BERNSTEIN-TYPE APPROXIMATION OF SMOOTH FUNCTIONS

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1. INTRODUCTION

The Bernstein polynomials are generally used for the approximation of continuous functions, uniformly, on a closed interval of interest. The Bernstein polynomials were introduced to provide a simple proof of the Weierstrass approximation theorem. See Lorentz (1986), chapter 1, and Pinkus (2000). The Bernstein polynomials are worth of being applied to practical contexts, if they are able to agree with a reasonable number of derivatives of the function to approximate. In any case, the Bernstein polynomials can be regarded as a relevant unifying concept in approximation theory. See also Korovkin (1960), chapter 1, Davis (1963), chapter 6, Feller (1971), chapter 7, Rivlin (1981), chapter 1, Cheney (1982), chapters 1 to 4, and Timan (1994), chapter 1. Here, following the Bernstein polynomials, we propose new Bernstein-type approximations based on the binomial distribution and the multivariate binomial distribution. The Bernstein-type approximations can be viewed as the generalizations of the Bernstein polynomials obtained by considering a convenient approximation coefficient in linear kernels. The Bernstein-type approximations are shown to be uniformly convergent in the sense of Weierstrass. The Bernstein-type approximations are shown to yield a degree of approximation that is better than the degree of approximation of the Bernstein polynomials.

In section 2, we overview the main theoretical features of the Bernstein polynomials, focussing on the binomial distribution. We propose the Bernstein-type approximations. In section 3, we study the multivariate Bernstein polynomials that are defined by the multivariate binomial distribution. We propose the multivariate Bernstein-type approximations. In section 4, we study the degrees of approximation by the Bernstein polynomials and the Bernstein-type approximations. In section 5, we study the Bernstein-type estimates for smooth functions of population means. In section 6, we discuss the results of a simulation study on examples of smooth functions of means. Finally, in section 7, we conclude the contribution with some remarks.

We refer to Serfling (1980), Barndorff-Nielsen and Cox (1989), and Sen and Singer (1993), chapter 3, for asymptotics and results in classical theory of statistical inference. We also refer to Aigner (1997), for results on combinatorial theory.

2. BINOMIAL DISTRIBUTION

2.1. *Bernstein polynomials*

Let $\mathbf{P}_m$ be the space of polynomials $P(x)$ of degree at most m, for all real numbers x. Let g be a bounded, real-valued function defined on the closed interval $[0,1]$. The Bernstein polynomial $B_m(g;x)$ of order m for the function g is defined as

$$B_m(g;x) = \sum_{v=0}^{m} g(m^{-1}v) \binom{m}{v} x^v (1-x)^{m-v} , \tag{1}$$

where m is a positive integer, and $x \in [0,1]$. See Lorentz (1986), chapter 1. Point x in (1) works as a probability, in the binomial distribution, where $x \in [0,1]$. It is seen that $B_m(g;x) \in \mathbf{P}_m$, for every $x \in [0,1]$.

The Bernstein polynomial $B_m(g;x)$, where $x \in [0,1]$, is a well known linear positive operator. The general approximation theory for the Bernstein polynomial $B_m(g;x)$ as a linear positive operator, where $x \in [0,1]$, is provided by Korovkin (1960), chapters 1 to 4. See also Appendix (8.1).

The Bernstein polynomial $B_m(g;x)$, where $x \in [0,1]$, was introduced to prove the Weierstrass approximation theorem. See Pinkus (2000). In particular, if $g(x)$ is continuous on $x \in [0,1]$, then we have that

$$\lim_{m \to \infty} B_m(g;x) = g(x) , \tag{2}$$

uniformly at any point $x \in [0,1]$. The basic proofs of the uniform convergence (2) are detailed in Rivlin (1981), chapter 1, and Lorentz (1986), chapter 1. See also Korovkin (1960), chapters 1 to 4, Davis (1963), chapter 6, Feller (1971), chapter 7, and Cheney (1982), chapters 1 to 4.

2.2. *Bernstein-type approximations*

The Bernstein polynomial $B_m(g;x)$ is given by (1), where $x \in [0,1]$. The Bernstein-type approximation $B_m^{(s)}(g;x)$ of order m for the function $g(x)$ is defined as

$$B_m^{(s)}(g;x) = \sum_{v=0}^{m} g(m^{-s}(m^{-1}v - x) + x) \binom{m}{v} x^v (1-x)^{m-v} , \tag{3}$$

where $s > -1/2$ is a convenient approximation coefficient, m is a positive integer, and $x \in [0,1]$. It is seen that $B_m^{(s)}(g;x) \in \mathbf{P}_m$, where $s > -1/2$, for every $x \in [0,1]$.

The Bernstein polynomial $B_m(g;x)$ can be obtained as $B_m^{(0)}(g;x)$, by setting $s=0$ in the definition (3) of $B_m^{(s)}(g;x)$, for every $x\in[0,1]$.

The Bernstein-type approximation $B_m^{(s)}(g;x)$, where $s>-1/2$, and $x\in[0,1]$, is a linear positive operator, with the properties outlined in Appendix (8.1).

If $g(x)$ is continuous on $x\in[0,1]$, then we have that $B_m^{(s)}(g;x)\to g(x)$, where $s>-1/2$ is fixed, as $m\to\infty$, uniformly at any point $x\in[0,1]$. In Appendix (8.2), under the condition $s>-1/2$, we provide a proof of this uniform convergence.

The uniform convergence (2) of the Bernstein polynomial $B_m(g;x)$, for every $x\in[0,1]$, can also be proved by setting $s=0$ in $B_m^{(s)}(g;x)$, given by (3), for every $x\in[0,1]$, in the proof in Appendix (8.2).

3. MULTIVARIATE BINOMIAL DISTRIBUTION

3.1. *Multivariate Bernstein polynomials*

Let g be a bounded, real-valued function defined on the closed k-dimensional cube $[0,1]^k$. We let $\mathrm{x}=(x_1,\dots,x_k)^T$, where $\mathrm{x}\in[0,1]^k$. The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ for the function g is defined as

$$B_{\mathrm{m}}(g;\mathrm{x})=\sum_{v_1=0}^{m_1}\cdots\sum_{v_k=0}^{m_k} g\begin{pmatrix} m_1^{-1}v_1\\ \vdots\\ m_k^{-1}v_k\end{pmatrix}\binom{m_1}{v_1}\cdots\binom{m_k}{v_k} x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_k^{v_k}(1-x_k)^{m_k-v_k}, \tag{4}$$

where $\mathrm{m}=(m_1,\dots,m_k)^T$ are positive integers, and $\mathrm{x}\in[0,1]^k$. See Lorentz (1986), chapter 2. Points $x_1,\dots,x_k$ in (4) work as probabilities, in a multivariate binomial distribution characterized by the product of k mutually independent binomial distributions, where $\mathrm{x}\in[0,1]^k$. It is seen that the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})\in\mathrm{P}_m$, where $m=\sum_{i=1}^{k} m_i$ is the total degree in $B_{\mathrm{m}}(g;\mathrm{x})$, for every $\mathrm{x}\in[0,1]^k$.

The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, is a linear positive operator, where $\mathrm{x}\in[0,1]^k$. See Appendix (8.1).

The definition of the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, where $\mathrm{x}\in[0,1]^k$, is suggested in Lorentz (1986), chapter 2, without proving its uniform convergence. It can be shown that the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ converges to $g(\mathrm{x})$ uniformly, at any k-dimensional point of continuity $\mathrm{x}\in[0,1]^k$, as $m_i\to\infty$, where $i=1,\ldots,k$.

3.2. *Multivariate Bernstein-type approximations*

The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ is given by (4), where $\mathrm{x}\in[0,1]^k$. The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$ for the function $g(\mathrm{x})$ is defined as

$$B_{\mathrm{m}}^{(s)}(g;\mathrm{x})=\sum_{v_1=0}^{m_1}\cdots\sum_{v_k=0}^{m_k} g\begin{pmatrix} m_1^{-s}(m_1^{-1}v_1-x_1)+x_1\\ \vdots\\ m_k^{-s}(m_k^{-1}v_k-x_k)+x_k\end{pmatrix}$$

$$\cdot\binom{m_1}{v_1}\cdots\binom{m_k}{v_k}x_1^{v_1}(1-x_1)^{m_1-v_1}\cdots x_k^{v_k}(1-x_k)^{m_k-v_k}, \qquad (5)$$

where $s>-1/2$ is a convenient approximation coefficient, $\mathrm{m}=(m_1,\ldots,m_k)^T$ are positive integers, and $\mathrm{x}\in[0,1]^k$. It is seen that the multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})\in\mathbf{P}_m$, where $m=\sum_{i=1}^{k}m_i$ is the total degree in $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, where $s>-1/2$, for every $\mathrm{x}\in[0,1]^k$.

The multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ can be obtained as $B_{\mathrm{m}}^{(0)}(g;\mathrm{x})$, by setting $s=0$ in the definition (5) of $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, for every $\mathrm{x}\in[0,1]^k$.

The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, where $s>-1/2$, and $\mathrm{x}\in[0,1]^k$, is a linear positive operator, with the properties outlined in Appendix (8.1).

The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$ converges to $g(\mathrm{x})$ uniformly, where $s>-1/2$ is fixed, at any k-dimensional point of continuity $\mathrm{x}\in[0,1]^k$, as $m_i\to\infty$, where $i=1,\ldots,k$. In Appendix (8.2), under the condition $s>-1/2$, we provide a proof of this uniform convergence.

The uniform convergence of the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, for every $\mathrm{x}\in[0,1]^k$, can be proved by setting $s=0$ in $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (5), for every $\mathrm{x}\in[0,1]^k$, in the proof in Appendix (8.2).

4. DEGREE OF APPROXIMATION

4.1. *Bernstein polynomials*

Let $\omega(\delta)$ be the modulus of continuity of the real-valued function g, for every $\delta > 0$. The modulus of continuity $\omega(\delta)$ of the function $g(x)$, where $x \in [0,1]$, is defined as the maximum of $\left| g(x_0) - g(x) \right|$, for $\left| x_0 - x \right| < \delta$, where $x_0, x \in [0,1]$. If the function g is continuous, then $\omega(\delta) \to 0$, as $\delta \to 0$.

Setting $\delta = m^{-1/2}$, for every $x \in [0,1]$, it can be shown that the Bernstein polynomial $B_m(g;x)$, given by (1), has degree of approximation

$$\left| B_m(g;x) - g(x) \right| \le \left(1 + \frac{1}{4} m^{-2} \right) \omega(m^{-1/2}) . \tag{6}$$

We let $\left| \mathrm{x} \right| = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$, where $\mathrm{x} \in [0,1]^k$. The modulus of continuity $\omega(\delta)$ of the real-valued function $g(\mathrm{x})$, $\mathrm{x} \in [0,1]^k$, is defined as the maximum of $\left| g(\mathrm{x}_0) - g(\mathrm{x}) \right|$, for $\left| \mathrm{x}_0 \text{-} \mathrm{x} \right| < \delta$, where $\mathrm{x}_0, \mathrm{x} \in [0,1]^k$. If the function g is continuous, then $\omega(\delta) \to 0$, as $\delta \to 0$.

Setting $\delta = m^{-1/2}$, for every $\mathrm{x} \in [0,1]^k$, it can be shown that the multivariate Bernstein polynomial $B_{\mathrm{m}}(\mathrm{g;x})$, given by (4), has degree of approximation

$$\left| B_{\mathbf{m}}(g;\mathrm{x}) - g(\mathrm{x}) \right| \le \left(1 + \frac{1}{4} m^{-1} \sum_{i=1}^{k} m_i^{-1} \right) \omega(m^{-1/2}) , \tag{7}$$

where $m = \sum_{i=1}^{k} m_i$.

We can observe that the coefficient $1 + m^{-2}/4$ in the upper bound (6) corresponds to the coefficient $1 + m^{-1} \sum_{i=1}^{k} m_i^{-1} / 4$ in the upper bound (7), by considering the dimension $k = 1$ in the upper bound (7).

4.2. *Bernstein-type approximations*

In Appendix (8.3), for every $x \in [0,1]$, it is shown that the Bernstein-type approximation $B_m^{(s)}(g;x)$, given by (3), where $s > -1/2$, has degree of approximation

$$\left| B_m^{(s)}(g;\mathrm{x}) - g(\mathrm{x}) \right| \leq \left(1 + \frac{1}{4} m^{-1} m^{-2s-1} \right) \omega(m^{-1/2}) . \tag{8}$$

The upper bound (6) can be obtained from the upper bound (8), by setting $s = 0$.

In Appendix (8.3), for every $\mathrm{x} \in [0,1]^k$, it is also shown that the multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (5), where $s > -1/2$, has degree of approximation

$$\left| B_{\mathrm{m}}^{(s)}(g;\mathrm{x}) - g(\mathrm{x}) \right| \leq \left(1 + \frac{1}{4} m^{-1} \sum_{i=1}^{k} m_i^{-2s-1} \right) \omega(m^{-1/2}) , \tag{9}$$

where $m = \sum_{i=1}^{k} m_i$.

The upper bound (7) can be obtained from the upper bound (9), by setting $s = 0$. We can observe that the coefficient in the upper bound (8) corresponds to the dimension $k = 1$ for the coefficient in the upper bound (9).

4.3. *A comparison*

Given a convenient value for the approximation coefficient s, the Bernstein-type approximations $B_m^{(s)}(g;x)$ and $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (3) and (5), where $s > -1/2$, can typically outperform the corresponding Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, given by (1) and (4), for any function g to approximate, for every $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$, respectively.

The value for s, where $s > -1/2$, can only modify the coefficients in the degrees of approximation (8) and (9), without affecting their modulus of continuity $\omega(m^{-1/2})$, for any fixed m and any fixed $\mathrm{m} = (m_1, \ldots, m_k)^T$, respectively.

Very large values of s do not bring any advantage, with typical examples of application of the Bernstein-type approximations $B_m^{(s)}(g;x)$ and $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, defined by (3) and (5), where $s > -1/2$, $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$. Convergence to unity of the coefficients that determine the degrees of approximation (8) and (9) is rather fast, as s increases.

In Figure 1 we compare the Bernstein polynomial $B_m(g;x)$, given by (1), with the Bernstein-type approximation $B_m^{(s)}(g;x)$, given by (3), where $m = 2$, the function g is defined as $g(x) = x^4$, $g(x) = x^3$, and $g(x) = x^2$, for values of x in the interval $[0.01, 0.99]$, and $\mathrm{s} = 0.5, 0.6, 0.8, 1.1, 1.5, 2$. The value $s = 2$ gives the best performance; the value $s = 0.5$ gives the worst performance. The performance of the Bernstein-type approximation (3) can be even more substantial for values of m larger than the chosen value $m = 2$.

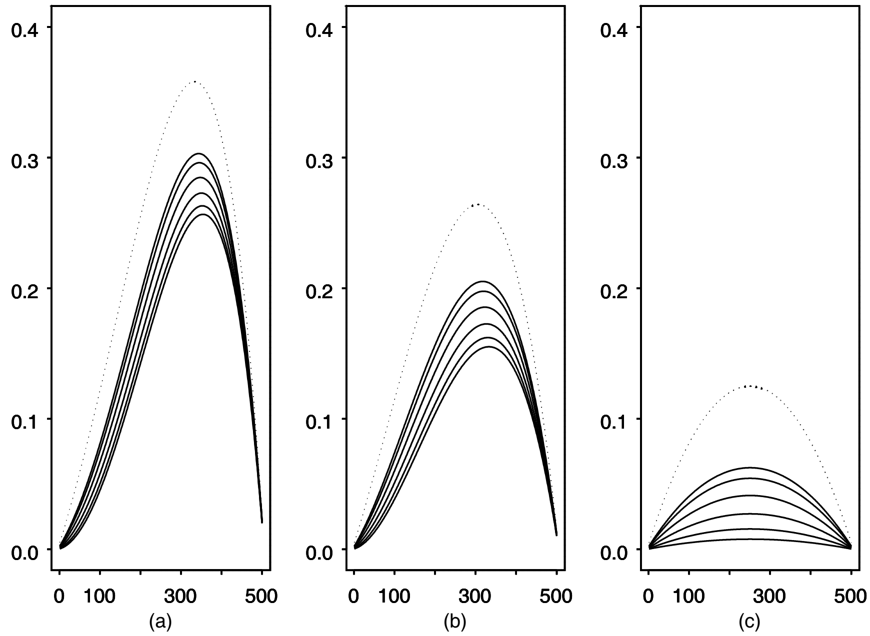

Figure 1 – The difference between $B_m(g;x) - g(x)$ (dotted line), and the differences $B_m^{(s)}(g;x) - g(x)$, (solid lines), for the Bernstein polynomial $B_m(g;x)$, given by (1), and the Bernstein-type approximations $B_m^{(s)}(g;x)$, given by (3), for 500 equidistant values of x that range in the interval $[0.01, 0.99]$, where $s = 0.5, 0.6, 0.8, 1.1, 1.5, 2$, and $m = 2$; $g(x) = x^4$, panel (a), $g(x) = x^3$, panel (b), $g(x) = x^2$, panel (c).

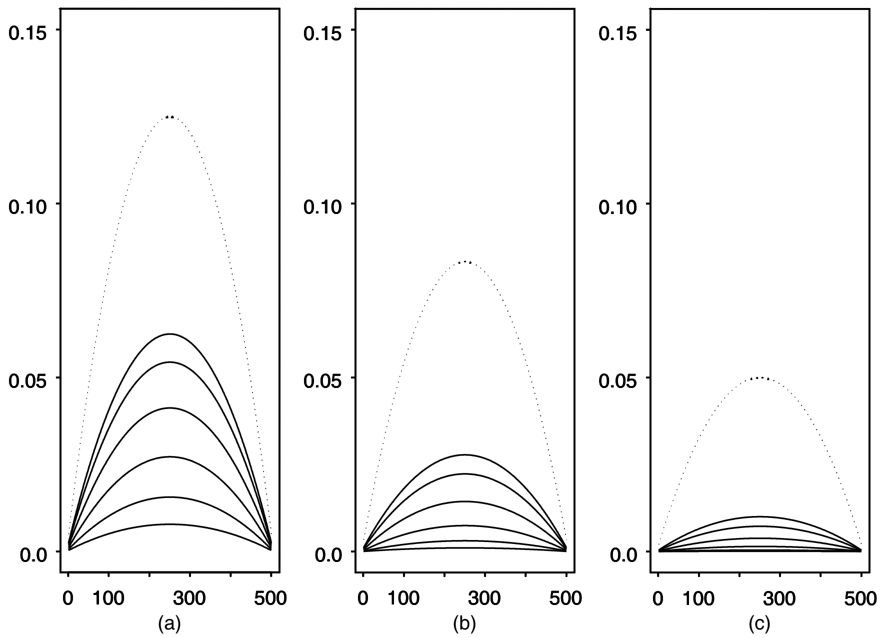

Figure 2 – The difference between $B_m(g;x) - g(x)$, (dotted line), and the differences $B_m^{(s)}(g;x) - g(x)$, (solid lines), for the Bernstein polynomial $B_m(g;x)$, given by (1), and the Bernstein-type approximations $B_m^{(s)}(g;x)$, given by (3), where $s = 0.5, 0.6, 0.8, 1.1, 1.5, 2$, and $g(x) = x^2 + x$, for 500 equidistant values of x that range in the interval $[0.01, 0.99]$; $m = 2$, panel (a), $m = 3$, panel (b), $m = 5$, panel (c).

In Figure 2, we show the performance of the Bernstein-type approximation $B_m^{(s)}(g;x)$ given by (3), where the function g is defined as $g(x) = x^2 + x$, for values of x in the interval $[0.01, 0.99]$, and $s = 0.5, 0.6, 0.8, 1.1, 1.5, 2$. The Bernstein-type approximation $B_m^{(s)}(g;x)$, given by (3), clearly performs better than the Bernstein polynomial $B_m(g;x)$, given by (1), as m increases, from $m = 2$ to $m = 5$, for values of x in the interval $[0.01, 0.99]$, and $s = 0.5, 0.6, 0.8, 1.1, 1.5, 2$.

5. ESTIMATION OF SMOOTH FUNCTIONS OF MEANS

5.1. *Bernstein-type estimators*

The Bernstein-type approximations $B_m^{(s)}(g;x)$ and $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (3) and (5), respectively, where $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$, can be used for estimating smooth functions of population means in the statistical inference from a random sample of n independent and identically distributed (i.i.d.) random observations.

Let X be a univariate random variable with values $x \in [0,1]$, with distribution function F, and finite mean $\mu = E[X]$. We want to estimate a population characteristic θ of the form $\theta = g(\mu)$, where g is a smooth function $g : [0,1] \to R^1$. The natural estimator of θ is $\hat{\theta} = g(\bar{x})$, where $\bar{x}$ is the sample mean, calculated on a random sample of n i.i.d. observations X_j, $j = 1, \ldots, n$, of X. That is, $\bar{x} = n^{-1} \sum_{j=1}^{n} X_j$. An alternative estimator of $\theta = g(\mu)$ can be obtained as the Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$ defined as

$$B_m^{(s)}(g;\bar{x}) = \sum_{v=0}^{m} g(m^{-s}(m^{-1}v - \bar{x}) + \bar{x}) \binom{m}{v} \bar{x}^{v} (1-\bar{x})^{m-v}, \tag{10}$$

where $s > -1/2$. The Bernstein-type estimator (10) follows from the definition (3) of $B_m^{(s)}(g;x)$, by substituting the argument $x \in [0,1]$ with the sample mean $\bar{x}$, where $\bar{x}$ ranges in $[0,1]$.

Let X be a k-variate random variable with values $\mathrm{x} \in [0,1]^k$, where $\mathrm{X} = (X_1, \ldots, X_k)^T$, with distribution function F, and finite k-variate mean $\mu = E[\mathrm{X}]$, $\mu = (\mu_1, \ldots, \mu_k)^T$. We want to estimate $\theta = g(\mu)$, where $g : [0,1]^k \to R^1$. The natural estimator of θ is $\hat{\theta} = g(\bar{\mathrm{x}})$, where $\bar{\mathrm{x}} = (\bar{x}_1, \ldots, \bar{x}_k)^T$ is the k-variate sample mean on a random sample of n i.i.d. observations X_i,

$i=1,\ldots,n$, of X, $\overline{x}_i = n^{-1}\sum_{j=1}^{n} X_{ij}$, $i=1,\ldots,k$. An alternative estimator of $\theta = g(\mu)$ can be obtained as the multivariate Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ defined as

$$B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}}) = \sum_{v_1=0}^{m_1}\cdots\sum_{v_k=0}^{m_k} g\begin{pmatrix} m_1^{-s}(m_1^{-1}v_1 - \overline{x}_1) + \overline{x}_1 \\ \vdots \\ m_k^{-s}(m_k^{-1}v_k - \overline{x}_k) + \overline{x}_k \end{pmatrix}$$

$$\cdot\binom{m_1}{v_1}\cdots\binom{m_k}{v_k}\overline{x}_1^{v_1}(1-\overline{x}_1)^{m_1-v_1}\cdots\overline{x}_k^{v_k}(1-\overline{x}_k)^{m_k-v_k}, \tag{11}$$

where $s > -1/2$. The multivariate Bernstein-type estimator (11) follows the definition (5) of $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, by substituting the argument $\mathrm{x}\in[0,1]^k$ with the sample $\overline{\mathrm{x}}$, where $\overline{\mathrm{x}}$ ranges in $[0,1]^k$.

5.2. *Orders of error of Bernstein-type estimators*

In Appendix (8.4), it is shown that the Bernstein-type estimator $B_m^{(s)}(g;\overline{x})$, given by (10), where $s > -1/2$, can be an accurate substitute for the natural estimator $g(\overline{x})$. In particular, if we consider the sample size *n* as fixed, then we have that

$$B_m^{(s)}(g;\overline{x}) = g(\overline{x}) + O(m^{-2s-1}), \tag{12}$$

where $s > -1/2$, as $m \to \infty$.

In Appendix (8.4), it is also shown that the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, given by (11), where $s > -1/2$, and $\mathrm{m} = (m_1,\ldots,m_k)^T$, can be an accurate substitute for the natural estimator $g(\overline{\mathrm{x}})$. In particular, if we consider the sample size *n* as fixed, then we have that

$$B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}}) = g(\overline{\mathrm{x}}) + \sum_{i=1}^{k} O(m_i^{-2s-1}), \tag{13}$$

where $s > -1/2$, as $m_i \to \infty$, $i=1,\ldots,k$.

We know that $\overline{x} = \mu + O_p(n^{-1/2})$, as $n\to\infty$. We also know that $g(\overline{x}) = g(\mu) + O_p(n^{-1/2})$, as $n\to\infty$.

In Appendix (8.4), it is shown that the Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$, given by (10), where $s > -1/2$, is a consistent estimator of $g(\mu)$, as $m \to \infty$, and $n \to \infty$. In particular, it is shown that

$$B_m^{(s)}(g;\bar{x}) = g(\mu) + O(m^{-2s-1}) + O_p(n^{-1/2}), \tag{14}$$

where $s > -1/2$, as $m \to \infty$, and $n \to \infty$.

We know that $\bar{\mathrm{x}} = \mu + \mathrm{O}_\mathrm{p}(n^{-1/2})$, where $\bar{x}_i = \mu_i + O_p(n^{-1/2})$, for every $i = 1,\ldots,k$, as $n \to \infty$. We also know that $g(\bar{\mathrm{x}}) = g(\mu) + O_p(n^{-1/2})$, as $n \to \infty$.

In Appendix (8.4), it is also shown that the multivariate Bernstein-type estimator $B_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}})$, given by (11), where $s > -1/2$, and $\mathrm{m} = (m_1,\ldots,m_k)^T$, is a consistent estimator of $g(\mu)$, as $m_i \to \infty$, where $i = 1,\ldots,k$, and $n \to \infty$. In particular, it is shown that

$$B_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}}) = g(\mu) + \sum_{i=1}^{k} O(m_i^{-2s-1}) + O_p(n^{-1/2}), \tag{15}$$

where $s > -1/2$, as $m_i \to \infty$, $i = 1,\ldots,k$, and $n \to \infty$.

5.3. *Asymptotic normality of Bernstein-type estimators*

The Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$ is defined by (10), where $s > -1/2$, and m is a positive integer. We denote by σ^2 the asymptotic variance of $n^{1/2} g(\bar{x})$, as $n \to \infty$. That is,

$$\sigma^2 = \{g'(\mu)\}^2 E[(X-\mu)^2],$$

where $g'(x) = (dx)^{-1} dg(x)$, $x \in [0,1]$. We suppose that $m^{-2s-1} \le n^{-1/2}$. Then, the distribution of the Bernstein-type estimator $B_m^{(s)}(g;\bar{x})$ is asymptotically normal,

$$n^{1/2}\{B_m^{(s)}(g;\bar{x}) - g(\mu)\} \xrightarrow{d} N(0,\sigma^2), \tag{16}$$

where $s > -1/2$, as $m \to \infty$, and $n \to \infty$. See Appendix (8.5).

The Bernstein-type estimator $B_\mathrm{m}^{(s)}(g;\bar{\mathrm{x}})$ is defined by (11), where $s > -1/2$, and $\mathrm{m} = (m_1,\ldots,m_k)^T$ are positive integers. We denote by σ^2 the asymptotic variance of $n^{1/2} g(\bar{\mathrm{x}})$, as $n \to \infty$. That is,

$$\sigma^2 = \sum_{i=1}^{k}\sum_{j=1}^{k} (\partial x_i)^{-1} \partial g(x_1,\ldots,x_i,\ldots,x_k)\big|_{\mathrm{x}=\mu} (\partial x_j)^{-1} \partial g(x_1,\ldots,x_j,\ldots,x_k)\big|_{\mathrm{x}=\mu} \cdot E[(X_i-\mu)(X_j-\mu)] .$$

We suppose that $m_i^{-2s-1} \le n^{-1/2}$, for every $i = 1,\ldots,k$. Then, the distribution of the Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ is asymptotically normal,

$$n^{1/2}\{B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}}) - g(\mu)\} \xrightarrow{d} N(0,\sigma^2)\,, \tag{17}$$

where $s > -1/2$, as $m_i \to \infty$, $i = 1,\ldots,k$, and $n \to \infty$. See Appendix (8.5).

6. A SIMULATION STUDY

In Figure 3, we report on a small Monte Carlo experiment concerning with the effectiveness of the Bernstein-type estimator $B_m^{(s)}(g;\overline{x})$, given by (10), where $m = 3,4,4,5$, and $s = 0.5,0.5,0.6,2$, for approximating the smooth function of means $g(x) = \overline{x}^2 + \overline{x}$. Independent samples of size $n = 4,6,10,16$, from the uniform distribution on the interval $(0,1)$, were simulated. The values of the Bernstein-type estimator $B_m^{(s)}(g;\overline{x})$ were practically indistinguishable from the values of the natural estimator $g(\overline{x})$. We denote by $\hat{\sigma}_n^2$ the Monte Carlo variance of

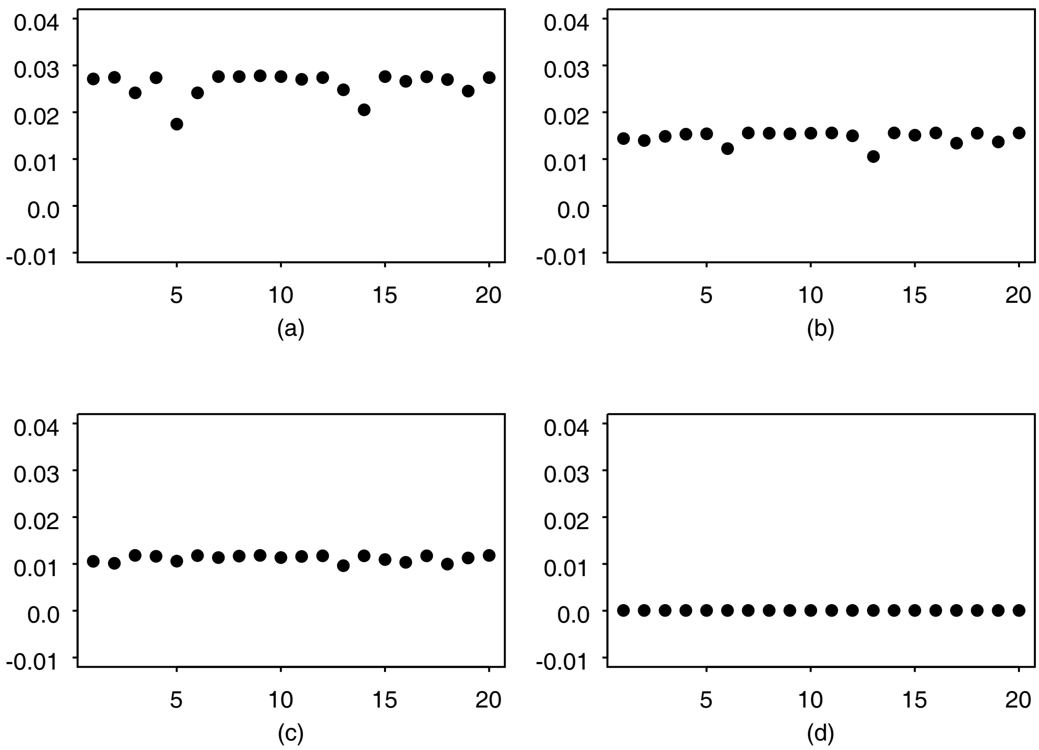

Figure 3 – The difference $B_m^{(s)}(g;\overline{x}) - g(\overline{x})$, (symbol ●), where the Bernstein-type estimates $B_m^{(s)}(g;\overline{x})$ are obtained as (10), and $g(\overline{x}) = \overline{x}^2 + \overline{x}$, for 20 samples of size n, from the uniform distribution on the interval $(0,1)$; $s = 0.5$, $m = 3$, and $n = 4$, in panel (a), $s = 0.5$, $m = 4$, and $n = 6$, in panel (b), $s = 0.6$, $m = 4$, and $n = 10$, in panel (c), and $s = 2$, $m = 5$, and $n = 16$, in panel (d).

the Bernstein-type estimator $B_m^{(s)}(g;\overline{x})$, given by (10). The variance of $B_3^{(0.5)}(\overline{x}^2+\overline{x};\overline{x})$ was $\hat{\sigma}_4^2=0.070565$, the variance of $B_4^{(0.5)}(\overline{x}^2+\overline{x};\overline{x})$ was $\hat{\sigma}_6^2=0.056010$, the variance of $B_4^{(0.6)}(\overline{x}^2+\overline{x};\overline{x})$ was $\hat{\sigma}_{10}^2=0.038841$, and the variance of $B_5^{(2)}(\overline{x}^2+\overline{x};\overline{x})$ was $\hat{\sigma}_{16}^2=0.018470$.

In Figure 4, we also report on an equivalent Monte Carlo experiment concerning with the effectiveness of the multivariate Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, given by (11), where $m_1=m_2=3$, $m_1=m_2=4$, $m_1=m_2=4$, and $m_1=m_2=5$, and $s=0.5,0.5,0.6,2$, for approximating the ratio of means $g(\overline{\mathrm{x}})=(\overline{x}_2)^{-1}\overline{x}_1$. Independent samples, of size $n=4,6,10,16$, from a bivariate distribution with independent uniform marginals on the interval $(0,1)$, were simulated. The multivariate Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ practically took the same values as its natural counterpart $g(\overline{\mathrm{x}})$. We denote by $\hat{\sigma}_n^2$ the Monte Carlo variance of the multivariate Bernstein-type estimator $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, given by (11). The variance of $B_{(3,3)^T}^{(0.5)}((\overline{x}_2)^{-1}\overline{x}_1;\overline{\mathrm{x}})$ was $\hat{\sigma}_4^2=0.629584$, the variance of

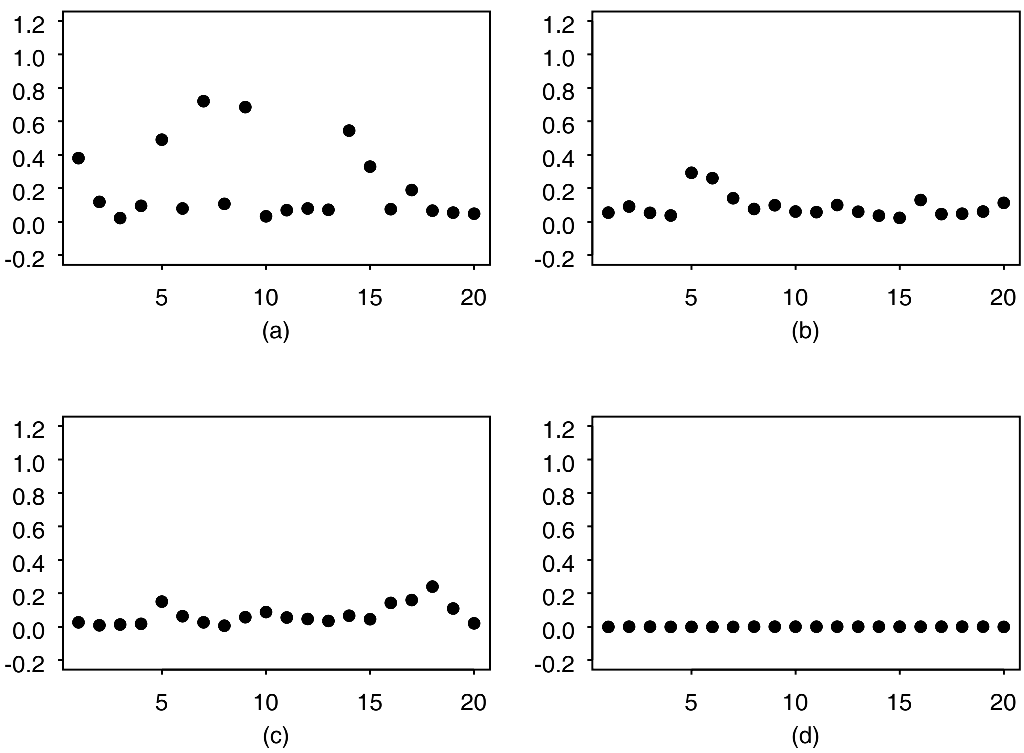

Figure 4 – The difference $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})-g(\overline{\mathrm{x}})$, (symbol ●), where the Bernstein-type estimates $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ are obtained as (11), and $g(\overline{\mathrm{x}})=(\overline{x}_2)^{-1}\overline{x}_1$, for 20 samples of size n, from independent uniform distributions on the interval $(0,1)$; $s=0.5$, $m_1=m_2=3$, and $n=4$, in panel (a), $s=0.5$, $m_1=m_2=4$, and $n=6$, in panel (b), $s=0.6$, $m_1=m_2=4$, and $n=10$, in panel (c), and $s=2$, $m_1=m_2=5$, and $n=6$, and $n=16$, in panel (d).

$B^{(0.5)}_{(4,4)^T}((\overline{x}_2)^{-1}\overline{x}_1;\overline{\mathrm{x}})$ was $\hat{\sigma}^2_6 = 0.106994$, the variance of $B^{(0.6)}_{(4,4)^T}((\overline{x}_2)^{-1}\overline{x}_1;\overline{\mathrm{x}})$ was $\hat{\sigma}^2_{10} = 0.101995$, and the variance of $B^{(2)}_{(5,5)^T}((\overline{x}_2)^{-1}\overline{x}_1;\overline{\mathrm{x}})$ was $\hat{\sigma}^2_{16} = 0.064750$.

7. CONCLUDING REMARKS

1) Alternative definitions for the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$, given by (4), where $\mathrm{x} \in [0,1]^k$, and the multivariate Bernstein-type approximation $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, given by (5), where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$, can be obtained by replacing the multivariate binomial distribution by other multivariate discrete distributions. The multinomial distribution, the multinomial-related distributions, and the other multivariate discrete distributions are reviewed in Johnson, Kotz and Balakrishnan (1997), chapters 35 to 42. See also Feller (1968), chapter 6, Johnson and Kotz (1977), chapter 1 and 2, and Sveshnikov (1978), chapters 1 and 2.

2) In the multivariate Bernstein-type approximation $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, given by (5), where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$, we can use a different approximation coefficient for each component. More precisely, we can use $\mathrm{s} = (s_1,\ldots,s_k)^T$ in a straightforward generalization $B^{(\mathrm{s})}_{\mathrm{m}}(g;\mathrm{x})$ of the multivariate Bernstein polynomial $B_{\mathrm{m}}(g;\mathrm{x})$ given by (4), where $\mathrm{x} \in [0,1]^k$, following the definition (5) of $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$.

3) The Bernstein-type approximations $B^{(s)}_m(g;x)$ and $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, given by (3) and (5), respectively, where $s > -1/2$, $x \in [0,1]$, and $\mathrm{x} \in [0,1]^k$, can be studied in order to approximate derivatives of functions, monotone functions, convex functions, functions of bounded variation, discontinuous functions, and integrable functions. See Lorentz (1986), chapters 1 and 2.

4) The Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$ and the Bernstein-type approximations $B^{(s)}_m(g;x)$ and $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, where $s > -1/2$, given by (1) and (4), and (3) and (5), admit extensions on bounded intervals $x \in [\alpha,\beta]$, and bounded regions $\mathrm{x} \in [\alpha_1,\beta_1] \times \cdots \times [\alpha_k,\beta_k]$, and extensions on unbounded intervals $x \in [\alpha,+\infty)$, and unbounded regions $\mathrm{x} \in [\alpha_1,+\infty) \times \cdots \times [\alpha_k,+\infty)$. See Lorentz (1986), chapter 2.

5) In the Bernstein-type approximations $B^{(s)}_m(g;x)$ and $B^{(s)}_{\mathrm{m}}(g;\mathrm{x})$, given by (3) and (5), where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$, the linear kernels $m^{-s}(m^{-1}v - x) + x$ and $m_i^{-s}(m_i^{-1}v_i - x_i) + x_i$ can be substituted by nonlinear kernels $h^{(s)}(m,v;x)$

and $h^{(s)}(m_i, v_i; x_i)$, where $s > -1/2$, $x \in [0,1]$, and $v_i = 0,1,\ldots,m_i$, $i = 1,\ldots,k$, $\mathrm{x} = (x_1,\ldots,x_k)^T \in [0,1]^k$.

6) Following Babu, Canty and Chaubey (2002), the Bernstein-type approximation $B_m^{(s)}(g;x)$, given by (3), where $s > -1/2$, and $x \in [0,1]$, can be used for distribution and density estimation. Let F_n be the empirical distribution function on a random sample of n i.i.d. observations X_i, $i = 1,\ldots,n$, from a distribution function F, $F_n(x) = n^{-1} \sum_{i=1}^{n} I(X_i \le x)$, $x \in [0,1]$, where $I(A)$ denotes the indicator function of the set A. The Bernstein-type estimator of F can be defined as $B_m^{(s)}(F_n;x)$, $s > -1/2$, $x \in [0,1]$. Let f be the density of F. The Bernstein-type estimator of f can be defined as $m B_{m-1}^{(s)}(f_n;x)$, $s > -1/2$, $x \in [0,1]$, where $f_n(m^{-1}v) = F_n(m^{-1}(v+1)) - F_n(m^{-1}v)$, $v = 0,1,\ldots,m$, and $f_n(0) = 0$.

7) The Bernstein-type approximations $B_m^{(s)}(g;\overline{x})$ and $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$, given by (10) and (11), where $s > -1/2$, respectively, can be regarded as examples of random functions. See Sveshnikov (1978), chapter 7, and Gikhman and Skorokhod (1996), chapter 4.

8. APPENDIX

8.1. *Basic properties of the Bernstein-type approximations (3) and (5)*

The Bernstein-type approximations $B_m^{(s)}(g;x)$ and $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, given by (3) and (5), where $s > -1/2$, and $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$, are linear positive operators. Let δ be a finite constant. Let g, g_1, and g_2 be functions, $g(x)$, $g_1(x)$, and $g_2(x)$, $x \in [0,1]$. We have

$$B_m^{(s)}(\delta g;x) = \delta B_m^{(s)}(g;x),$$
$$B_m^{(s)}(g_1 + g_2;x) = B_m^{(s)}(g_1;x) + B_m^{(s)}(g_2;x),$$

$x \in [0,1]$. If $g_1(x) \le g_2(x)$, for all $x \in [0,1]$, we have

$$B_m^{(s)}(g_1;x) \le B_m^{(s)}(g_2;x),$$

$x \in [0,1]$. Multivariate versions of these properties hold for $B_{\mathrm{m}}^{(s)}(g;\mathrm{x})$, where $s > -1/2$, $\mathrm{g}(\mathrm{x})$, and $\mathrm{x} \in [0,1]^k$. The corresponding properties for the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, given by (1) and (4), where $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$, can also be obtained by setting $s = 0$ above.

8.2. *Uniform convergence of the Bernstein-type approximations (3) and (5)*

The uniform norm $\|g\|$ of the function $g(x)$, where $x \in [0,1]$, is defined as

$$\| g \| = \max_{x \in [0,1]} | g(x) | .$$

The Bernstein-type approximation $B_m^{(s)}(g;x)$, where $s > -1/2$, and $x \in [0,1]$, is given by (3). We want to show that, given any constant $\varepsilon > 0$, there exists a positive integer m_0, such that

$$\left| B_{m_0}^{(s)}(g;x) - g(x) \right| < \varepsilon , \tag{18}$$

for every $x \in [0,1]$.

For every $x \in [0,1]$, the Bernstein-type approximation $B_m^{(s)}(1;x)$ is

$$B_m^{(s)}(1;x) = 1 . \tag{19}$$

We define the function $\mu_1(x)$ as $\mu_1(x) = x$, and the function $\mu_2(x) = x$ as $\mu_2(x) = x^2$. The Bernstein-type approximation $B_m^{(s)}(\mu_1(x);x)$ is

$$B_m^{(s)}(\mu_1(x);x) = x . \tag{20}$$

The Bernstein-type approximation $B_m^{(s)}(\mu_2(x);x)$ is

$$\begin{aligned} B_m^{(s)}(\mu_2(x);x) &= \sum_{v=0}^{m} \{ m^{-s} (m^{-1} v - x) + x \}^2 \binom{m}{v} x^v (1-x)^{m-v} \\ &= m^{-2s-1} x(1-x) + x^2 . \end{aligned} \tag{21}$$

Suppose that $\| g \| = M$. We take $x_0 \in [0,1]$. We have

$$-2M \le g(x_0) - g(x) \le 2M , \tag{22}$$

where $x_0, x \in [0,1]$. The function g is continuous; given $\varepsilon_1 > 0$, there exists a constant $\delta > 0$, such that

$$-\varepsilon_1 < g(x_0) - g(x) < \varepsilon_1 , \tag{23}$$

for $| x_0 - x | < \delta$, and $x_0, x \in [0,1]$. From (22) and (23), it follows that

$$-\varepsilon_1 - \frac{2M}{\delta^2}(|x_0 - x|)^2 \le g(x_0) - g(x) \le \varepsilon_1 + \frac{2M}{\delta^2}(|x_0 - x|)^2 , \tag{24}$$

for $x_0, x \in [0,1]$. If $|x_0 - x| < \delta$, (23) implies (24), $x_0, x \in [0,1]$. If $|x_0 - x| \geq \delta$, then $\delta^{-2}(|x_0 - x|)^2 \geq 1$, and (22) implies (24), $x_0, x \in [0,1]$. Following Appendix (8.1), (24) becomes

$$-\varepsilon_1 - \frac{2M}{\delta^2} B_m^{(s)}((|x_0 - x|)^2; x) \leq B_m^{(s)}(g; x) - g(x) \leq \varepsilon_1 + \frac{2M}{\delta^2} B_m^{(s)}((|x_0 - x|)^2; x), \tag{25}$$

for $x_0, x \in [0,1]$. We observe that $(|x_0 - x|)^2 = x_0^2 + x^2 - 2x_0 x$, $x_0, x \in [0,1]$. The Bernstein-type approximations $B_m^{(s)}(1; x)$, $B_m^{(s)}(\mu_1(x); x)$, and $B_m^{(s)}(\mu_2(x); x)$, given by (19), (20) and (21), imply in (25) that

$$B_m^{(s)}((|x_0 - x|)^2; x) = m^{-2s-1} x(1-x),$$

$x_0, x \in [0,1]$. That is,

$$B_m^{(s)}((|x_0 - x|)^2; x) = O(m^{-2s-1}),$$

as $m \to \infty$, $x \in [0,1]$. The condition $s > -1/2$ is required for the uniform convergence. In fact, we can observe that $0 \leq x(1-x) \leq 1/4$, $x \in [0,1]$. Then, (25) becomes

$$\left| B_m^{(s)}(g; x) - g(x) \right| \leq \varepsilon_1 + m^{-2s-1} \frac{M}{2\delta^2}, \tag{26}$$

$x \in [0,1]$. Setting $\varepsilon_1 = \varepsilon / 2$, for any $m_0 > (\delta^2 \varepsilon)^{-1} M$, the uniform convergence (18) is proved.

The convergence $B_m^{(s)}(g; x) \to g(x)$, where $s > -1/2$, is uniform, at any point of continuity $x \in [0,1]$, as $m \to \infty$, in the sense that the upper bound (26) for the uniform norm does not depend on x, $x \in [0,1]$.

The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g; \mathrm{x})$, where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$, is given by (5). We observe that k is fixed and does not depend on m. Considering the uniform norm $\| g \|$ of the function $g(\mathrm{x})$, $\mathrm{x} \in [0,1]^k$, defined as

$$\| g \| = \max_{\mathrm{x} \in [0,1]^k} |g(\mathrm{x})|.$$

we want to show that, given any constant $\varepsilon > 0$, there exist positive integers $\mathrm{m}_0 = (m_{01}, \ldots, m_{0k})^T$, such that

$$\left| B_{\mathrm{m}_0}^{(s)}(g;\mathrm{x}) - g(\mathrm{x}) \right| < \varepsilon , \tag{27}$$

for every $\mathrm{x} \in [0,1]^k$.

For every $\mathrm{x} \in [0,1]^k$, the multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(1;\mathrm{x})$ is

$$B_{\mathrm{m}}^{(s)}(1;\mathrm{x}) = 1 , \tag{28}$$

where $s > -1/2$. We define the functions $\mu_1(\mathrm{x}) = \sum_{i=1}^{k} x_i$ and $\mu_2(\mathrm{x}) = \sum_{i=1}^{k} x_i^2$. The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(\mu_1(\mathrm{x});\mathrm{x})$ is

$$B_{\mathrm{m}}^{(s)}(\mu_1(\mathrm{x});\mathrm{x}) = \sum_{i=1}^{k} x_i , \tag{29}$$

and the multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(\mu_2(\mathrm{x});\mathrm{x})$ is

$$B_{\mathrm{m}}^{(s)}(\mu_2(\mathrm{x});\mathrm{x}) = \sum_{v_1=1}^{m_1} \cdots \sum_{v_k=1}^{m_k} \sum_{i=1}^{k} \{ m_i^{-s}(m_i^{-1} v_i - x_i) + x_i \}^2$$

$$\cdot \binom{m_1}{v_1} \cdots \binom{m_k}{v_k} x_1^{v_1}(1-x_1)^{m_1 - v_1} \cdots x_k^{v_k}(1-x_k)^{m_k - v_k}$$

$$= \sum_{i=1}^{k} m_i^{-2s-1} x_i(1-x_i) + x_i^2 . \tag{30}$$

Suppose that $\| g \| = M$. We take $\mathrm{x}_0 = (x_{01},\ldots,x_{0k})^T \in [0,1]^k$. We observe that

$$(|\mathrm{x}_0 - \mathrm{x}|)^2 = \sum_{i=1}^{k} (x_{0i}^2 + x_i^2 - 2x_{0i}x_i) ,$$

$\mathrm{x}_0, \mathrm{x} \in [0,1]^k$. The uniform convergence (27) follows from the result

$$B_{\mathrm{m}}^{(s)}((|\mathrm{x}_0 \text{-} \mathrm{x}|)^2;\mathrm{x}) = \sum_{i=1}^{k} m_i^{-2s-1} x_i(1-x_i) ,$$

$\mathrm{x}_0, \mathrm{x} \in [0,1]^k$. That is,

$$B_{\mathrm{m}}^{(s)}((|\mathrm{x}_0\text{-}\mathrm{x}|)^2;\mathrm{x}) = \sum_{i=1}^{k} O(m_i^{-2s-1}),$$

as $m_i \to \infty$, where $i = 1,\ldots,k$, $\mathrm{x}_0,\mathrm{x} \in [0,1]^k$. Under the condition $s > -1/2$, the convergence $B_{\mathrm{m}}^{(s)}(\mathrm{g;x}) \to g(\mathrm{x})$ is uniform at any point of continuity $\mathrm{x} \in [0,1]^k$, as $m_i \to \infty$, where $i = 1,\ldots,k$.

The uniform convergence of the Bernstein polynomials $B_m(g;x)$ and $B_{\mathrm{m}}(g;\mathrm{x})$, given by (1) and (4), where $x \in [0,1]$ and $\mathrm{x}_0,\mathrm{x} \in [0,1]^k$, can also be obtained by setting $s = 0$ above.

8.3. *Degrees (8) and (9) of approximation by the Bernstein-type approximations (3) and (5)*

For every $\delta > 0$, we denote by $\lambda(x_0, x;\delta)$ the maximum integer less than or equal to $\delta^{-1}|x_0 - x|$, where $x_0, x \in [0,1]$. We recall the definition of modulus of continuity $\omega(\delta)$, where $\delta > 0$. We have

$$|g(x_0) - g(x)| \le \omega(\delta)\{1 + \lambda(x_0, x;\delta)\}, \tag{31}$$

$x_0, x \in [0,1]$.

The Bernstein-type approximation $B_m^{(s)}(g;x)$ is given by (3), where $s > -1/2$, and $x \in [0,1]$. Then, we have

$$\begin{aligned}
\left|B_m^{(s)}(g;x) - g(x)\right| &\le \sum_{v=0}^{m} \left| g(m^{-s}(m^{-1}v - x) + x) - g(x)\right| \binom{m}{v} x^v (1-x)^{m-v} \\
&\le \omega(\delta) \sum_{v=0}^{m} \{1 + \lambda(x_0, x;\delta)\} \binom{m}{v} x^v (1-x)^{m-v} \\
&\le \omega(\delta) \sum_{v=0}^{m} \{1 + \delta^{-1}\left| m^{-s}(m^{-1}v - x)\right|\} \binom{m}{v} x^v (1-x)^{m-v} \\
&\le \omega(\delta) \sum_{v=0}^{m} \{1 + \delta^{-2} m^{-2s-2}(v - mx)^2\} \binom{m}{v} x^v (1-x)^{m-v}
\end{aligned}$$

$x \in [0,1]$. It follows that

$$\left| B_m^{(s)}(g;x) - g(x)\right| \le \omega(\delta)\{1 + \delta^{-2} m^{-2s-1} x(1-x)\},$$

$x \in [0,1]$. We observe that $0 \le x(1-x) \le 1/4$, $x \in [0,1]$. Setting $\delta = m^{-1/2}$, we finally have the degree of approximation (8).

For every $\delta > 0$, we denote by $\lambda(\mathrm{x}_0, \mathrm{x}; \delta)$ the maximum integer less than or equal to $\delta^{-1}|\mathrm{x}_0 - \mathrm{x}|$, where $\mathrm{x}_0, \mathrm{x} \in [0,1]^k$. We have

$$|g(\mathrm{x}_0) - g(\mathrm{x})| \le \omega(\delta)\{1 + \lambda(\mathrm{x}_0, \mathrm{x}; \delta)\},$$

where $\omega(\delta)$ is the modulus of continuity, $\delta > 0$, and $\mathrm{x}_0, \mathrm{x} \in [0,1]^k$.

The multivariate Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g; \mathrm{x})$ is given by (5), where $s > -1/2$, and $\mathrm{x} \in [0,1]^k$. We have

$$\left| B_{\mathrm{m}}^{(s)}(g; \mathrm{x}) - g(\mathrm{x}) \right| \le \sum_{v_1=0}^{m_1} \cdots \sum_{v_k=0}^{m_k} \left| g\begin{pmatrix} m_1^{-s}(m_1^{-1}v_1 - x_1) + x_1 \\ \vdots \\ m_k^{-s}(m_k^{-1}v_k - x_k) + x_k \end{pmatrix} - g\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \right|$$

$$\cdot \binom{m_1}{v_1} \cdots \binom{m_k}{v_k} x_1^{v_1}(1-x_1)^{m_1-v_1} \cdots x_k^{v_k}(1-x_k)^{m_k-v_k}$$

$$\le \omega(\delta) \sum_{v_1=0}^{m_1} \cdots \sum_{v_k=0}^{m_k} \left\{ 1 + \delta^{-2} \sum_{i=1}^{k} m_i^{-2s-2}(v_i - m_i x_i)^2 \right\}$$

$$\cdot \binom{m_1}{v_1} \cdots \binom{m_k}{v_k} x_1^{v_1}(1-x_1)^{m_1-v_1} \cdots x_k^{v_k}(1-x_k)^{m_k-v_k},$$

$\mathrm{x} \in [0,1]^k$. Thus, we have

$$\left| B_{\mathrm{m}}^{(s)}(g; \mathrm{x}) - g(\mathrm{x}) \right| \le \omega(\delta) \left\{ 1 + \delta^{-2} \sum_{i=1}^{k} m_i^{-2s-1} x_i (1 - x_i) \right\},$$

$\mathrm{x} \in [0,1]^k$. We can observe that $0 \le x_i(1-x_i) \le 1/4$, $i = 1, \ldots, k$, $\mathrm{x} \in [0,1]^k$. Setting $\delta = m^{-1/2}$, where $m = \sum_{i=1}^{k} m_i$, we finally have the degree of approximation (9).

Degrees (6) and (7) of approximation by the Bernstein polynomials $B_m(g; x)$ and $B_{\mathrm{m}}(g; \mathrm{x})$, given by (1) and (4), where $x \in [0,1]$ and $\mathrm{x} \in [0,1]^k$, can be proved by setting $s = 0$ above.

8.4. *Orders of error in (12) and (13), and orders of error in probability in (14) and (15)*

The Bernstein-type approximation $B_m^{(s)}(g; \bar{x})$ is given by (10), where $s > -1/2$. We consider n as fixed. Let $g'(x) = (dx)^{-1} dg(x)$ and

$g''(x) = (dx)^{-2} d^2 g(x)$ be the first two derivatives of the function $g(x)$, $x \in [0,1]$. By Taylor expanding the function $g(m^{-s}(m^{-1}v - \bar{x}) + \bar{x})$ around $\bar{x}$, for every $v = 0,1,\ldots,m$, we have

$$
\begin{aligned}
B_m^{(s)}(g;\bar{x}) &= g(\bar{x}) \\
&\quad + \frac{1}{2} g'(\bar{x}) \sum_{v=0}^{m} m^{-s}(m^{-1}v - \bar{x}) \binom{m}{v} \bar{x}^v (1-\bar{x})^{m-v} \\
&\quad + \frac{1}{2} g''(\bar{x}) \sum_{v=0}^{m} m^{-2s}(m^{-1}v - \bar{x})^2 \binom{m}{v} \bar{x}^v (1-\bar{x})^{m-v} \\
&\quad + \cdots \\
&= g(\bar{x}) + \frac{1}{2} g''(\bar{x}) m^{-2s-1} \bar{x}(1-\bar{x}) + O(m^{-3s-2}),
\end{aligned}
$$

where $s > -1/2$, as $m \to \infty$. Order $O(m^{-2s-1})$ of error in (12), $s > -1/2$, as $m \to \infty$, with n fixed, is thus proved.

The Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\mathrm{\bar{x}})$ is given by (11), where $s > -1/2$. We consider n as fixed. By Taylor expanding the function

$$
g\begin{pmatrix} m_1^{-s}(m_1^{-1} v_1 - \bar{x}_1) + \bar{x}_1 \\ \vdots \\ m_k^{-s}(m_k^{-1} v_k - \bar{x}_k) + \bar{x}_k \end{pmatrix}
$$

around $\mathrm{x} = (\bar{x}_1,\ldots,\bar{x}_k)^T$, for every $v_i = 1,\ldots,m_i$, $i = 1,\ldots,k$, we can prove the order $\sum_{i=1}^{k} O(m_i^{-2s-1})$ of error in (13), $s > -1/2$, as $m_i \to \infty$, where $i = 1,\ldots,k$, with n fixed.

The Bernstein-type approximation $B_m^{(s)}(g,\bar{x})$ is given by (10), where $s > -1/2$. We observe that $\bar{x} - \mu = O_p(n^{-1/2})$, as $n \to \infty$, and $(\bar{x} - \mu)^2 = O_p(n^{-1})$, as $n \to \infty$. By Taylor expanding the function $g(m^{-s}(m^{-1}v - \bar{x}) + \bar{x})$ around μ, for every $v = 0,1,\ldots,m$, we have

$$
\begin{aligned}
B_m^{(s)}(g;\overline{x}) &= g(\mu) \\
&+ g'(\mu)\sum_{v=0}^{m}\{m^{-s}(m^{-1}v-\overline{x})+(\overline{x}-\mu)\}\binom{m}{v}\overline{x}^{v}(1-\overline{x})^{m-v} \\
&+ \frac{1}{2}g''(\mu)\sum_{v=0}^{m}\{m^{-2s}(m^{-1}v-\overline{x})+(\overline{x}-\mu)\}^{2}\binom{m}{v}\overline{x}^{v}(1-\overline{x})^{m-v} \\
&+\cdots \\
&= g(\mu) \\
&+ g'(\mu)(\overline{x}-\mu) \\
&+ \frac{1}{2}g''(\mu)\{m^{-2s-1}(\mu(1-\mu)+O_p(n^{-1/2}))+(\overline{x}-\mu)^2\} \\
&+\cdots \\
&= g(\mu)+O_p(n^{-1/2})+O(m^{-2s-1})+O_p(m^{-2s-1}n^{-1/2})+O_p(n^{-1}) \\
&= g(\mu)+O(m^{-2s-1})+O_p(n^{-1/2}),
\end{aligned}
$$

where $s>-1/2$, as $m\to\infty$, and $n\to\infty$. Order $O(m^{-2s-1})+O_p(n^{-1/2})$ of error in probability in (14), as $m\to\infty$, and $n\to\infty$, is thus proved.

The Bernstein-type approximation $B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})$ is given by (11), where $s>-1/2$. We consider n as fixed. By Taylor expanding the function

$$
g\begin{pmatrix} m_1^{-s}(m_1^{-1}v_1-\overline{x}_1)+\overline{x}_1 \\ \vdots \\ m_k^{-s}(m_k^{-1}v_k-\overline{x}_k)+\overline{x}_k \end{pmatrix}
$$

around $\mu=(\mu_1,\ldots,\mu_k)^T$, for every $v_i=1,\ldots,m_i$, $i=1,\ldots,k$, we can prove the order $\sum_{i=1}^{k}O(m_i^{-2s-1})+O_p(n^{-1/2})$ of error in probability in (15), $s>-1/2$, as $m_i\to\infty$, where $i=1,\ldots,k$, and $n\to\infty$.

8.5. *Asymptotic normality in (16) and (17)*

Following (12) and (14), we have

$$
n^{1/2}\{B_m^{(s)}(g;\overline{x})-g(\mu)\}=n^{1/2}\{g(\overline{x})-g(\mu)\},
$$

where $s>-1/2$, as $m\to\infty$, and $n\to\infty$. We consider the Taylor expansion of $g(\overline{x})$ around μ. An application of the Central Limit Theorem then shows the asymptotic normality in (16), as $m\to\infty$, and $n\to\infty$.

Following (13) and (15), we have

$$n^{1/2}\{B_{\mathrm{m}}^{(s)}(g;\overline{\mathrm{x}})-g(\mu)\}=n^{1/2}\{g(\overline{\mathrm{x}})-g(\mu)\},$$

where $s>-1/2$, $\mathrm{m}=(m_1,\ldots,m_k)^T$, as $m_i\to\infty$, where $i=1,\ldots,k$, and $n\to\infty$. We consider the Taylor expansion of $g(\overline{\mathrm{x}})$ around $\mu=(\mu_1,\ldots,\mu_k)^T$. An application of the Central Limit Theorem then shows the asymptotic normality in (17), as $m_i\to\infty$, where $i=1,\ldots,k$, and $n\to\infty$.

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REFERENCES

M. AIGNER (1997), *Combinatorial Theory*, Reprint of the 1979 Edition, Springer-Verlag, Berlin.

G.J. BABU, A.J. CANTY, Y.P. CHAUBEY (2002), *Application of Bernstein polynomials for smooth estimation of a distribution and density function*, "Journal of Statistical Planning and Inference", 105, 377-392.

O.E. BARNDORFF-NIELSEN, D.R. COX (1989), *Asymptotic Techniques for Use in Statistics*, Chapman and Hall, London.

R.A. BECKER, J.M. CHAMBERS, A.R. WILKS (1988), The *New S Language*, Wadsworth and Brooks/ Cole, Pacific Grove, California.

E.W. CHENEY (1982), *Introduction to Approximation Theory*, 2nd Edition, AMS Chelsea Publishing, American Mathematical Society, Providence, Rhode Island.

P.J. DAVIS (1963), *Interpolation and Approximation*, Blaisdell Publishing Company, Waltham, Massachusetts.

W. FELLER (1968), *An Introduction to Probability Theory and Its Applications*, Volume I, 3rd Edition, Wiley and Sons, New York.

W. FELLER (1971), An *Introduction to Probability Theory and Its Applications*, Volume II, 2nd Edition, Wiley and Sons, New York.

I.I. GIKHMAN, A.V. SKOROKHOD (1996), *Introduction to the Theory of Random Processes*, Reprint of the 1969 Edition, Dover Publications, Mineola, New York.

P. HALL (1992), *The Bootstrap and Edgeworth Expansion*, Springer-Verlag, New York.

N.L. JOHNSON, S. KOTZ (1977), *Urn Models and Their Application*, Wiley and Sons, New York.

N.L. JOHNSON, S. KOTZ, S. BALAKRISHNAN (1997), *Discrete Multivariate Distributions*, Wiley and Sons, New York.

P.P. KOROVKIN (1960), *Linear Operators and Approximation Theory*, Hindustan Publishing Corporation, Delhi.

G.G. LORENTZ (1986), *Bernstein Polynomials*, Chelsea Publishing Company, New York.

A. PINKUS (2000), *Weierstrass and approximation theory*, "Journal of Approximation Theory", 107, 1-66.

R.J. RIVLIN (1981), *An Introduction to the Approximation of Functions*, Reprint of the 1969 Edition, Dover Publications, Mineola, New York.

P.K. SEN, J.M. SINGER (1993), *Large Sample Methods in Statistics*, Chapman and Hall, New York.

R.J. SERFLING (1980), *Approximation Theorems of Mathematical Statistics*, Wiley and Sons, New York.

A.A. SVESHNIKOV (1978), *Problems in Probability Theory*, Mathematical Statistics and Theory of Random Functions, Reprint of the 1968 Edition, Dover Publications, New York.
A.F. TIMAN (1994), *Theory of Approximation of Functions of a Real Variable*, Reprint of the 1963 Edition, Dover Publications, Mineola, New York.

RIASSUNTO

Approssimazioni del tipo di Bernstein di funzioni regolari

Viene proposta e studiata un'approssimazione del tipo di Bernstein per funzioni regolari. Proponiamo un'approssimazione del tipo di Bernstein con definizioni che direttamente applicano la distribuzione binomiale e la distribuzione binomiale multivariata. Le approssimazioni del tipo di Bernstein generalizzano i corrispondenti polinomi di Bernstein, considerando definizioni che dipendono da un conveniente coefficiente di approssimazione in nuclei lineari. Nelle approssimazioni del tipo di Bernstein, studiamo la convergenza uniforme ed il grado di approssimazione. Vengono anche proposti e studiati stimatori del tipo di Bernstein per funzioni regolari di medie nella popolazione.

SUMMARY

Bernstein-type approximations of smooth functions

The Bernstein-type approximation for smooth functions is proposed and studied. We propose the Bernstein-type approximation with definitions that directly apply the binomial distribution and the multivariate binomial distribution. The Bernstein-type approximations generalize the corresponding Bernstein polynomials, by considering definitions that depend on a convenient approximation coefficient in linear kernels. In the Bernstein-type approximations, we study the uniform convergence and the degree of approximation. The Bernstein-type estimators of smooth functions of population means are also proposed and studied.