat{h}(t)_{TK} = \sqrt{\frac{|\Sigma^*_{h(t)}|}{|\Sigma|}}\, e^{n[\delta^*_{h(t)}(\hat{\alpha}_{\delta^*}, \hat{\beta}_{\delta^*}, \hat{\lambda}_{\delta^*}) - \delta(\hat{\alpha}_{\delta}, \hat{\beta}_{\delta}, \hat{\lambda}_{\delta})]}.,$$

4.2. *Gibbs sampling method*

In this sub-section, we propose to use the Gibbs sampling method to draw the random sample from the joint posterior distribution so that the sample based inference can be performed. The detail study about MCMC techniques can be found in Robert and Casella (2004) and Gelman *et al.* (2013). For implementing the Gibbs sampling method, the full conditional posterior densities of α, β and λ are, respectively given by

$$\pi_1(\beta \mid \alpha, \mathbf{y}, \mathbf{d}) = \beta^{m+a_2-1} e^{-\beta\left(b_2 + \sum_{i=1}^{n} d_i y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{(1-d_i)}, \tag{17}$$

$$\pi_2(\lambda \mid \alpha, \mathbf{y}, \mathbf{d}) = \lambda^{n-m+a_3-1} e^{-\lambda\left(b_3 + \sum_{i=1}^{n} (1-d_i) y_i^{-\alpha}\right)} \prod_{i=1}^{n} \left(1 - e^{-\beta y_i^{-\alpha}}\right)^{d_i}, \tag{18}$$

$$\pi_3(\alpha \mid \beta, \lambda, \mathbf{y}, \mathbf{d}) = \alpha^{n+a_1-1} e^{-\alpha\left(b_1+\sum_{i=1}^{n}\ln y_i\right)} e^{-\left(\beta\sum_{i=1}^{n} d_i y_i^{-\alpha}+\lambda\sum_{i=1}^{n}(1-d_i) y_i^{-\alpha}\right)}$$
$$\prod_{i=1}^{n}(1-e^{-\lambda y_i^{-\alpha}})^{d_i}\prod_{i=1}^{n}(1-e^{-\beta y_i^{-\alpha}})^{(1-d_i)}. \tag{19}$$

We use following Gibbs sampler algorithm to generate samples from the full conditional posterior densities (17), (18) and (19).

Step 1: Start with initial guess of α, β and λ say $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$.

Step 2: Set $j = 1$.

Step 3: Generate $\beta^{(j)}$ from $\pi_1(\beta \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (17) using MH algorithm with normal proposal density.

Step 4: Generate $\lambda^{(j)}$ from $\pi_2(\lambda \mid \alpha^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (18) using MH algorithm with normal proposal density.

Step 5: Generate $\alpha^{(j)}$ from $\pi_1(\alpha \mid \beta^{(j-1)}, \lambda^{(j-1)}, \mathbf{y}, \mathbf{d})$ in (19) using MH algorithm with normal proposal density.

Step 6: Set $j = j+1$ and repeat steps 3-5 for all $j = 1, 2, \ldots, M$ to obtain MCMC samples

$$\left(\alpha^{(1)}, \beta^{(1)}, \lambda^{(1)}\right), \left(\alpha^{(2)}, \beta^{(2)}, \lambda^{(2)}\right), \ldots, \left(\alpha^{(M)}, \beta^{(M)}, \lambda^{(M)}\right).$$

Now, the approximate Bayes estimator of $\phi(\alpha, \beta, \lambda)$, can be obtained as

$$\hat{\phi}_{GS}(\alpha, \beta, \lambda) = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\phi\left(\alpha^{(j)}, \beta^{(j)}, \lambda^{(j)}\right), \tag{20}$$

where, M_0 is the burn-in period i.e. a number of iterations in Markov chain before the stationary distribution is achieved. Thus, taking, $\phi(\alpha, \beta, \lambda) = \alpha, \beta$ and λ, the Bayes estimators of the parameters α, β and λ under SELF, respectively, are given by

$$\hat{\alpha}_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\alpha^{(j)}, \;\; \hat{\beta}_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\beta^{(j)}, \text{ and}$$
$$\hat{\lambda}_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\lambda^{(j)}.$$

Also, the Bayes estimators of the survival and failure rate functions, respectively, are given by

$$\hat{S}(t)_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\left(1-e^{-\beta^{(j)}t^{-\alpha^{(j)}}}\right); \; t > 0, \text{ and}$$

$$\hat{h}(t)_{GS} = \frac{1}{M-M_0}\sum_{j=M_0+1}^{M}\frac{\alpha^{(j)}\beta^{(j)}t^{-(\alpha^{(j)}+1)}}{e^{\beta^{(j)}t^{-\alpha^{(j)}}}-1}\ ;\ t>0.$$

4.3. HPD credible intervals

Now, we construct the HPD credible interval of the unknown parameters using the generated MCMC samples.

Let $\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(M-M_0)}$ denote the ordered values of $\alpha^{(M_0+1)}, \alpha^{(M_0+2)}, \ldots, \alpha^{(M)}$. Then using the algorithm proposed by Chen and Shao (1999), the 100(1-ξ)%, where, $0<\xi<1$, HPD credible interval for the parameter α is given by

$$\left(\alpha_{(j)}, \alpha_{(j+[(1-\xi)(M-M_0)])}\right),$$

where, j is chosen such that

$$\alpha_{(j+[(1-\xi)(M-M_0)])} - \alpha_{(j)} = \min_{1\le i\xi\le(M-M_0)}\left(\alpha_{(i+[(1-\xi)(M-M_0)])} - \alpha_{(i)}\right);\ j=1,2,\ldots,(M-M_0),$$

here, $[x]$ is the largest integer less than or equal to x. Similarly, 100(1-ξ)% HPD credible intervals for β and λ can be constructed.

5. Numerical explorations

Here, we conduct a simulation study to compare proposed estimators developed in the previous sections. All the computations are performed using statistical software R, see, R Core Team (2018). We consider five different sample sizes n= 20, 30, 40, 50 and 60 in this simulation study. In all the cases true value of $\lambda = 1.0$, two different value of β=0.5, 1.5 and two different values of α=0.5, 2 are used. For Bayesian computation non-informative as well as gamma informative priors under SELF are considered. In case of informative priors following values of hyper-parameters $(a_1, b_1, a_2, b_2, a_3, b_3)$ are taken so that prior means are exactly equal to the true values of the parameters: $(2,4,2,4,2,2), (2,4,3,2,2,2), (4,2,2,4,2,2)$ and $(4,2,3,2,2,2)$.

For each case the ML and Bayes estimates of the unknown parameters, survival and failure rate functions are computed. The mission time $t = 0.80$ is taken for survival and failure rate functions. For Bayesian estimators, T-K approximation and Gibbs sampling methods are considered. The 95% asymptotic confidence intervals based on expected Fisher information matrix and HPD credible intervals based on Gibbs sampling method are constructed. The integrals associated with expected Fisher information matrix are solved using the `integrate` function of software R. We take $M = 10,000$ with burn-in period $M_0 = 2,000$ for Gibbs sampling method. The whole process was simulated 1,000 times and the average absolute biases (AB) with the corresponding mean squared errors (MSE) are computed for different estimators. Also, the average length (AL) and the coverage probabilities (CP) of 95% asymptotic confidence and HPD credible intervals are calculated. The results of the simulation study are reported in Tables 2, 3, 4, 5, 6, 7, 8 and 9 (Appendix B).

In simulation tables, the short notations TK stands for Tierney-Kadane method, GS stand for Gibbs sampling method, P1 for non-informative prior and P2 for gamma informative prior. From these results the following conclusions are made:

(i) As the sample size increases the AB and MSE of the ML and Bayes estimators of the parameters and reliability characteristics decrease in all the cases.

(ii) As value of the failure time parameter β increases, the AB and MSE decrease.

(iii) Bayes estimates are better than MLEs in terms of both AB and MSEs as they include prior information. Also, Gibbs sampling method is better than T-K approximation method in respect of both AB as well as MSEs.

(iv) The average length of all intervals decreases as the sample size n increases. On an average the HPD credible intervals have shorter average length than the asymptotic confidence intervals.

(v) For classical estimation, the coverage probabilities attain their prescribed confidence levels quite satisfactorily. While, in case of Bayesian estimation, coverage probabilities attain their prescribed nominal level only when true value of the shape parameter $\alpha = 0.5$. When value of α and sample size n increase, they do not attain their nominal levels in most of the cases.

6. Real data analysis

In this section, we illustrate estimation procedures discussed in the previous sections with the help of a real data example. This data set was originally reported in Lawless (2003). These data show remission times (in weeks) of a group of 30 patients with leukemia who received a similar treatment. The observations with + sign are censored times. Data are: (1, 1, 2, 4, 4, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31+, 42, 45+, 50+, 57, 60, 71+, 85+, 91.).

Before going further, we fit randomly censored IW, generalized inverted exponential (GIE), gamma and Weibull distributions for this data set. We calculate MLEs of the unknown parameters together with some useful measure of goodness-of-fit tests, namely, the negative log likelihood function $-\ln L$, the Akaike information criterion defined by $\text{AIC} = 2 \times k - 2 \times \ln L$, proposed by Akaike (1974) and the Bayesian information criterion defined by $\text{BIC} = k \times \ln(n) - 2 \times \ln L$, proposed by Schwarz (1978), where k is the number of parameters in the model, n is the number of observations in the given data set, L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (K-S) statistic with its p-value. The best distribution corresponds to the lowest –lnL, AIC, BIC and K-S statistic and the highest p values. The K-S statistic with its p-value are obtained using `ks.test` function of statistical software R.

The results of the MLEs and measures of goodness-of-fit tests are reported in Table 10 (Appendix B).

These results show that randomly censored IW distribution is the best choice for the considered data set. We also consider the Kaplan-Meier (KM) product limit estimator for fitting the randomly censored data through the graphs. The KM product limit estimator for survival function was proposed by Kaplan and Meier (1958) and is given by

$$\hat{S}(t) = \prod_{y_i \le t} \left(1 - \frac{1}{n_i}\right)^{d_i},$$

where, n_i is the number of items survived at time y_i and $d_i = 1$ if item failed, 0 otherwise. The graphs of KM estimator and estimated survival functions of the considered models are given in Figure 2 (Appendix A). Figure 2 shows that the estimate of survival function for IW distribution is quite close to that proposed by KM estimator. Thus, the KM estimator also supports the choice of IW distribution to represent this data set.

In Table 11 (Appendix B), we present result of estimation procedures studied in this article based on the above real data set. We obtain the ML and Bayes estimates of the unknown parameters and reliability characteristics. For reliability characteristics, we take median of the data as the mission time i.e. $t = 13.5$.

Since, we do not have any prior information about the parameters, the Bayes estimates are computed using non-informative priors under SELF. The Bayes estimates of parameters are computed using TK approximation and Gibbs sampling methods. For Gibbs sampling method with the MH algorithm, we generate Markov chain taking $M = 1{,}00{,}000$. We discard first $M_0 = 20{,}000$ observations as burn-in-period and take every 10^{th} observation as iid observation of generated MCMC samples of α, β and λ. Also, we check the convergence for their stationary distributions using graphical diagnostic tools like trace, auto correlation function (ACF) and histogram with Gaussian kernel density plots. Figure 3 in the Appendix A shows the trace, ACF and histogram with Gaussian kernel plots for the parameters.

The trace plots indicate a random scatter about the mean value (represented by solid line) and show the fine mixing of the chains for all parameters. ACF plots show that chains have very low autocorrelations. The histogram plots of the generated MCMC samples show that the marginal distributions of the parameters are almost symmetrical i.e. we can take the mean as the best estimate for the parameters. In fact, these plots are hallmarks of rapid MCMC convergence.

7. Concluding remarks

The inverse Weibull distribution is an important lifetime model for representing the unimodal behavior of the failure rate function. In this article, we considered the classical and Bayesian estimation procedures for the parameters and reliability characteristics of IW distribution under random censoring model. The MLEs of the unknown parameters and reliability characteristics were derived. Asymptotic confidence intervals for the parameters based on expected Fisher information are also obtained. ETT for randomly censored experiment were computed. The Bayes estimators of the parameters and reliability characteristics under SELF were approximated using T-K approximation and Gibbs sampling methods. The performance of these estimators was examined by extensive simulation study. The Bayes estimates based on gamma informative priors using Gibbs sampling method showed minimum average absolute biases and mean squared errors than both the ML and the Bayes estimates under non-informative priors. We recommend the Bayes estimators when some prior information about parameters is available or using non-informative priors.

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Appendix

A. Figures

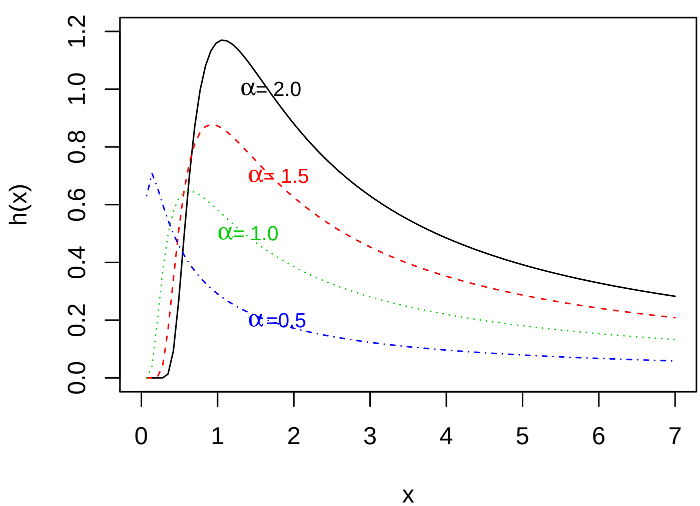

Figure 1 – The plot of the failure rate function of IW distribution with β=1.

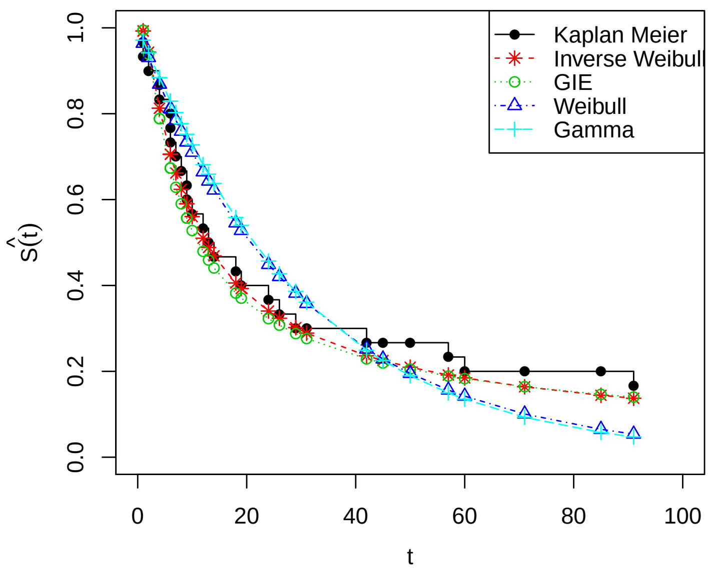

Figure 2 – The plot of estimates of survival functions of considered models.

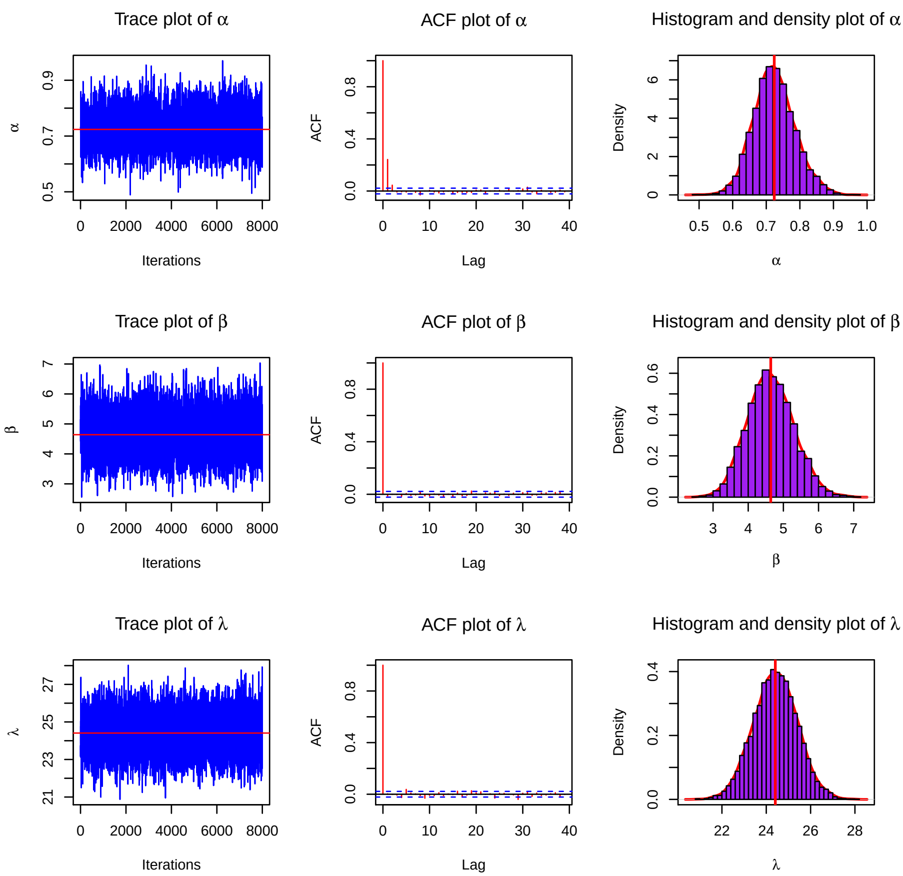

Figure 3 – MCMC dignostic plots of the parameters.

B. Tables

Table 1
Expected time on test (ETT) and the observed time on test (OBTT).

λ	n	$\alpha=2, \beta=0.5$			$\alpha=2, \beta=1$			$\alpha=2, \beta=2$		
		ETT	OBTT		ETT	OBTT		ETT	OBTT	
			AB	MSE		AB	MSE		AB	MSE
0.5	20	1.7601	0.0582	0.6400	2.0871	0.0833	0.5272	2.4592	0.0509	0.5224
	30	1.9620	0.0482	0.6377	2.3279	0.0573	0.6017	2.7482	0.0805	0.6903
	40	2.1175	0.0702	0.678	2.5132	0.0859	0.6609	2.9702	0.0889	0.4232
	50	2.2456	0.0912	0.8631	2.6659	0.0867	0.6270	3.1530	0.0751	0.6062
	60	2.3556	0.0718	0.9476	2.7969	0.0857	0.7214	3.3096	0.0775	0.7966
1	20	2.0871	0.0659	0.644	2.4891	0.0887	0.6800	2.9515	0.0673	0.6545
	30	2.3279	0.0891	0.619	2.7747	0.0753	0.6754	3.2921	0.0696	0.7035
	40	2.5132	0.0829	0.6797	2.9946	0.0764	0.6559	3.5542	0.0700	0.7217
	50	2.6659	0.0830	0.7519	3.1758	0.0961	0.7262	3.7702	0.0894	0.7541
	60	2.7969	0.0878	0.8634	3.3313	0.0801	0.7952	3.9554	0.0994	0.7428
2	20	2.4592	0.0191	0.6023	2.9515	0.0869	0.6880	3.5202	0.0864	0.5600
	30	2.7482	0.0935	0.7368	3.2921	0.0779	0.6381	3.9240	0.0965	0.5507
	40	2.9702	0.0819	0.7145	3.5542	0.0817	0.7594	4.2350	0.0904	0.7119
	50	3.1530	0.0836	0.7161	3.7702	0.0983	0.7037	4.4913	0.0824	0.7524
	60	3.3096	0.0680	0.7428	3.9554	0.0941	0.7269	4.7111	0.0836	0.7905

Table 2
The ML and Bayes estimates of parameters and reliability characteristics when α=0.5, β=0.5, λ=1, t=0.8, $S(t)$= 0.4282, $h(t)$= 0.4665.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.0856	0.0132	0.0037	0.0274	0.2759	0.3763	0.0813	0.0101	0.1097	0.0213
	TK P1	0.0844	0.0128	0.1246	0.0254	0.2721	0.1604	0.0755	0.0089	0.1078	0.0212
20	TK P2	0.0703	0.0086	0.1060	0.0178	0.2101	0.0805	0.0646	0.0064	0.0871	0.012
	GS P1	0.0545	0.0044	0.1027	0.0183	0.2322	0.1088	0.0617	0.0061	0.0626	0.0061
	GS P2	0.0492	0.0037	0.0905	0.0136	0.1823	0.0593	0.055	0.0047	0.0536	0.0043
	MLE	0.0597	0.0060	0.0989	0.0154	0.2037	0.0764	0.0609	0.0058	0.0758	0.0096
	TK P1	0.0631	0.0067	0.1011	0.0169	0.2056	0.0755	0.0616	0.0061	0.0822	0.0114
30	TK P2	0.0529	0.0046	0.0854	0.0116	0.1788	0.0566	0.0520	0.0042	0.0654	0.0070
	GS P1	0.0476	0.0034	0.0829	0.0124	0.1812	0.0588	0.0496	0.0041	0.0543	0.0044
	GS P2	0.0454	0.003	0.0754	0.0096	0.1596	0.0457	0.045	0.0032	0.0497	0.0036
	MLE	0.0513	0.0045	0.0869	0.0119	0.1756	0.0534	0.0538	0.0045	0.0667	0.0075
	TK P1	0.052	0.0048	0.0849	0.0118	0.1741	0.0527	0.0043	0.0428	0.0664	0.008
40	TK P2	0.0465	0.0036	0.0773	0.0095	0.1591	0.0432	0.0478	0.0036	0.0595	0.0059
	GS P1	0.0482	0.0032	0.0727	0.0091	0.1561	0.0434	0.0428	0.003	0.0543	0.0041
	GS P2	0.0457	0.0029	0.0671	0.0076	0.1432	0.0362	0.0402	0.0026	0.0495	0.0035
	MLE	0.0460	0.0036	0.0789	0.0096	0.1493	0.037	0.0485	0.0036	0.006	0.0551
	TK P1	0.0481	0.0039	0.0794	0.0098	0.1531	0.039	0.0484	0.0036	0.0626	0.0063
50	TK P2	0.0424	0.0030	0.0719	0.0080	0.1382	0.0317	0.0439	0.0030	0.0050	0.0502
	GS P1	0.0479	0.0031	0.0668	0.0077	0.1376	0.0330	0.0392	0.0025	0.0523	0.0039
	GS P2	0.0456	0.0028	0.0629	0.0065	0.1255	0.0273	0.0370	0.0022	0.0035	0.2159
	MLE	0.0427	0.0029	0.0714	0.0081	0.1385	0.0321	0.0437	0.0030	0.0546	0.0048
	TK P1	0.0402	0.0026	0.0664	0.0070	0.1302	0.0283	0.0402	0.0025	0.0508	0.0041
60	TK P2	0.0396	0.0026	0.0698	0.0077	0.1308	0.0283	0.0421	0.0027	0.0513	0.0043
	GS P1	0.0459	0.0029	0.0632	0.0065	0.1188	0.0246	0.0367	0.0021	0.0514	0.0036
	GS P2	0.0483	0.0031	0.0634	0.0067	0.1199	0.0248	0.0366	0.0021	0.0539	0.0039

Table 3
The average length (AL) and coverage probability (CP) of 95% asymptotic confidence and HPD credible intervals of parameters when α=0.5, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	MLE	0.3671	0.935	0.5843	0.920	1.2637	0.943
20	GS P1	0.2281	0.930	0.5834	0.932	0.3498	0.935
	GS P2	0.2153	0.941	0.3299	0.941	0.7012	0.945
	MLE	0.2851	0.930	0.4827	0.935	0.9586	0.938
30	GS P1	0.1784	0.938	0.2902	0.941	0.6215	0.944
	GS P2	0.1718	0.932	0.2814	0.946	0.5878	0.952
	MLE	0.2433	0.947	0.4135	0.930	0.8181	0.938
40	GS P1	0.1511	0.941	0.2556	0.941	0.5355	0.94
	GS P2	0.1471	0.946	0.2452	0.945	0.5157	0.941
	MLE	0.2159	0.940	0.3718	0.943	0.7231	0.940
50	GS P1	0.1338	0.940	0.2296	0.947	0.4791	0.940
	GS P2	0.1312	0.948	0.2231	0.948	0.464	0.9453
	MLE	0.1954	0.949	0.3426	0.936	0.6567	0.943
60	GS P1	0.1204	0.948	0.2123	0.948	0.4344	0.942
	GS P2	0.1189	0.949	0.2072	0.947	0.4255	0.952

Table 4
ML and Bayes estimates of parameters and reliability characteristics when $\alpha=0.5$, $\beta=1.5$, $\lambda=1$, $t=0.8$, $S(t)=0.8131$, $h(t)=0.2410$.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.0819	0.0121	0.3834	0.4124	0.2143	0.0775	0.0649	0.0065	0.0750	0.0095
	TK P1	0.0817	0.012	0.3941	0.3528	0.2256	0.0928	0.0645	0.0063	0.0728	0.0090
20	TK P2	0.0715	0.0089	0.2898	0.1526	0.1847	0.0563	0.0532	0.0044	0.0607	0.0062
	GS P1	0.0760	0.0109	0.3318	0.2129	0.2004	0.0729	0.0615	0.0057	0.0622	0.0063
	GS P2	0.0659	0.0078	0.2558	0.1110	0.1651	0.0451	0.0507	0.0039	0.0520	0.0044
	MLE	0.0620	0.0066	0.2756	0.1413	0.1711	0.0508	0.0521	0.0043	0.0561	0.0053
	TK P1	0.0618	0.0068	0.2843	0.1541	0.1680	0.0479	0.0520	0.0042	0.0564	0.0054
30	TK P2	0.0574	0.0056	0.2381	0.0993	0.1553	0.0411	0.0456	0.0033	0.0490	0.0041
	GS P1	0.0577	0.0060	0.2527	0.1155	0.1523	0.0398	0.0492	0.0037	0.0482	0.0038
	GS P2	0.0538	0.0052	0.2157	0.0788	0.1411	0.0344	0.0435	0.0030	0.0422	0.0030
	MLE	0.0532	0.0048	0.2301	0.0948	0.1396	0.0318	0.0440	0.0030	0.0465	0.0035
	TK P1	0.0532	0.0049	0.2296	0.0918	0.1491	0.0374	0.0445	0.003	0.0481	0.0038
40	TK P2	0.0503	0.0043	0.2069	0.0738	0.1300	0.0275	0.0394	0.0024	0.0419	0.0029
	GS P1	0.0498	0.0042	0.2094	0.074	0.1359	0.0322	0.0420	0.0027	0.0410	0.0027
	GS P2	0.0477	0.0039	0.1882	0.0596	0.1186	0.0233	0.0375	0.0022	0.0354	0.0020
	MLE	0.0464	0.0036	0.2072	0.0746	0.1244	0.0258	0.0405	0.0026	0.0430	0.0030
	TK P1	0.0453	0.0034	0.2056	0.0705	0.1309	0.0278	0.0398	0.0024	0.0421	0.0029
50	TK P2	0.0443	0.0033	0.1909	0.0621	0.1174	0.023	0.0373	0.0022	0.0398	0.0026
	GS P1	0.0423	0.0030	0.1885	0.0584	0.1196	0.0244	0.0376	0.0021	0.0357	0.0021
	GS P2	0.0426	0.0031	0.1763	0.0523	0.1104	0.0206	0.0354	0.0019	0.0340	0.0019
	MLE	0.0432	0.0031	0.1772	0.052	0.1185	0.0226	0.0354	0.0020	0.0370	0.0021
	TK P1	0.0405	0.0027	0.1874	0.0577	0.1135	0.0209	0.0363	0.0021	0.0381	0.0023
60	TK P2	0.0417	0.0028	0.1657	0.0452	0.1130	0.0205	0.0332	0.0017	0.0347	0.0019
	GS P1	0.0378	0.0024	0.1739	0.0493	0.1044	0.0183	0.0346	0.0019	0.0324	0.0016
	GS P2	0.0388	0.0025	0.1538	0.0389	0.1042	0.0178	0.0314	0.0015	0.0292	0.0013

Table 5
Average length (AL) and coverage probability (CP) of 95% asymptotic confidence/HPD credible intervals of parameters when $\alpha=0.5$, $\beta=1.5$, $\lambda=1$.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	MLE	0.3642	0.952	1.7017	0.953	0.9861	0.929
20	GS P1	0.2497	0.939	1.1026	0.937	0.6654	0.929
	GS P2	0.2395	0.976	0.9716	0.945	0.6184	0.931
	MLE	0.2840	0.954	1.2835	0.943	0.8124	0.942
30	GS P1	0.1969	0.940	0.8683	0.941	0.5399	0.942
	GS P2	0.1909	0.952	0.8055	0.957	0.5249	0.943
	MLE	0.2418	0.935	1.1032	0.961	0.6874	0.95
40	GS P1	0.1684	0.939	0.7378	0.944	0.4715	0.943
	GS P2	0.1633	0.947	0.7133	0.946	0.4521	0.955
	MLE	0.2152	0.949	0.9663	0.947	0.6188	0.951
50	GS P1	0.1486	0.942	0.6615	0.942	0.4219	0.949
	GS P2	0.1459	0.946	0.6368	0.955	0.4111	0.952
	MLE	0.1955	0.945	0.8734	0.965	0.5618	0.936
60	GS P1	0.1339	0.944	0.6076	0.937	0.3819	0.945
	GS P2	0.1328	0.948	0.582	0.955	0.3756	0.955

Table 6
The ML and Bayes estimates of parameters and reliability characteristics when α=2, β=0.5, λ=1, t=0.8, $S(t)$= 0.5422 , $h(t)$= 1.6493.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.3528	0.2323	0.1287	0.0273	0.2571	0.1577	0.0791	0.0101	0.4018	0.3022
	TK P1	0.3438	0.2205	0.1265	0.0268	0.2642	0.2276	0.0763	0.0093	0.3984	0.2968
20	TK P2	0.2699	0.1255	0.0967	0.0149	0.2165	0.0806	0.0618	0.0060	0.2953	0.1534
	GS P1	0.2906	0.1184	0.1826	0.0514	0.2578	0.1249	0.0879	0.0115	0.3748	0.1815
	GS P2	0.2492	0.086	0.1448	0.0309	0.2149	0.0807	0.0731	0.0078	0.3102	0.1249
	MLE	0.2529	0.1106	0.0997	0.0164	0.2059	0.0768	0.0611	0.0061	0.2933	0.1521
	TK P1	0.2489	0.1067	0.0986	0.0162	0.2081	0.0797	0.0596	0.0059	0.292	0.1507
30	TK P2	0.2103	0.0734	0.0869	0.0122	0.1788	0.0579	0.0554	0.0048	0.2396	0.0968
	GS P1	0.2946	0.1127	0.1866	0.0476	0.2455	0.1027	0.0875	0.0104	0.3945	0.1876
	GS P2	0.2686	0.0932	0.1661	0.037	0.2178	0.0783	0.0798	0.0088	0.3611	0.1532
	MLE	0.2164	0.0801	0.0904	0.0123	0.1784	0.0535	0.0558	0.0048	0.2562	0.1081
	TK P1	0.2136	0.0778	0.0898	0.0122	0.1796	0.0549	0.0548	0.0046	0.2551	0.1073
40	TK P2	0.1858	0.0571	0.0763	0.0091	0.1545	0.0388	0.048	0.0035	0.2106	0.0739
	GS P1	0.3059	0.1149	0.1898	0.0465	0.2436	0.0927	0.0885	0.0103	0.4102	0.1931
	GS P2	0.2871	0.1011	0.1718	0.0373	0.2183	0.0711	0.0808	0.0085	0.3831	0.1664
	MLE	0.1839	0.0592	0.0766	0.0092	0.1531	0.0402	0.0477	0.0035	0.2164	0.0811
	TK P1	0.1822	0.0578	0.0760	0.0091	0.1533	0.0410	0.0470	0.0035	0.2159	0.0807
50	TK P2	0.1625	0.0465	0.0699	0.0077	0.1368	0.0307	0.0436	0.003	0.1892	0.0637
	GS P1	0.3188	0.1185	0.1927	0.0452	0.2367	0.0828	0.0892	0.0099	0.4249	0.1992
	GS P2	0.2999	0.1052	0.1821	0.0398	0.2148	0.0665	0.0855	0.009	0.4044	0.1793
	MLE	0.1677	0.045	0.0719	0.008	0.1349	0.0289	0.0441	0.003	0.2006	0.0642
	TK P1	0.1662	0.0441	0.0715	0.008	0.1349	0.0292	0.0436	0.0029	0.2001	0.0639
60	TK P2	0.1458	0.0364	0.0621	0.0059	0.127	0.0273	0.0386	0.0023	0.1715	0.0487
	GS P1	0.3255	0.121	0.1933	0.045	0.2272	0.0721	0.0891	0.0097	0.431	0.203
	GS P2	0.3104	0.1098	0.1824	0.0385	0.2277	0.0706	0.0853	0.0087	0.4125	0.1831

Table 7
The average length (AL) and coverage probability (CP) of 95% asymptotic confidence/HPD credible intervals of parameters when α=2, β=0.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	MLE	1.4658	0.918	0.5896	0.929	1.2104	0.933
20	GS P1	0.8695	0.938	0.3921	0.932	0.7789	0.939
	GS P2	0.8107	0.941	0.3579	0.935	0.7164	0.942
	MLE	1.1445	0.948	0.4825	0.928	0.9524	0.926
30	GS P1	0.6727	0.946	0.321	0.935	0.6383	0.936
	GS P2	0.6475	0.952	0.305	0.941	0.6041	0.94
	MLE	0.9762	0.939	0.4165	0.934	0.8203	0.943
40	GS P1	0.5659	0.94	0.2717	0.936	0.5553	0.944
	GS P2	0.5487	0.943	0.2606	0.943	0.5298	0.948
	MLE	0.8637	0.941	0.3722	0.936	0.7243	0.937
50	GS P1	0.4904	0.943	0.2339	0.938	0.4926	0.941
	GS P2	0.482	0.944	0.229	0.941	0.4724	0.952
	MLE	0.7808	0.949	0.34	0.935	0.6517	0.947
60	GS P1	0.4355	0.952	0.2011	0.94	0.4427	0.947
	GS P2	0.4307	0.951	0.1989	0.943	0.4372	0.951

Table 8
The ML and Bayes estimates of parameters and reliability characteristics when α=2, β=1.5, λ=1, t=0.8, $S(t)$= 0.9040, $h(t)$= 0.6220.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$		$\hat{S}(t)$		$\hat{h}(t)$	
		AB	MSE	AB	MSE	AB	MSE	AB	MSE	AB	MSE
	MLE	0.3267	0.1896	0.3681	0.2911	0.2091	0.0725	0.0432	0.0028	0.2198	0.079
	TK P1	0.3194	0.1812	0.3747	0.3139	0.2071	0.0902	0.0435	0.003	0.213	0.0756
20	TK P2	0.2702	0.1226	0.2853	0.1458	0.1809	0.0552	0.039	0.0024	0.1809	0.0538
	GS P1	0.2805	0.1381	0.2451	0.1128	0.1793	0.0574	0.0341	0.0018	0.1482	0.0352
	GS P2	0.253	0.1118	0.1933	0.0641	0.1538	0.039	0.0304	0.0015	0.1244	0.0252
	MLE	0.2542	0.1134	0.2796	0.1319	0.167	0.0451	0.0376	0.0022	0.1865	0.0551
	TK P1	0.2503	0.1095	0.2817	0.1354	0.1659	0.0446	0.0374	0.0022	0.1826	0.0537
30	TK P2	0.2338	0.0939	0.2263	0.0962	0.1581	0.0417	0.0306	0.0015	0.146	0.036
	GS P1	0.2266	0.0878	0.1999	0.0674	0.1546	0.0394	0.0287	0.0013	0.1265	0.0255
	GS P2	0.2168	0.0814	0.1588	0.0447	0.1471	0.036	0.0238	0.0009	0.1015	0.0169
	MLE	0.2083	0.0718	0.2435	0.1039	0.1404	0.0313	0.0325	0.0016	0.1571	0.039
	TK P1	0.206	0.07	0.2448	0.1059	0.1398	0.0311	0.0323	0.0016	0.1542	0.0378
40	TK P2	0.2033	0.0698	0.2025	0.0694	0.1366	0.0312	0.0285	0.0013	0.1313	0.0271
	GS P1	0.1886	0.0597	0.1763	0.0544	0.1478	0.0342	0.0252	0.001	0.1079	0.0185
	GS P2	0.1911	0.0612	0.148	0.0361	0.1399	0.0316	0.0224	0.0008	0.0927	0.0133
	MLE	0.1874	0.0584	0.2089	0.0766	0.1293	0.0274	0.029	0.0013	0.1391	0.0319
	TK P1	0.1853	0.0569	0.2096	0.0776	0.1289	0.0272	0.0293	0.0014	0.1378	0.0314
50	TK P2	0.1748	0.0489	0.1983	0.067	0.1178	0.0221	0.026	0.0011	0.1192	0.0228
	GS P1	0.1684	0.047	0.1554	0.0421	0.1469	0.0333	0.0225	0.0008	0.0978	0.0154
	GS P2	0.1619	0.0426	0.1827	0.056	0.1283	0.0259	0.0207	0.0007	0.0853	0.0114
	MLE	0.174	0.0479	0.1854	0.0585	0.1174	0.0224	0.026	0.001	0.124	0.0242
	TK P1	0.1724	0.0469	0.1859	0.059	0.117	0.0223	0.0259	0.0011	0.1225	0.0238
60	TK P2	0.1595	0.0419	0.1715	0.0478	0.1097	0.019	0.0242	0.0009	0.1144	0.0205
	GS P1	0.1549	0.0387	0.1429	0.0334	0.14	0.0295	0.0204	0.0006	0.0885	0.0119
	GS P2	0.1419	0.0333	0.131	0.0283	0.1338	0.0263	0.0191	0.0006	0.0821	0.0105

Table 9
The average length (AL) and the coverage probability (CP) of 95% asymptotic confidence/HPD credible intervals of parameters when α=2, β=1.5, λ=1.

n	Method	$\hat{\alpha}$		$\hat{\beta}$		$\hat{\lambda}$	
		AL	CP	AL	CP	AL	CP
	MLE	1.4535	0.948	1.6405	0.955	1.0017	0.931
20	GS P1	0.9936	0.946	1.0499	0.94	0.6852	0.936
	GS P2	0.9293	0.949	0.9477	0.945	0.6452	0.945
	MLE	1.1502	0.957	1.2786	0.945	0.7987	0.937
30	GS P1	0.7845	0.955	0.8503	0.929	0.5575	0.94
	GS P2	0.7476	0.952	0.7969	0.958	0.5387	0.942
	MLE	0.9687	0.954	1.1014	0.948	0.6938	0.958
40	GS P1	0.6611	0.945	0.7416	0.923	0.4875	0.944
	GS P2	0.6452	0.948	0.701	0.959	0.4722	0.948
	MLE	0.8667	0.946	0.9628	0.931	0.62	0.943
50	GS P1	0.5901	0.945	0.6541	0.912	0.4373	0.943
	GS P2	0.574	0.946	0.6338	0.949	0.4216	0.952
	MLE	0.7814	0.945	0.8769	0.934	0.5626	0.938
60	GS P1	0.5319	0.945	0.5994	0.938	0.3976	0.94
	GS P2	0.5196	0.951	0.5781	0.947	0.3889	0.945

Table 10
Summary fit of the real data set of remission times (in weeks) of 30 leukemia patients.

Model	MLE	-lnL	AIC	BIC	K-S Test	
					Statistic	*p*-value
Inverse Weibull distribution	$\hat{\alpha}$=0.7774 $\hat{\beta}$=4.9231 $\hat{\lambda}$=33.3523	137.7351	281.4701	285.6737	0.1373	0.624
Generalized inverted exponential distribution	$\hat{\alpha}$=0.6619 $\hat{\beta}$= 4.7952 $\hat{\lambda}$=63.1718	138.2794	282.5587	286.7623	0.1599	0.4272
Weibull distribution	$\hat{\alpha}$=0.9714 $\hat{\beta}$=0.0365 $\hat{\lambda}$=0.0073	140.4595	286.9191	291.1227	0.1556	0.4621
Gamma distribution	$\hat{\alpha}$=1.0441 $\hat{\beta}$=0.0346 $\hat{\lambda}$=0.0072	140.4587	286.9175	291.1211	0.1710	0.3444

Table 11
The ML and Bayes estimates of the parameters and reliability characteristics and 95% confidence/HPD credible interval estimates corresponding to the real data set of remission times (in weeks) of 30 leukemia patients.

Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{S}(t)$	$\hat{h}(t)$
MLE	0.7774 (0.5768, 0.9781)	4.9231 (2.7112, 7.1349)	33.3524 (2.3046, 64.4001)	0.4784	0.0409
Bayes (TK)	0.7759	4.9254	36.4505	0.4760	0.0410
Bayes (MCMC)	0.7238 (0.6146, 0.8573)	4.6407 (3.3702, 5.8947)	24.4085 (22.4474, 26.2966)	0.5048	0.0370

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Abstract

This article deals with the estimation of the parameters and reliability characteristics in inverse Weibull (IW) distribution based on the random censoring model. The censoring distribution is also taken as an IW distribution. Maximum likelihood estimators of the parameters, survival and failure rate functions are derived. Asymptotic confidence intervals of the parameters based on the Fisher information matrix are constructed. Bayes estimators of the parameters, survival and failure rate functions under squared error loss function using non-informative and gamma informative priors are developed. Furthermore, Bayes estimates are obtained using Tierney-Kadane's approximation method and Markov chain Monte Carlo (MCMC) techniques. Also, highest posterior density (HPD) credible intervals of the parameters based on MCMC techniques are constructed. A simulation study is conducted to compare the performance of various estimates. Finally, a randomly censored real data set supports the estimation procedures developed in this article.

Keywords: Random censoring; Inverse Weibull distribution; Maximum likelihood estimation; Expected Fisher information; Expected time on test; MCMC technique.