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ON THE ESTIMATION OF PARAMETERS AND RELIABILITY FUNCTIONS OF A NEW TWO-PARAMETER LIFETIME DISTRIBUTION BASED ON TYPE II CENSORING

Ajit Chaturvedi
Department of Statistics, University of Delhi, Delhi 110007, India
Taruna Kumari[1]
Department of Statistics, University of Delhi, Delhi 110007, India
Vijay Kumar Pandey
Department of Statistics, Bareilly College, Bareilly 243001, India

1. INTRODUCTION AND PRELIMINARIES

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item or system, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength set-up is the probability $\mathscr{P} = P(X > Y)$, which represents the reliability of an item or system of random strength X subject to random stress Y. Estimation of $\mathscr{P} = P(X > Y)$, when the random variables (rvs) X and Y follow a specified distribution has been extensively discussed by many authors in the literature. Some recent contributions on the topic, to name but a few, can be found in the papers by Chaturvedi and Pathak (2012), Chaturvedi *et al.* (2016), Chaturvedi and Kumari (2015), Chaturvedi and Kumari (2017), Chaturvedi and Kumari (2019).

A number of lifetime models have been proposed for the analysis of life time data in literature by various authors, see Mann *et al.* (1974), Lawless, 1982, Martz and Waller (1982), Sinha (1986), Johnson *et al.* (1994), Kotz *et al.* (2003) etc. We know exponential and Weibull models cover either constant or monotone (increasing or decreasing) type of the hazard rates. But non-increasing or non-decreasing (non-monotone) hazard functions such as unimodal and bathtub shaped functions also arise in practice because at present scenario non-monotone hazard rate is useful in field of science, engineering, medical, ecology and space explorations. Therefore, handling lifetime data for non-monotonic hazard rates seems to be a growing interest. Also, in survival analysis

[1] Corresponding Author. E-mail: taruna1986@yahoo.com

applications, the hazard rate function may frequently present a unimodal shape. It is well known that the log-normal distribution is a popular model for the survival time when the hazard rate function is unimodal, see Nelson (1982). There are some other models that have unimodal shape hazard rate function, like the log-logistic, Burr type XII, Burr type III distribution and the inverse Weibull distributions.

Here it is pertinent to the current study to briefly discuss Abd-Elrahman (2017). The Bilal(θ) distribution was introduced by Abd-Elrahman (2013), as a member of the families of distributions for the median of a random sample drawn from an arbitrary lifetime distributions. Unfortunately, the failure rate function related to this distribution is monotonically increasing with finite limit. Therefore, Abd-Elrahman (2017) proposed a generalization of Bilal(θ) distribution. The author generalized the Bilal(θ) distribution by using the transformation: $X=\left(\frac{Y-\delta}{\theta}\right)^{\lambda}$, $(Y>\delta)$, the parameter δ is a threshold parameter, while θ and λ are the scale and the shape parameters, respectively. Without loss of generality, the parameter δ is set to zero. So obtained proposed model they referred to as the generalized Bilal i.e., $GB(\theta,\lambda)$ distribution. The probability density function (pdf), cumulative distribution function (cdf), hazard rate function and the reliability function of $GB(\theta,\lambda)$ are given by

$$f(x;\theta,\lambda)=\frac{6\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1}\exp\left(-2\left(\frac{x}{\theta}\right)^{\lambda}\right)\left(1-\exp\left(-\left(\frac{x}{\theta}\right)^{\lambda}\right)\right),\quad x\geq 0,\ \theta,\lambda>0, \tag{1}$$

$$F(x;\theta,\lambda)=1-\exp\left(-2\left(\frac{x}{\theta}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x}{\theta}\right)^{\lambda}\right)\right),\quad x\geq 0,\ \theta,\lambda>0, \tag{2}$$

$$h(x;\theta,\lambda)=\frac{6\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1}\left(\frac{1-\exp\left(-\left(\frac{x}{\theta}\right)^{\lambda}\right)}{3-2\exp\left(-\left(\frac{x}{\theta}\right)^{\lambda}\right)}\right),\quad x\geq 0,\ \theta,\lambda>0, \tag{3}$$

and

$$R(t;\theta,\lambda)=\exp\left(-2\left(\frac{t}{\theta}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{t}{\theta}\right)^{\lambda}\right)\right)\theta,\quad \lambda>0, \tag{4}$$

respectively.

Also, in order to study the estimation of the stress-strength parameter $\mathscr{P}=P(X>Y)$, where X and Y are independent rvs from (1), with common shape parameter $\lambda_1=\lambda_2=\lambda$ and scale parameters, $\theta_1>0$ and $\theta_2>0$, respectively. The pdf of X and Y are given by

$$f_1(x;\theta_1,\lambda)=\frac{6\lambda}{\theta_1}\left(\frac{x}{\theta_1}\right)^{\lambda-1}\exp\left(-2\left(\frac{x}{\theta_1}\right)^{\lambda}\right)\left(1-\exp\left(-\left(\frac{x}{\theta_1}\right)^{\lambda}\right)\right),\quad x\geq 0,\ \theta_1,\lambda>0,$$

and

$$f_2(y;\theta_2,\lambda)=\frac{6\lambda}{\theta_2}\left(\frac{y}{\theta_2}\right)^{\lambda-1}\exp\left(-2\left(\frac{y}{\theta_2}\right)^{\lambda}\right)\left(1-\exp\left(-\left(\frac{y}{\theta_2}\right)^{\lambda}\right)\right),$$
$$y\geq 0,\ \theta_2,\lambda>0.$$

Then, the stress-strength reliability is given by

$$\begin{aligned}\mathscr{P}=P(X>Y)&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}f_1(x;\theta_1,\lambda)f_2(y;\theta_2,\lambda)dxdy\\&=18\int_0^1 z(1-z)^{2\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+1}dz-12\int_0^1 z(1-z)^{3\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+1}dz\\&=\frac{\left(19\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+6\right)}{\left(\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+1\right)\left(2\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+3\right)\left(3\left(\frac{\theta_2}{\theta_1}\right)^{\lambda}+2\right)}.\end{aligned}\tag{5}$$

Since we know that the selection of loss function is an important part of Bayesian estimation procedures and SELF is frequently used, see Gep and Tiao (1973) and Berger (1985), due to its mathematical simplicity and relevance with classical procedures. But SELF is not suitable where the losses are not symmetric. LLF is frequently used when losses are asymmetric. It was originally introduced by Varian (1975) and got a lot of popularity due to Zellner (1986). The mathematical form of LLF for estimating α through its estimator $\widehat{\alpha}$ may simply be expressed as

$$L(\Delta)=b\left[\exp(a\Delta)-a\Delta-1\right],\quad a\neq 0, b>0,\tag{6}$$

where $\Delta=\widehat{\alpha}-\alpha$, a and b are respectively shape and scale parameters of the loss function given in (6), its asymmetric nature depends on shape parameter a. When value of a is less than zero, LLF gives more weight to under estimation against over estimation and the situation is reverse when value assigned to a is greater than zero. If a tends to zero LLF tends to SELF., viz,

$$L(\Delta)\propto\Delta^2.$$

Also, without loss of generality scale parameter b of LLF can be taken to 1.0. Under the LLF (6), the Bayes estimator (BE) of α is given by

$$\widehat{\alpha}=-\frac{1}{a}\ln E(\exp(-a\alpha)).\tag{7}$$

Maximum likelihood estimation, Bayes estimation and Lindley's approximation of the parameters under different loss functions has been discussed by several authors like

Raqab *et al.* (2008), Singh *et al.* (2008), Singh *et al.* (2013) and the references therein, to cite a few.

The paper is organized as follows. In Section 2, we provide the MLEs, BEs and Lindley's approximation of the parameters λ and θ. In Section 3, we proposed the MLE, BEs and Lindley's approximation for the Bayes estimators of $R(t)$. Next, in Section 4, we give the MLE and BEs of $\mathscr{P}$. Furthermore in Section 5, simulation is performed. Finally, in Section 6, discussions are made and conclusions are presented.

2. Estimators of the parameters λ and θ

Suppose n items are put on a test and the test is terminated after the first r ordered observations are recorded. Let $X = (x_1, x_2, x_3, \ldots, x_r)$, where $x_1 \leq x_2 \leq \cdots \leq x_r$, $0 < r \leq n$, the lifetimes of first r failures. Obviously, $(n-r)$ items survived until x_r. We assume that the lifetimes of these components follow the distribution expressed in (1). Hence the likelihood function of this setup can be written as

$$L = \frac{n!}{(n-r)!}\left(\frac{6\lambda}{\theta}\right)^r \prod_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right)$$
$$\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right)\right]^{n-r}. \tag{8}$$

2.1. Maximum likelihood estimators

The log likelihood function of the distribution given in (1) under the above said set-up is given by

$$\begin{aligned}\ln L = {} & \ln\left(\frac{n!}{(n-r)!}\right) + r\ln 6 + r\ln\lambda - r\ln\theta \\ & -2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda} + \sum_{i=1}^{r}\ln\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right) \\ & +(\lambda-1)\sum_{i=1}^{r}\ln\left(\frac{x_i}{\theta}\right) - 2(n-r)\left(\frac{x_r}{\theta}\right)^{\lambda} \\ & +(n-r)\ln\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right).\end{aligned} \tag{9}$$

To find the likelihood equations, we differentiate (9) with respect to the parameters

λ and θ, respectively and equate them to zero. The resulting equations are as follows:

$$
\begin{aligned}
0=\frac{\partial \ln L}{\partial \lambda} &= \frac{r}{\lambda}-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda}\ln\left(\frac{x_i}{\theta}\right)+\sum_{i=1}^{r}\frac{\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\left(\frac{x_i}{\theta}\right)^{\lambda}\ln\left(\frac{x_i}{\theta}\right)}{\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right)} \\
&\quad +\sum_{i=1}^{r}\ln\left(\frac{x_i}{\theta}\right)-2(n-r)\left(\frac{x_r}{\theta}\right)^{\lambda}\ln\left(\frac{x_r}{\theta}\right) \\
&\quad +\frac{2(n-r)\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\left(\frac{x_r}{\theta}\right)^{\lambda}\ln\left(\frac{x_r}{\theta}\right)}{\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right)},
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
0=\frac{\partial \ln L}{\partial \theta} &= -\frac{r}{\theta}+2\frac{\lambda}{\theta}\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda}-\frac{\lambda}{\theta}\sum_{i=1}^{r}\frac{\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\left(\frac{x_i}{\theta}\right)^{\lambda}}{\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right)}-r\frac{(\lambda-1)}{\theta} \\
&\quad +2(n-r)\frac{\lambda}{\theta}\left(\frac{x_r}{\theta}\right)^{\lambda}-\frac{2(n-r)\lambda\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\left(\frac{x_r}{\theta}\right)^{\lambda}}{\theta\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right)}.
\end{aligned}
\tag{11}
$$

Equations (10) and (11) are nonlinear and analytical solutions are not possible. Hence to obtain the solutions from these equations, we have used the nlm function available in *R Software*.

2.2. *Bayes estimators*

To obtain the BEs of λ and θ, we have considered independent non-informative type of priors, $\pi_1(\lambda)$ and $\pi_2(\theta)$, given as

$$\pi_1(\lambda)=\frac{1}{k},\quad 0<\lambda<k, \tag{12}$$

$$\pi_2(\theta)=\frac{1}{\theta},\quad 0<\theta. \tag{13}$$

Joint posterior density of λ and θ, obtained with the help of Bayes theorem which combines (8), (12) and (13), is given by

$$
\begin{aligned}
h(\lambda,\theta|\underset{\sim}{x}) &= \frac{\lambda^r}{N_1\theta^{r+1}}\prod_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right) \\
&\quad \cdot\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right)\right]^{n-r},
\end{aligned}
\tag{14}
$$

where

$$N_1=\int_0^k\int_0^\infty \frac{\lambda^r}{\theta^{r+1}}\prod_{i=1}^r\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^r\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^r\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$.\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r}d\lambda d\theta. \tag{15}$$

Marginal posterior densities of λ and θ obtained by integrating (14), with respect to the rest parameters, are given by

$$h(\lambda|\underset{\sim}{x})=\frac{\lambda^r N_2}{N_1},\ \ 0<\lambda<k, \tag{16}$$

$$h(\theta|\underset{\sim}{x})=\frac{N_3}{\theta^{r+1}N_1},\ \ 0<\theta, \tag{17}$$

where N_1 is given by (15) and; N_2 and N_3 are given by

$$N_2=\int_0^\infty \frac{1}{\theta^{r+1}}\prod_{i=1}^r\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^r\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^r\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$.\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r}d\theta, \tag{18}$$

and

$$N_3=\int_0^k \lambda^r\prod_{i=1}^r\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^r\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^r\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$.\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r}d\lambda. \tag{19}$$

In order to obtain the BEs of the parameters under SELF, we utilize the fact that the BEs of the parameters under SELF are nothing but the posterior mean of the corresponding parameters. Hence, the BE, $\widehat{\lambda}_{BS}$ of λ, under SELF, is given by

$$\widehat{\lambda}_{BS}=E(\lambda|\underset{\sim}{x})=\frac{1}{N_1}\int_0^k \lambda h(\lambda|\underset{\sim}{x})d\lambda$$
$$=\frac{N_4}{N_1}, \tag{20}$$

where N_1 is given in (15) and N_4 is given by

$$N_4=\int_0^k\int_0^\infty \frac{\lambda^{r+1}}{\theta^{r+1}}\prod_{i=1}^r\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^r\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^r\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$.\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r}d\lambda d\theta. \tag{21}$$

In a similar way, we have obtained the BE, $\widehat{\theta}_{BS}$ of θ, under SELF, which is given by

$$\widehat{\theta}_{BS} = \frac{N_5}{N_1}, \tag{22}$$

where, N_1 is given in (15) and

$$N_5 = \int_0^k \int_0^\infty \frac{\lambda^r}{\theta^r} \prod_{i=1}^r \left(\frac{x_i}{\theta}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^r \left(\frac{x_i}{\theta}\right)^\lambda\right) \prod_{i=1}^r \left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r} d\lambda d\theta. \tag{23}$$

Using (7), the BE, $\widehat{\lambda}_{LX}$ of the parameter λ, under LLF, is given by

$$\widehat{\lambda}_{LX} = -\frac{1}{a}\ln E(\exp(-a\lambda)), \tag{24}$$

where,

$$E(\exp(-a\lambda)) = \int_0^k \int_0^\infty \frac{\exp(-a\lambda)\lambda^r}{N_1\theta^{r+1}} \exp\left(-2\sum_{i=1}^r \left(\frac{x_i}{\theta}\right)^\lambda\right) \prod_{i=1}^r \left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$\cdot\prod_{i=1}^r \left(\frac{x_i}{\theta}\right)^{\lambda-1}\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r} d\lambda d\theta. \tag{25}$$

Thus using (24) and (25), the BE of λ is given by

$$\widehat{\lambda}_{LX} = -\frac{1}{a}\ln\left(\frac{N_6}{N_1}\right), \tag{26}$$

where

$$N_6 = \int_0^k \int_0^\infty \frac{\exp(-a\lambda)\lambda^r}{\theta^{r+1}} \exp\left(-2\sum_{i=1}^r \left(\frac{x_i}{\theta}\right)^\lambda\right) \prod_{i=1}^r \left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)$$
$$\cdot\prod_{i=1}^r \left(\frac{x_i}{\theta}\right)^{\lambda-1}\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r} d\lambda d\theta. \tag{27}$$

Proceeding on the similar lines the BE $\widehat{\theta}_{LX}$ of θ, under LLF, is given by

$$\widehat{\theta}_{LX} = -\frac{1}{a}\ln\left(\frac{N_7}{N_1}\right), \tag{28}$$

where

$$N_7 = \int_0^k \int_0^\infty \frac{\exp(-a\theta)\lambda^r}{\theta^{r+1}} \exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right) \cdot \prod_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda-1}\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r} d\lambda d\theta. \tag{29}$$

2.3. *Lindley's approximation*

In this subsection, we consider the Lindley's approximation technique for the estimation of λ and θ. Consider the posterior expectation $I(x)$ expressible in the form of ratio of integral as given below

$$I(\underline{x}) = E[U(\lambda,\theta)|\underline{x}] = \frac{\int_{(\lambda,\theta)} U(\lambda,\theta)e^{L(\lambda,\theta)+\rho(\lambda,\theta)}d(\lambda,\theta)}{\int_{(\lambda,\theta)} e^{L(\lambda,\theta)+\rho(\lambda,\theta)}d(\lambda,\theta)}, \tag{30}$$

where

$$\begin{aligned} U(\lambda,\theta) &= \text{a function of } \lambda \text{ and } \theta \text{ only}, \\ L(\lambda,\theta) &= \text{Log likelihood}, \\ \rho(\lambda,\theta) &= \text{Log of joint prior of } \lambda \text{ and } \theta. \end{aligned}$$

If sample size n is sufficiently large, according to Lindley (1980), it can now be approximately evaluated as

$$\begin{aligned} I(x) = U(\widehat{\lambda},\widehat{\theta}) &+ \frac{1}{2}\Big[\left(\widehat{U}_{\lambda\lambda}+2\widehat{U}_{\lambda}\widehat{\rho}_{\lambda}\right)\widehat{\sigma}_{\lambda\lambda}+\left(\widehat{U}_{\theta\lambda}+2\widehat{U}_{\theta}\widehat{\rho}_{\lambda}\right)\widehat{\sigma}_{\theta\lambda} \\ &\qquad +\left(\widehat{U}_{\lambda\theta}+2\widehat{U}_{\lambda}\widehat{\rho}_{\theta}\right)\widehat{\sigma}_{\lambda\theta}+\left(\widehat{U}_{\theta\theta}+2\widehat{U}_{\theta}\widehat{\rho}_{\theta}\right)\widehat{\sigma}_{\theta\theta}\Big] \\ &+ \frac{1}{2}\Big[\left(\widehat{U}_{\lambda}\widehat{\sigma}_{\lambda\lambda}+\widehat{U}_{\theta}\widehat{\sigma}_{\lambda\theta}\right)\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda}+\widehat{L}_{\lambda\theta\lambda}\widehat{\sigma}_{\lambda\theta}+\widehat{L}_{\theta\lambda\lambda}\widehat{\sigma}_{\theta\lambda}+\widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta}\right) \\ &\qquad +\left(\widehat{U}_{\lambda}\widehat{\sigma}_{\theta\lambda}+\widehat{U}_{\theta}\widehat{\sigma}_{\theta\theta}\right)\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda}+\widehat{L}_{\lambda\theta\theta}\widehat{\sigma}_{\lambda\theta}+\widehat{L}_{\theta\lambda\theta}\widehat{\sigma}_{\theta\lambda}+\widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta}\right)\Big] \end{aligned} \tag{31}$$

where

$$\widehat{\lambda} = \text{MLE of } \lambda, \quad \widehat{\theta} = \text{MLE of } \theta, \quad \widehat{\rho}_{\lambda} = \frac{\partial \rho(\lambda,\theta)}{\partial \lambda} \quad \widehat{\rho}_{\theta} = \frac{\partial \rho(\lambda,\theta)}{\partial \theta}$$

$$\widehat{U}_{\lambda} = \frac{\partial U(\lambda,\theta)}{\partial \lambda}, \quad \widehat{U}_{\theta} = \frac{\partial U(\lambda,\theta)}{\partial \theta}, \quad \widehat{U}_{\theta\lambda} = \frac{\partial^2 U(\lambda,\theta)}{\partial \theta \partial \lambda}, \quad \widehat{U}_{\theta\theta} = \frac{\partial^2 U(\lambda,\theta)}{\partial \theta \partial \theta},$$

$$\widehat{U}_{\lambda\theta} = \frac{\partial^2 U(\lambda,\theta)}{\partial \lambda \partial \theta}, \quad \widehat{U}_{\lambda\lambda} = \frac{\partial^2 U(\lambda,\theta)}{\partial \lambda \partial \lambda}, \quad \widehat{\sigma}_{\theta\theta} = -\frac{1}{\widehat{L}_{\theta\theta}}, \quad \widehat{\sigma}_{\lambda\lambda} = -\frac{1}{\widehat{L}_{\lambda\lambda}}$$

$$\widehat{L}_{\lambda\lambda\lambda} = \frac{\partial^3 U(\lambda,\theta)}{\partial \lambda \partial \lambda \partial \lambda}, \quad \widehat{L}_{\lambda\lambda\theta} = \frac{\partial^3 U(\lambda,\theta)}{\partial \lambda \partial \lambda \partial \theta}, \quad \widehat{L}_{\theta\theta\lambda} = \frac{\partial^3 U(\lambda,\theta)}{\partial \theta \partial \theta \partial \lambda}, \quad \widehat{L}_{\theta\theta\theta} = \frac{\partial^3 U(\lambda,\theta)}{\partial \theta \partial \theta \partial \theta}$$

etc.

More elaborately, here $U_{\lambda\lambda}$ denotes the second derivative of the function $U(\lambda,\theta)$ with respect to λ and $\widehat{U}_{\lambda\lambda}$ represents the same expression evaluated at $\lambda = \widehat{\lambda}$ and $\theta = \widehat{\theta}$. All other quantities can be defined in similar manner.

Thus, using Lindley's approximation the Bayes estimate of λ, under SELF is obtained as:

$U(\lambda,\theta) = \lambda$, $\widehat{U}_{\lambda} = 1$, $\widehat{U}_{\theta} = 0$, $\widehat{U}_{\lambda\lambda} = \widehat{U}_{\lambda\theta} = \widehat{U}_{\theta\lambda} = \widehat{U}_{\theta\theta} = 0$, $\widehat{\sigma}_{\theta\lambda} = \widehat{\sigma}_{\theta\lambda} = 0$,
$\widehat{\rho}_{\lambda} = 0$, $\widehat{\rho}_{\theta} = -\dfrac{1}{\theta}$.

So,

$$\widehat{\lambda}_{LS} = \widehat{\lambda} + \frac{1}{2}\widehat{\sigma}_{\lambda\lambda}\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta}\right). \tag{32}$$

Also, the Bayes estimate of θ, under SELF is obtained as:

$U(\lambda,\theta) = \theta$, $\widehat{U}_{\lambda} = 0$, $\widehat{U}_{\theta} = 1$, $\widehat{U}_{\lambda\lambda} = \widehat{U}_{\lambda\theta} = \widehat{U}_{\theta\lambda} = \widehat{U}_{\theta\theta} = 0$, $\widehat{\sigma}_{\lambda\theta} = \widehat{\sigma}_{\theta\lambda} = 0$,
$\widehat{\rho}_{\lambda} = 0$, $\widehat{\rho}_{\theta} = -\dfrac{1}{\theta}$.

So,

$$\widehat{\theta}_{LS} = \widehat{\theta} - \frac{1}{\widehat{\theta}}\widehat{\sigma}_{\theta\theta} + \frac{1}{2}\widehat{\sigma}_{\theta\theta}\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta}\right). \tag{33}$$

Next, using Lindley's approximation the Bayes estimate of λ, under LLF is obtained as:

$$U(\lambda,\theta) = \exp(-a\,\lambda),\; \widehat{U}_{\lambda} = -a\exp(-a\,\widehat{\lambda}),\; \widehat{U}_{\lambda\lambda} = a^2\exp(-a\,\widehat{\lambda}),$$

$$\widehat{U}_{\theta} = \widehat{U}_{\lambda\theta} = \widehat{U}_{\theta\lambda} = \widehat{U}_{\theta\theta} = 0,\; \widehat{\sigma}_{\theta\lambda} = \widehat{\sigma}_{\theta\lambda} = 0,\; \widehat{\rho}_{\lambda} = 0,\; \widehat{\rho}_{\theta} = -\frac{1}{\theta},$$

$$E\left(\exp(-a\,\lambda)\right) = \exp(-a\,\widehat{\lambda}) + \frac{1}{2}a^2\exp(-a\,\widehat{\lambda})\widehat{\sigma}_{\lambda\lambda}$$
$$-\frac{1}{2}a\exp(-a\,\widehat{\lambda})\widehat{\sigma}_{\lambda\lambda}\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta}\right).$$

Therefore,

$$\widehat{\lambda}_{LL} = -\frac{1}{a}\ln\left[\exp(-a\,\widehat{\lambda}) - \frac{1}{2}a\exp(-a\,\widehat{\lambda})\widehat{\sigma}_{\lambda\lambda}\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta} - a\right)\right]. \tag{34}$$

Also, the Bayes estimate of θ, under LLF is obtained as

$$U(\lambda,\theta) = \exp(-a\,\theta),\ \widehat{U}_{\theta} = -a\exp(-a\,\widehat{\theta}),\ \widehat{U}_{\theta\theta} = a^2\exp(-a\,\widehat{\theta}),$$
$$\widehat{U}_{\lambda} = \widehat{U}_{\lambda\theta} = \widehat{U}_{\theta\lambda} = \widehat{U}_{\lambda\lambda} = 0,\ \widehat{\sigma}_{\theta\lambda} = \widehat{\sigma}_{\theta\lambda} = 0,\ \widehat{\rho}_{\lambda} = 0,\ \widehat{\rho}_{\theta} = -\frac{1}{\theta},$$
$$E(\exp(-a\,\theta)) = \exp(-a\,\widehat{\theta}) + \frac{1}{2}\exp(-a\,\widehat{\lambda})\widehat{\sigma}_{\theta\theta}\left(a^2 + 2\frac{a}{\widehat{\theta}}\right)$$
$$- \frac{1}{2}a\exp(-a\,\widehat{\theta})\widehat{\sigma}_{\theta\theta}\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta}\right).$$

Therefore,

$$\widehat{\theta}_{LL} = -\frac{1}{a}\ln\left[\exp(-a\,\widehat{\theta}) - \frac{1}{2}a\exp(-a\,\widehat{\theta})\widehat{\sigma}_{\theta\theta}\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta} - a - \frac{2}{\widehat{\theta}}\right)\right]. \tag{35}$$

3. Estimators of the reliability function $R(t)$

3.1. Maximum likelihood estimator

If $\widetilde{\lambda}$ and $\widetilde{\theta}$ denotes the MLE of λ and θ based on type II censoring scheme then by using the standard likelihood theory the MLE of the reliability function $R(t)$ is given by

$$\widetilde{R}(t) = \exp\left(-2\left(\frac{t}{\widetilde{\theta}}\right)^{\widetilde{\lambda}}\right)\left(3 - 2\exp\left(-\left(\frac{t}{\widetilde{\theta}}\right)^{\widetilde{\lambda}}\right)\right). \tag{36}$$

3.2. Bayes estimators

The BE of $R(t)$, under SELF, is given by

$$\widehat{R}(t)_{BS} = \int_0^k\int_0^\infty R(t)h(\lambda,\theta|\underline{x})d\lambda d\theta = \frac{N_8}{N_1}, \tag{37}$$

where, N_1 is given in (15) and N_8 is given by

$$N_8 = \int_0^k\int_0^\infty \exp\left(-2\left(\frac{t}{\theta}\right)^{\lambda}\right)\left(3 - 2\exp\left(-\left(\frac{t}{\theta}\right)^{\lambda}\right)\right)$$
$$\cdot\frac{\lambda^r}{\theta^{r+1}}\prod_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\prod_{i=1}^{r}\left(1 - \exp\left(-\left(\frac{x_i}{\theta}\right)^{\lambda}\right)\right)$$
$$\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\left(3 - 2\exp\left(-\left(\frac{x_r}{\theta}\right)^{\lambda}\right)\right)\right]^{n-r} d\lambda d\theta. \tag{38}$$

Proceeding on the similar lines as in Section 2, the BE of $R(t)$, under LLF, is given by

$$\begin{aligned}\widehat{R}(t)_{LX} &= -\frac{1}{a}\ln E(\exp(-aR(t)))\\ &= -\frac{1}{a}\ln\left(\frac{N_9}{N_1}\right), \end{aligned} \tag{39}$$

where, N_1 is given in (15) and N_9 is given by

$$\begin{aligned} N_9 = &\int_0^k\int_0^\infty \exp\left\{-a\exp\left(-2\left(\frac{t}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{t}{\theta}\right)^\lambda\right)\right)\right\}\\ &\cdot\frac{\lambda^r}{\theta^{r+1}}\prod_{i=1}^r\left(\frac{x_i}{\theta}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^r\left(\frac{x_i}{\theta}\right)^\lambda\right)\prod_{i=1}^r\left(1-\exp\left(-\left(\frac{x_i}{\theta}\right)^\lambda\right)\right)\\ &\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta}\right)^\lambda\right)\right)\right]^{n-r} d\lambda d\theta. \end{aligned} \tag{40}$$

3.3. *Lindley's approximation*

In this subsection, we consider the Lindley's approximation technique for the estimation of reliability function $R(t)$. Proceeding in a similar manner as in Section 2, we have obtained the Lindley's approximation for the Bayes estimate of $R(t)$, under SELF, which is obtained as

$$U(\lambda,\theta) = \exp\left(-2\left(\frac{t}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{t}{\theta}\right)^\lambda\right)\right),\ \widehat{\sigma}_{\theta\lambda}=\widehat{\sigma}_{\theta\lambda}=0,\ \widehat{\rho}_\lambda=0,\ \widehat{\rho}_\theta=-\frac{1}{\theta}.$$

So,

$$\begin{aligned}\widehat{R}(t)_{LS} = &U(\widehat{\lambda},\widehat{\theta}) + \frac{1}{2}\left(\widehat{U}_{\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \left(\widehat{U}_{\theta\theta} - \frac{2}{\widehat{\theta}}\widehat{U}_\theta\right)\widehat{\sigma}_{\theta\theta}\right)\\ &+\frac{1}{2}\left[\widehat{U}_\lambda\widehat{\sigma}_{\lambda\lambda}\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda}+\widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta}\right)+\widehat{U}_\theta\widehat{\sigma}_{\theta\theta}\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda}+\widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta}\right)\right]. \end{aligned} \tag{41}$$

The Lindley's approximation for the Bayes estimate of $R(t)$, under SELF, is given as

$$\begin{aligned} &U(\lambda,\theta) = \exp\left[-a\exp\left(-2\left(\frac{t}{\theta}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{t}{\theta}\right)^\lambda\right)\right)\right],\\ &\widehat{\sigma}_{\theta\lambda} = \widehat{\sigma}_{\theta\lambda} = 0,\ \widehat{\rho}_\lambda = 0,\ \widehat{\rho}_\theta = -\frac{1}{\theta}. \end{aligned}$$

So,

$$\widehat{R}(t)_{LL} = -\frac{1}{a}\ln\left[U(\widehat{\lambda},\widehat{\theta}) + \frac{1}{2}\left(\widehat{U}_{\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \left(\widehat{U}_{\theta\theta} - \frac{2}{\widehat{\theta}}\widehat{U}_{\theta}\right)\widehat{\sigma}_{\theta\theta}\right)\right.$$
$$\left. + \frac{1}{2}\left[\widehat{U}_{\lambda}\widehat{\sigma}_{\lambda\lambda}\left(\widehat{L}_{\lambda\lambda\lambda}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\lambda}\widehat{\sigma}_{\theta\theta}\right) + \widehat{U}_{\theta}\widehat{\sigma}_{\theta\theta}\left(\widehat{L}_{\lambda\lambda\theta}\widehat{\sigma}_{\lambda\lambda} + \widehat{L}_{\theta\theta\theta}\widehat{\sigma}_{\theta\theta}\right)\right]\right]. \tag{42}$$

4. ESTIMATORS OF THE RELIABILITY FUNCTION $\mathscr{P}$

Let us suppose that n items on X and m items on Y are put on a test. Also, suppose $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be two independent type II censored samples from $GB(\theta_1, \lambda)$ and $GB(\theta_2, \lambda)$. Then the joint likelihood function is given by

$$L = \frac{n!}{(n-r)!}\left(\frac{6\lambda}{\theta_1}\right)^r \prod_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right)$$
$$\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)\right]^{n-r}$$
$$\cdot\frac{m!}{(m-s)!}\left(\frac{6\lambda}{\theta_2}\right)^s \prod_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\prod_{i=1}^{s}\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)$$
$$\cdot\left[\exp\left(-2\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right)\right]^{m-s}. \tag{43}$$

4.1. *Maximum likelihood estimator*

To obtain the MLE of $\mathscr{P}$ based on type II censored data for both variables, we first evaluate the MLE of λ, θ_1 and θ_2. Using (43), the log likelihood function given by

$$\ln L = \ln\left(\frac{n!}{(n-r)!}\right) + r\ln 6 + r\ln\lambda - r\ln\theta_1 - 2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}$$
$$+\sum_{i=1}^{r}\ln\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right) + (\lambda-1)\sum_{i=1}^{r}\ln\left(\frac{x_i}{\theta_1}\right)$$
$$-2(n-r)\left(\frac{x_r}{\theta_1}\right)^{\lambda} + (n-r)\ln\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)$$

$$
\begin{aligned}
&+\ln\left(\frac{m!}{(m-s)!}\right)+s\ln 6+s\ln\lambda-s\ln\theta_2-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda}\\
&+\sum_{i=1}^{s}\ln\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)+(\lambda-1)\sum_{i=1}^{s}\ln\left(\frac{y_i}{\theta_2}\right)\\
&-2(m-s)\left(\frac{y_s}{\theta_2}\right)^{\lambda}+(m-s)\ln\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right).
\end{aligned}
\tag{44}
$$

To find the normal equations, we differentiate (44) with respect to the parameters λ, θ_1 and θ_2, respectively and equate them to zero. The resulting equations are as follows:

$$
\begin{aligned}
0=&\frac{\partial\ln L}{\partial\lambda}\\
=&\frac{r}{\lambda}-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}\ln\left(\frac{x_i}{\theta_1}\right)+\sum_{i=1}^{r}\frac{\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\left(\frac{x_i}{\theta_1}\right)^{\lambda}\ln\left(\frac{x_i}{\theta_1}\right)}{\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right)}+\sum_{i=1}^{r}\ln\left(\frac{x_i}{\theta_1}\right)\\
&-2(n-r)\left(\frac{x_r}{\theta_1}\right)^{\lambda}\ln\left(\frac{x_r}{\theta_1}\right)+\frac{2(n-r)\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\left(\frac{x_r}{\theta_1}\right)^{\lambda}\ln\left(\frac{x_r}{\theta_1}\right)}{\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)}\\
&+\frac{s}{\lambda}-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda}\ln\left(\frac{y_i}{\theta_2}\right)+\sum_{i=1}^{s}\frac{\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\left(\frac{y_i}{\theta_2}\right)^{\lambda}\ln\left(\frac{y_i}{\theta_2}\right)}{\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)}+\sum_{i=1}^{s}\ln\left(\frac{y_i}{\theta_2}\right)\\
&-2(m-s)\left(\frac{y_s}{\theta_2}\right)^{\lambda}\ln\left(\frac{y_s}{\theta_2}\right)+\frac{2(m-s)\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\left(\frac{y_s}{\theta_2}\right)^{\lambda}\ln\left(\frac{y_s}{\theta_2}\right)}{\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right)},
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
0=&\frac{\partial\ln L}{\partial\theta_1}\\
=&-\frac{r}{\theta_1}+2\frac{\lambda}{\theta_1}\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}-\frac{\lambda}{\theta_1}\sum_{i=1}^{r}\frac{\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\left(\frac{x_i}{\theta_1}\right)^{\lambda}}{\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right)}-r\frac{(\lambda-1)}{\theta_1}\\
&+2(n-r)\frac{\lambda}{\theta_1}\left(\frac{x_r}{\theta_1}\right)^{\lambda}+\frac{2(n-r)\lambda\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\left(\frac{x_r}{\theta_1}\right)^{\lambda}}{\theta_1\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
0 &= \frac{\partial \ln L}{\partial \theta_2} \\
&= -\frac{s}{\theta_2} + 2\frac{\lambda}{\theta_2}\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda} - \frac{\lambda}{\theta_2}\sum_{i=1}^{s}\frac{\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\left(\frac{y_i}{\theta_2}\right)^{\lambda}}{\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)} - s\frac{(\lambda-1)}{\theta_2} \\
&\quad + 2(m-s)\frac{\lambda}{\theta_2}\left(\frac{y_s}{\theta_2}\right)^{\lambda} + \frac{2(m-s)\lambda\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\left(\frac{y_s}{\theta_2}\right)^{\lambda}}{\theta_2\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right)}.
\end{aligned}
\tag{47}
$$

Thus, if $\tilde{\lambda}$, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ denotes the MLE of λ, θ_1 and θ_2 based on type II censoring then by using the standard likelihood theory the MLE of the stress-strength function $\mathscr{P}$ is given by

$$
\mathscr{P} = \frac{\left(19\left(\frac{\tilde{\theta}_2}{\tilde{\theta}_1}\right)^{\tilde{\lambda}}+6\right)}{\left(\left(\frac{\tilde{\theta}_2}{\tilde{\theta}_1}\right)^{\tilde{\lambda}}+1\right)\left(2\left(\frac{\tilde{\theta}_2}{\tilde{\theta}_1}\right)^{\tilde{\lambda}}+3\right)\left(3\left(\frac{\tilde{\theta}_2}{\tilde{\theta}_1}\right)^{\tilde{\lambda}}+2\right)}. \tag{48}
$$

4.2. *Bayes estimators*

To obtain the BEs of $\mathscr{P}$, we consider independent non-informative type of priors, $\pi_1(\lambda)$, $\pi_2(\theta_1)$ and $\pi_2(\theta_2)$, given as:

$$
\pi_1(\lambda) = \frac{1}{k}, \quad 0 < \lambda < k, \tag{49}
$$

$$
\pi_2(\theta_1) = \frac{1}{\theta_1}, \quad 0 < \theta_1, \tag{50}
$$

$$
\pi_3(\theta_2) = \frac{1}{\theta_2}, \quad 0 < \theta_2. \tag{51}
$$

Joint posterior density of λ, θ_1 and θ_2, obtained with the help of Bayes theorem which combines (49), (50), (51) and (43), is given by

$$
\begin{aligned}
&h\left(\lambda,\theta_1,\theta_2|\underset{\sim}{x},\underset{\sim}{y}\right)\\
&=\frac{\lambda^{r+s}}{T_1\theta_1^{r+1}\theta_2^{s+1}}\prod_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right)\\
&\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)\right]^{n-r}\\
&\cdot\prod_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\prod_{i=1}^{s}\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)\\
&\cdot\left[\exp\left(-2\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right)\right]^{m-s},
\end{aligned}\tag{52}
$$

where, T_1 is given by

$$
\begin{aligned}
T_1=&\int_0^k\int_0^\infty\int_0^\infty\frac{\lambda^{r+s}}{T_1\theta_1^{r+1}\theta_2^{s+1}}\prod_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\\
&\cdot\prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^{\lambda}\right)\right)\\
&\cdot\left[\exp\left(-2\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^{\lambda}\right)\right)\right]^{n-r}\\
&\cdot\prod_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda-1}\exp\left(-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\prod_{i=1}^{s}\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^{\lambda}\right)\right)\\
&\cdot\left[\exp\left(-2\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^{\lambda}\right)\right)\right]^{m-s}d\lambda d\theta_1 d\theta_2.
\end{aligned}\tag{53}
$$

The BE of $\mathscr{P}$, under SELF, is given by

$$
\widehat{\mathscr{P}}_{BS}=\int_0^k\int_0^\infty\int_0^\infty\mathscr{P}\ h(\lambda,\theta_,\theta_2|\underset{\sim}{x},\underset{\sim}{y})d\lambda d\theta_1 d\theta_2=\frac{T_2}{T_1},\tag{54}
$$

where, T_1 is given in (53) and T_2 is given by

$$
\begin{aligned}
T_2 = &\int_0^k \int_0^\infty \int_0^\infty \frac{\left(19\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 6\right)}{\left(\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 1\right)\left(2\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 3\right)\left(3\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 2\right)} \\
&\cdot \frac{\lambda^{r+s}}{T_1 \theta_1^{r+1} \theta_2^{s+1}} \prod_{i=1}^{r} \left(\frac{x_i}{\theta_1}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^\lambda\right) \prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^\lambda\right)\right) \\
&\cdot \left[\exp\left(-2\left(\frac{x_r}{\theta_1}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^\lambda\right)\right)\right]^{n-r} \\
&\cdot \prod_{i=1}^{s} \left(\frac{y_i}{\theta_2}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^\lambda\right) \prod_{i=1}^{s}\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^\lambda\right)\right) \\
&\cdot \left[\exp\left(-2\left(\frac{y_s}{\theta_2}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^\lambda\right)\right)\right]^{m-s} d\lambda d\theta_1 d\theta_2. \qquad (55)
\end{aligned}
$$

Proceeding on the similar lines as in Section 2 and 3, the BE of $\mathscr{P}$, under LLF, is given by

$$
\begin{aligned}
\widehat{\mathscr{P}}_{LX} &= -\frac{1}{a}\ln E(\exp(-a\mathscr{P})) \\
&= -\frac{1}{a}\ln\left(\frac{T_3}{T_1}\right), \qquad (56)
\end{aligned}
$$

where, T_1 is given in (53) and T_3 is given by

$$
\begin{aligned}
T_3 = &\int_0^k \int_0^\infty \int_0^\infty \exp\left\{\frac{-a\left(19\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 6\right)}{\left(\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 1\right)\left(2\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 3\right)\left(3\left(\frac{\theta_2}{\theta_1}\right)^\lambda + 2\right)}\right\} \\
&\cdot \frac{\lambda^{r+s}}{T_1 \theta_1^{r+1} \theta_2^{s+1}} \prod_{i=1}^{r} \left(\frac{x_i}{\theta_1}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{r}\left(\frac{x_i}{\theta_1}\right)^\lambda\right) \prod_{i=1}^{r}\left(1-\exp\left(-\left(\frac{x_i}{\theta_1}\right)^\lambda\right)\right) \\
&\cdot \left[\exp\left(-2\left(\frac{x_r}{\theta_1}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{x_r}{\theta_1}\right)^\lambda\right)\right)\right]^{n-r} \\
&\cdot \prod_{i=1}^{s} \left(\frac{y_i}{\theta_2}\right)^{\lambda-1} \exp\left(-2\sum_{i=1}^{s}\left(\frac{y_i}{\theta_2}\right)^\lambda\right) \prod_{i=1}^{s}\left(1-\exp\left(-\left(\frac{y_i}{\theta_2}\right)^\lambda\right)\right) \\
&\cdot \left[\exp\left(-2\left(\frac{y_s}{\theta_2}\right)^\lambda\right)\left(3-2\exp\left(-\left(\frac{y_s}{\theta_2}\right)^\lambda\right)\right)\right]^{m-s} d\lambda d\theta_1 d\theta_2. \qquad (57)
\end{aligned}
$$

Clearly, the BEs of the parameters λ and θ; and of reliability functions $R(t)$ and $\mathscr{P}$, do not result in nice closed forms due to involvement of multidimensional integrals. These expressions can not be simplified that is why, we obtain the estimates of the parameters λ and θ; and reliability functions $R(t)$ and $\mathscr{P}$, by using the `integrate` and `sapply` functions. Also, the expressions obtained to get the Bayes estimates of the parameters and reliability function using Lindley's approximation involves higher order derivatives. These derivatives are also not coming in nice closed form therefore, we evaluate these estimates using the `Deriv` function available in the `R software`.

5. Simulation study

Throughout this section comparisons are made on the basis of MSE. Also, for the random number generation from (1), we have used the algorithm given in Abd-Elrahman (2017).

To compare the performance of different estimators of the parameters λ and θ based on type II censoring scheme, we have conducted simulation experiments using Monte Carlo simulation technique. We have generated 1000 random samples from (1) each of size $n = 30$ for $(\lambda, \theta) = (1.4, 2.2), (2.0, 2.9)$ and $k = 3, 4$. For each sample we arranged the data in ascending order and considered a sample of first r $(r \leq n)$ observations. For different values of $r =$ 10, 15 and 20, using (9), (20), (22), (26), (28), (32), (33), (34) and (35), we have computed average maximum likelihood estimates, Bayes estimates and Lindley's estimates of λ and θ, their corresponding MSE, results are reported in Table 1.

Under the same set-up as discussed above, to compare the performances of the estimators of $R(t)$ for different values of t, using (9) (36), (37), (39), (41) and (42), we have computed average maximum likelihood estimates, Bayes estimates and Lindley's estimates of $R(t)$, their corresponding MSE. For $t=$ 1.0, 2.0, results are reported in Table 2.

Furthermore, to investigate performance of estimators of $\mathscr{P}$ based on type II censoring scheme, we have generated 1000 random samples from each of the populations X and Y with sizes $(n, m) =$ (30, 30) from (1) with $\lambda_1 = \lambda_2 = \lambda =$ 1.5, 2.0, 2.5, k=4.0, θ_1=1.6 and for different values of θ_2. Sample corresponding to both the populations are arranged in ascending order and first (r, s) observations are considered. For different pair of values (r, s), using (48), (54) and (56), we have computed average estimates with their corresponding MSEs are presented in Table 3.

From Table 1, we conclude that the best estimates of the parameters λ and θ are given by the estimators obtained under LLF as these estimates have the lowest MSE. Also from Table 2, we conclude that the best estimates of $R(t)$ are given by estimators obtained under SELF. Also, Table 1 and 2, depicts that the MSE of the estimators reduces as the sample size increases.

From Table 3, we conclude that for a moderate system estimates obtained under SELF are better then other. Also, for a system, which is not in good condition estimates obtained under LLF are better.

TABLE 1
Average estimates of the parameters λ and θ with their corresponding MSEs in paranthesis.

$\lambda\downarrow$	$\theta\downarrow$	$k\downarrow$	$r\downarrow$	$\tilde{\lambda}$	$\hat{\lambda}_{BS}$	$\hat{\lambda}_{LS}$	$\hat{\lambda}_{LX}$ $(a=1)$	$\hat{\lambda}_{LL}$ $(a=1)$
1.4	2.2	3	10	1.700 (0.415)	1.599 (0.201)	1.721 (0.434)	1.512 (0.155)	1.668 (0.361)
			15	1.564 (0.168)	1.536 (0.132)	1.573 (0.173)	1.479 (0.107)	1.530 (0.145)
			20	1.536 (0.117)	1.520 (0.102)	1.539 (0.119)	1.480 (0.086)	1.503 (0.100)
		4	10	1.671 (0.396)	1.634 (0.302)	1.693 (0.414)	1.530 (0.217)	1.641 (0.345)
			15	1.594 (0.205)	1.578 (0.187)	1.603 (0.211)	1.514 (0.145)	1.559 (0.176)
			20	1.518 (0.116)	1.508 (0.112)	1.521 (0.118)	1.467 (0.094)	1.485 (0.101)
2.0	2.9	3	10	2.428 (0.846)	2.026 (0.114)	2.456 (0.882)	1.923 (0.118)	2.351 (0.685)
			15	2.234 (0.343)	2.043 (0.102)	2.246 (0.351)	1.969 (0.098)	2.162 (0.275)
			20	2.195 (0.239)	2.065 (0.093)	2.200 (0.242)	2.008 (0.086)	2.127 (0.192)
		4	10	2.388 (0.809)	2.203 (0.318)	2.415 (0.842)	2.048 (0.240)	2.313 (0.657)
			15	2.277 (0.420)	2.179 (0.246)	2.289 (0.429)	2.077 (0.194)	2.201 (0.334)
			20	2.169 (0.237)	2.101 (0.164)	2.174 (0.240)	2.031 (0.138)	2.102 (0.193)
$\lambda\downarrow$	$\theta\downarrow$	$k\downarrow$	$r\downarrow$	$\tilde{\theta}$	$\hat{\theta}_{BS}$	$\hat{\theta}_{LS}$	$\hat{\theta}_{LX}$ $(a=1)$	$\hat{\theta}_{LL}$ $(a=1)$
1.4	2.2	3	10	2.092 (0.237)	2.393 (0.463)	2.123 (0.256)	2.170 (0.191)	2.074 (0.215)
			15	2.148 (0.106)	2.269 (0.135)	2.1667 (0.110)	2.192 (0.100)	2.134 (0.100)
			20	2.158 (0.069)	2.223 (0.076)	2.175 (0.070)	2.184 (0.068)	2.150 (0.067)
		4	10	2.122 (0.242)	2.420 (0.519)	2.155 (0.266)	2.184 (0.206)	2.102 (0.216)
			15	2.134 (0.112)	2.249 (0.140)	2.151 (0.116)	2.175 (0.107)	2.119 (0.107)
			20	2.158 (0.064)	2.223 (0.070)	2.1756 (0.066)	2.182 (0.063)	2.150 (0.062)
2.0	2.9	3	10	2.785 (0.204)	3.110 (0.285)	2.806 (0.210)	2.948 (0.143)	2.768 (0.191)
			15	2.845 (0.092)	2.987 (0.095)	2.857 (0.093)	2.931 (0.075)	2.831 (0.088)
			20	2.857 (0.060)	2.949 (0.054)	2.869 (0.060)	2.922 (0.050)	2.849 (0.059)
		4	10	2.813 (0.205)	3.069 (0.325)	2.834 (0.213)	2.904 (0.173)	2.795 (0.190)
			15	2.831 (0.098)	2.947 (0.107)	2.843 (0.099)	2.896 (0.091)	2.817 (0.095)
			20	2.857 (0.056)	2.945 (0.053)	2.869 (0.056)	2.921 (0.049)	2.848 (0.055)

TABLE 2

Average estimates of $R(t)$ with their corresponding MSEs in parenthesis.

λ	θ	t	$R(t)$	k	r	$\widetilde{R}(t)$	$\widehat{R}(t)_{BS}$	$\widehat{R}(t)_{LS}$	$\widehat{R}(t)_{LX}$ $(a=1.5)$	$\widehat{R}(t)_{LL}$ $(a=1.5)$
1.4	2.2	1.0	0.806	3	10	0.804 (0.006)	0.801 (0.005)	0.797 (0.005)	0.798 (0.005)	0.793 (0.006)
					15	0.805(0.004)	0.801 (0.004)	0.796 (0.004)	0.798 (0.004)	0.792 (0.004)
					20	0.812 (0.004)	0.807 (0.003)	0.804 (0.003)	0.804 (0.003)	0.800 (0.004)
				4	10	0.801 (0.006)	0.800 (0.005)	0.794 (0.006)	0.797 (0.006)	0.789 (0.006)
					15	0.808 (0.004)	0.805 (0.004)	0.799 (0.004)	0.802 (0.004)	0.795 (0.004)
					20	0.810 (0.003)	0.805 (0.003)	0.802 (0.003)	0.803 (0.003)	0.798 (0.003)
		2.0	0.376	3	10	0.303 (0.028)	0.340 (0.019)	0.310 (0.027)	0.327 (0.020)	0.304 (0.027)
					15	0.342 (0.014)	0.359 (0.012)	0.345 (0.013)	0.352 (0.012)	0.340 (0.014)
					20	0.360 (0.009)	0.370 (0.008)	0.363 (0.009)	0.365 (0.008)	0.359 (0.009)
				4	10	0.306 (0.029)	0.335 (0.021)	0.313 (0.027)	0.322 (0.022)	0.306 (0.027)
					15	0.346 (0.013)	0.361 (0.011)	0.349 (0.012)	0.354 (0.012)	0.344 (0.012)
					20	0.364 (0.008)	0.375 (0.008)	0.367 (0.008)	0.370 (0.008)	0.363 (0.008)
2.0	2.9	1.0	0.965	3	10	0.967 (6e-04)	0.954 (6e-04)	0.960 (7e-04)	0.954 (6e-04)	0.960 (7e-04)
					15	0.966 (6e-04)	0.956 (7e-04)	0.958 (8e-04)	0.955 (7e-04)	0.958 (8e-04)
					20	0.966 (5e-04)	0.958 (5e-04)	0.959 (6e-04)	0.957 (6e-04)	0.958 (6e-04)
				4	10	0.968 (6e-04)	0.959 (6e-04)	0.961 (7e-04)	0.958 (6e-04)	0.960 (7e-04)
					15	0.966 (5e-04)	0.958 (8e-04)	0.958 (7e-04)	0.958 (7e-04)	0.958 (7e-04)
					20	0.965 (5e-04)	0.958 (7e-04)	0.957 (7e-04)	0.957 (7e-04)	0.957 (7e-04)
		2.0	0.679	3	10	0.646 (0.016)	0.665 (0.009)	0.643 (0.0155)	0.660 (0.009)	0.636 (0.016)
					15	0.672 (0.006)	0.671 (0.006)	0.667 (0.006)	0.667 (0.007)	0.661 (0.006)
					20	0.678 (0.006)	0.679 (0.006)	0.674 (0.006)	0.675 (0.006)	0.669 (0.006)
				4	10	0.644 (0.014)	0.654 (0.010)	0.642 (0.013)	0.649 (0.011)	0.635 (0.014)
					15	0.672 (0.007)	0.674 (0.007)	0.667 (0.006)	0.669 (0.007)	0.662 (0.007)
					20	0.678 (0.006)	0.683 (0.007)	0.674 (0.006)	0.679 (0.007)	0.670 (0.006)

TABLE 3
Average estimates of $\mathscr{P}$ with their corresponding MSEs in parenthesis.

$\lambda\downarrow$	$r\downarrow$	$\theta_2\downarrow$	$\mathscr{P}$	$\widetilde{\mathscr{P}}$	$\widehat{\mathscr{P}}_{BS}$	$\widehat{\mathscr{P}}_{LX}$ $(a=1)$	$\widehat{\mathscr{P}}_{LX}$ $(a=2)$
1.5	(10,10)	1.2	0.655	0.671 (0.012)	0.658 (0.010)	0.653 (0.010)	0.649 (0.010)
		1.8	0.435	0.426 (0.013)	0.432 (0.011)	0.427 (0.011)	0.422 (0.011)
		2.2	0.330	0.321 (0.011)	0.335 (0.009)	0.331 (0.009)	0.326 (0.009)
	(15,15)	1.2	0.655	0.660 (0.008)	0.652 (0.008)	0.647 (0.008)	0.644 (0.008)
		1.8	0.435	0.432 (0.009)	0.436 (0.008)	0.432 (0.008)	0.429 (0.008)
		2.2	0.330	0.323 (0.008)	0.333 (0.007)	0.329 (0.007)	0.326 (0.007)
	(20,20)	1.2	0.655	0.660 (0.006)	0.653 (0.006)	0.651 (0.006)	0.648 (0.006)
		1.8	0.435	0.437 (0.007)	0.440 (0.006)	0.437 (0.007)	0.434 (0.006)
		2.2	0.330	0.320 (0.006)	0.326 (0.006)	0.323 (0.006)	0.321 (0.006)
2.0	(10,10)	1.2	0.702	0.716 (0.010)	0.699 (0.009)	0.695 (0.009)	0.690 (0.009)
		1.8	0.414	0.406 (0.012)	0.413 (0.010)	0.409 (0.010)	0.404 (0.010)
		2.2	0.279	0.262 (0.010)	0.281 (0.008)	0.278 (0.008)	0.273 (0.008)
	(15,15)	1.2	0.702	0.708 (0.008)	0.697 (0.007)	0.694 (0.007)	0.691 (0.007)
		1.8	0.414	0.410 (0.008)	0.414 (0.007)	0.411 (0.007)	0.407 (0.007)
		2.2	0.279	0.271 (0.007)	0.282 (0.006)	0.279 (0.006)	0.276 (0.006)
	(20,20)	1.2	0.702	0.709 (0.006)	0.700 (0.005)	0.698 (0.006)	0.696 (0.005)
		1.8	0.414	0.408 (0.007)	0.412 (0.006)	0.408 (0.007)	0.405 (0.006)
		2.2	0.279	0.274 (0.005)	0.282 (0.005)	0.280 (0.005)	0.277 (0.005)
2.5	(10,10)	1.2	0.746	0.758 (0.009)	0.734 (0.007)	0.730 (0.008)	0.726 (0.008)
		1.8	0.393	0.385 (0.013)	0.398 (0.010)	0.393 (0.010)	0.388 (0.010)
		2.2	0.233	0.213 (0.008)	0.240 (0.007)	0.235 (0.006)	0.232 (0.006)
	(15,15)	1.2	0.746	0.760 (0.007)	0.745 (0.006)	0.742 (0.006)	0.739 (0.006)
		1.8	0.393	0.390 (0.009)	0.396 (0.008)	0.392 (0.008)	0.389 (0.008)
		2.2	0.233	0.222 (0.006)	0.236 (0.00)	0.233 (0.006)	0.230 (0.006)
	(20,20)	1.2	0.746	0.752 (0.005)	0.740 (0.005)	0.738 (0.005)	0.735 (0.005)
		1.8	0.393	0.389 (0.007)	0.389 (0.007)	0.386 (0.007)	0.383 (0.007)
		2.2	0.233	0.225 (0.004)	0.230 (0.005)	0.227 (0.005)	0.225 (0.005)

6. Data analysis

In this section, we present different data sets which demonstrate the suitability of the GB distribution as a reliability model. We analyze two simulated data sets in Example 1 and two real data sets in Example 2.

EXAMPLE 1. Here, we analyze two simulated data sets. The first data set of size m=50 is generated from $GB(\theta_2 = 2.0,\ \lambda_2 = 1.5)$ and the second data set of size $n = 50$ is generated from $GB(\theta_1 = 3.0,\ \lambda_1 = 1.5)$. These two data sets are as follows:

Simulated data set 1 (Y):
0.328, 0.356, 0.675, 0.686, 0.695, 0.710, 0.712, 0.757, 0.910, 0.961, 0.984, 1.019, 1.087, 1.157, 1.259, 1.316, 1.405, 1.429, 1.449, 1.4450, 1.467, 1.500, 1.525, 1.538, 1.546, 1.574, 1.580, 1.638, 1.664, 1.667, 1.723, 1.726, 1.794, 1.815, 1.834, 1.839, 1.839, 1.855, 1.889, 1.894, 1.935, 1.937, 1.976, 2.019, 2.292, 2.312, 2.321, 2.532, 2.661, 2.836

Simulated data set 2 (X):
0.667, 0.766, 0.773, 0.898, 0.906, 1.026, 1.030, 1.114, 1.215, 1.497, 1.531, 1.585, 1.592, 1.763, 1.782, 1.946, 1.961, 1.991, 1.996, 2.077, 2.133, 2.144, 2.147, 2.160, 2.182, 2.220, 2.376, 2.402, 2.441, 2.470, 2.635, 2.712, 2.757, 2.837, 3.093, 3.184, 3.215, 3.297, 3.299, 3.351, 3.408, 3.555, 3.977, 3.993, 4.067, 4.185, 4.441, 4.594, 4.968, 5.740

By using the simulated data set 1 and set 2; and the expressions obtained in Section 1 and 2, respectively, we have calculated the maximum likelihood and Bayes estimates of the parameters $(\theta_2,\ \lambda_2)$, $(\theta_1,\ \lambda_1)$ and $R(t)$. Also, by considering simulated data set 1 as the stress population (Y) and simulated data set 2 as the strength population (X), we have calculated the maximum likelihood and Bayes estimates of $\mathscr{P}$ by using the expressions obtained in Section 3. For different values of r and (r, s) results are reported in Table 4.

EXAMPLE 2. In this example, we consider two real data sets and illustrate the inferential procedures discussed in the previous sections. The first data set of size $n = 69$ is a strength dataset originally reported by Bader and Priest (1982). This data represents the strength measured in Giga Pascal (GPA) for single carbon fibers and impregnated 1000 carbon fiber tows. The data set is as follows:

Real data set 1 (Y):
0.562, 0.564, 0.729, 0.802, 0.950, 1.053, 1.111, 1.115, 1.194, 1.208, 1.216, 1.247, 1.256, 1.271, 1.277, 1.305, 1.313, 1.348, 1.390, 1.429, 1.474, 1.490, 1.503, 1.520, 1.522, 1.524, 1.551, 1.551, 1.609, 1.632, 1.632, 1.676, 1.684, 1.685, 1.728, 1.740, 1.761, 1.764, 1.785, 1.804, 1.816, 1.824, 1.836, 1.879, 1.883, 1.892, 1.898, 1.934, 1.947, 1.976, 2.020, 2.023, 2.050, 2.059, 2.068, 2.071, 2.098, 2.130, 2.204, 2.262, 2.317, 2.334, 2.340, 2.346, 2.378, 2.483, 2.683, 2.835, 2.835

We fit the GB distribution to the above real data set 1 and compare its fitting with some well known reliability distributions, namely, exponential distribution with pdf given by $f(x) = \frac{1}{\lambda}\exp\left(\frac{-x}{\lambda}\right)\ x \geq 0,\ \lambda > 0$, Rayleigh distribution with pdf given by $f(x) = \frac{x}{\sigma^2}\exp\left(\frac{-x^2}{2\sigma^2}\right)\ x \geq 0,\ \sigma > 0$, Weibull distribution with pdf given by $f(x) =$

TABLE 4

Estimates based on simulated data set 1 and simulated data set 2 (Note: LY stands for Lindley's approximation).

Actual values ↓	r ↓		MLE		SELF		LY SELF		LLF (a=1)	LLF(a=2)		LY LLF (a=1)	LY LLF(a=2)
Estimates of λ, θ and $R(t)$ based on simulated data set 1 (Y), (k=3)													
	25	$\tilde{\lambda}_2 \rightarrow$	1.560	$\widehat{\lambda}_{2BS} \rightarrow$	1.575	$\widehat{\lambda}_{2LS} \rightarrow$	1.566	$\widehat{\lambda}_{2LX} \rightarrow$	1.535	1.496	$\widehat{\lambda}_{2LL} \rightarrow$	1.542	1.519
		$\tilde{\theta}_2 \rightarrow$	1.988	$\widehat{\theta}_{2BS} \rightarrow$	2.022	$\widehat{\theta}_{2LS} \rightarrow$	1.996	$\widehat{\theta}_{2LX} \rightarrow$	1.992	1.969	$\widehat{\theta}_{2LL} \rightarrow$	1.984	1.971
λ_2=1.5		$\tilde{R}(t) \rightarrow$	0.538	$\widehat{R}(t)_{BS} \rightarrow$	0.534	$\widehat{R}(t)_{LS} \rightarrow$	0.537	$\widehat{R}(t)_{LX} \rightarrow$	0.532	0.530	$\widehat{R}(t)_{LL} \rightarrow$	0.534	0.535
θ_2=2.0	15	$\tilde{\lambda}_2 \rightarrow$	1.324	$\widehat{\lambda}_{2BS} \rightarrow$	1.317	$\widehat{\lambda}_{2LS} \rightarrow$	1.337	$\widehat{\lambda}_{2LX} \rightarrow$	1.270	1.227	$\widehat{\lambda}_{2LL} \rightarrow$	1.319	1.302
t=1.5, $R(t)$=0.533		$\tilde{\theta}_2 \rightarrow$	2.280	$\widehat{\theta}_{2BS} \rightarrow$	2.492	$\widehat{\theta}_{2LS} \rightarrow$	2.305	$\widehat{\theta}_{2LX} \rightarrow$	2.335	2.242	$\widehat{\theta}_{2LL} \rightarrow$	2.270	2.236
		$\tilde{R}(t) \rightarrow$	0.594	$\widehat{R}(t)_{BS} \rightarrow$	0.599	$\widehat{R}(t)_{LS} \rightarrow$	0.595	$\widehat{R}(t)_{LX} \rightarrow$	0.595	0.592	$\widehat{R}(t)_{LL} \rightarrow$	0.592	0.592
Estimates of λ, θ and $R(t)$ based on simulated data set 2 (X), (k=3)													
	20	$\tilde{\lambda}_1 \rightarrow$	1.489	$\widehat{\lambda}_{1BS} \rightarrow$	1.471	$\widehat{\lambda}_{1LS} \rightarrow$	1.498	$\widehat{\lambda}_{1LX} \rightarrow$	1.436	1.401	$\widehat{\lambda}_{1LL} \rightarrow$	1.475	1.454
λ_1=1.5		$\tilde{\theta}_1 \rightarrow$	3.039	$\widehat{\theta}_{1BS} \rightarrow$	3.181	$\widehat{\theta}_{1LS} \rightarrow$	3.056	$\widehat{\theta}_{1LX} \rightarrow$	3.098	3.047	$\widehat{\theta}_{1LL} \rightarrow$	3.018	2.982
θ_2=3.0		$\tilde{R}(t) \rightarrow$	0.790	$\widehat{R}(t)_{BS} \rightarrow$	0.792	$\widehat{R}(t)_{LS} \rightarrow$	0.786	$\widehat{R}(t)_{LX} \rightarrow$	0.791	0.790	$\widehat{R}(t)_{LL} \rightarrow$	0.784	0.783
t=1.5, $R(t)$=0.787	15	$\tilde{\lambda}_1 \rightarrow$	1.480	$\widehat{\lambda}_{1BS} \rightarrow$	1.456	$\widehat{\lambda}_{1LS} \rightarrow$	1.494	$\widehat{\lambda}_{1LX} \rightarrow$	1.399	1.349	$\widehat{\lambda}_{1LL} \rightarrow$	1.472	1.451
		$\tilde{\theta}_1 \rightarrow$	3.033	$\widehat{\theta}_{1BS} \rightarrow$	3.422	$\widehat{\theta}_{1LS} \rightarrow$	3.061	$\widehat{\theta}_{1LX} \rightarrow$	3.054	2.956	$\widehat{\theta}_{1LL} \rightarrow$	3.011	2.966
		$\tilde{R}(t) \rightarrow$	0.788	$\widehat{R}(t)_{BS} \rightarrow$	0.820	$\widehat{R}(t)_{LS} \rightarrow$	0.784	$\widehat{R}(t)_{LX} \rightarrow$	0.748	0.766	$\widehat{R}(t)_{LL} \rightarrow$	0.782	0.782

Estimates of $\mathscr{P} = P(X > Y)$

	(r, s)	$\widetilde{\mathscr{P}}$ ↓	$\widehat{\mathscr{P}}_{BS}$ ↓	$\widehat{\mathscr{P}}_{LX}$ $(a=1)$ ↓	$\widehat{\mathscr{P}}_{LX}$ $(a=2)$ ↓
	(25,25)	0.701	0.713	0.710	0.708
$\mathscr{P}$=0.712	(20,25)	0.716	0.716	0.714	0.712
	(20,15)	0.679	0.669	0.669	0.661

$\frac{a}{b}\left(\frac{x}{b}\right)^{a-1}\exp\left(-\left(\frac{x}{b}\right)^{a}\right)$ $x \geq 0, a > 0, b > 0$, the generalized exponential distribution with pdf given by $f(x) = \frac{\alpha}{\sigma}\exp\left(\frac{-y}{\sigma}\right)\left(1-\exp\left(\frac{-y}{\sigma}\right)\right)^{\alpha-1}$ $y \geq 0$, $\alpha > 0$, $\sigma > 0$.

Maximum likelihood estimation method is used to estimate the parameters of the above distributions. These estimates, along with the data, are used to calculate estimated negative log likelihood function $-\ln L$, the Akaike information criterion (AIC), proposed by Akaike (1974), defined by $AIC = 2k' - 2\ln L$, Bayesian information criterion (BIC) proposed by Schwarz (1978), defined by $BIC = k'\ln n - 2\ln L$, where k' is the number of parameters in the reliability model, n is the number of observations in the given data set, and L is the maximized value of the likelihood function for the estimated model and Kolmogorov-Smirnov (K-S) test. The best distribution corresponds to lowest $-\ln L$, AIC, BIC and K-S statistic values with corresponding highest $\mathscr{P}$ value. Also, we draw quantile-quantile (Q-Q) plots of the above five reliability models which are given in Figure 1. A Q-Q plot depicts the points $\left\{F^{-1}\left(\frac{i-0.5}{n};\widehat{\theta}\right), x_{(i)}\right\}$, $i = 1, 2, \ldots, n$.

We have also plotted the empirical and theoretical cdf and fitted pdf for the real data set 1 to confirm the best fit of the above said data to the GB distribution in Figure 2.

Also, in order to show the accuracy of the obtained maximum likelihood estimates given in Table 5, for the fitted GB distribution, we have plotted the contour plot and log likelihood plots, given in Figure 3 and 4, respectively. Clearly, Figure 3 and 4, supports accuracy of the obtained estimates.

Table 5 gives the values of maximum likelihood estimates of the parameters of the considered reliability models, $-\ln L$, AIC, BIC, K-S statistic values and associated $\mathscr{P}$ values. This table also shows that the GB distribution is the best choice among the other commonly used reliability models in the literature for fitting lifetime data, since it has the smallest $-\ln L$, AIC, BIC (AIC and BIC are very close to the smallest values) and K-S statistic values and the corresponding highest $\mathscr{P}$ value. Also, Figure 1 and 2 supports the above findings.

TABLE 5
Summary fit to the real data set 1.

Distributions	Maximum likelihood estimates	$-\ln L$	AIC	BIC	K-S statistics D	*p value*
Generalized Bilal	λ=2.594, θ=1.941	49.261	102.522	106.990	0.044	0.999
Exponential	λ=1.701	105.670	213.338	215.572	0.393	4.3e-10
Rayleigh	σ=1.252	66.749	135.497	137.731	0.249	3e-04
Weibull	a=3.844, b=1.880	48.872	101.741	106.209	0.046	0.997
Generalized exponential	α=16.685, σ=0.506	55.514	115.032	119.500	0.103	0.428

The second real data set given below has been taken from Lawless (2003). Kumar *et al.* (2017) used this data for fitting the Nakagami distribution and shown that this data given the best fit to Nakagami distribution. Therefore, study of fitting this data to several distribution has been skipped. This data set originally reported by Schafft *et al.* (1987), represents hours to failure of 59 conductors of 400-micrometer length. All specimens

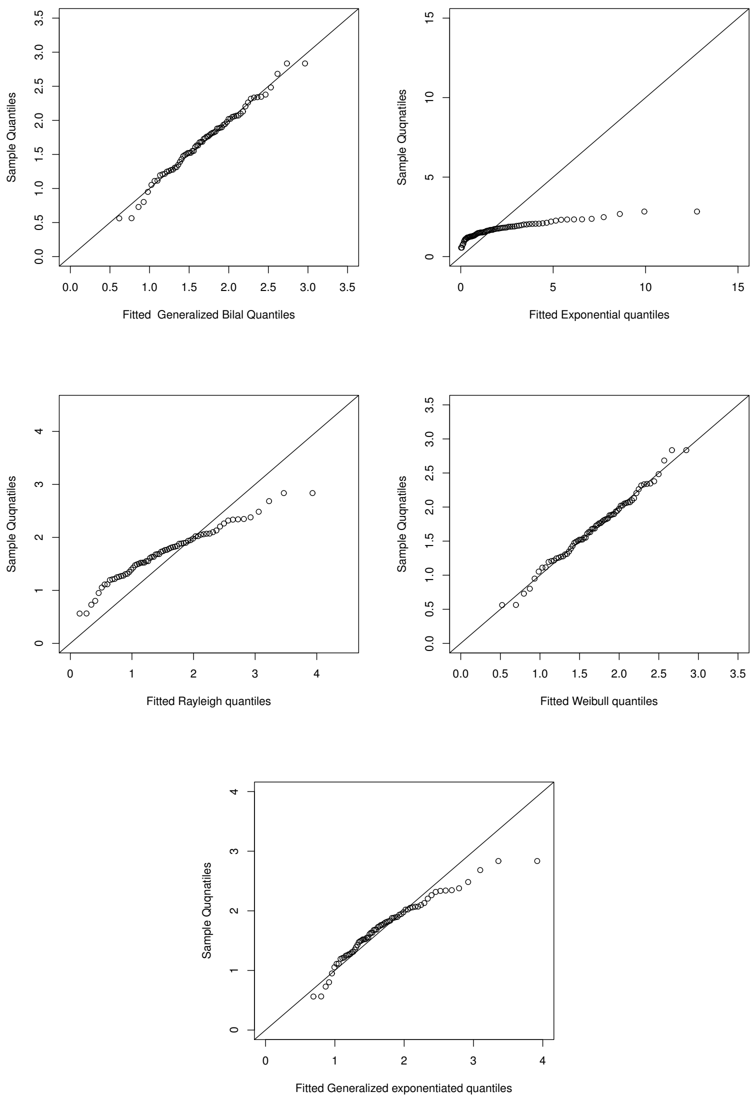

Figure 1 – Q-Q plot of various reliability models for the real data set 1.

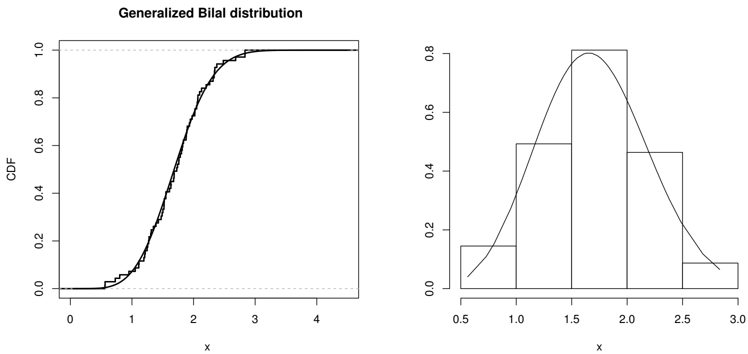

Figure 2 – Empirical and theoretical cdf (left) and fitted pdf plot (right) for real data set 1.

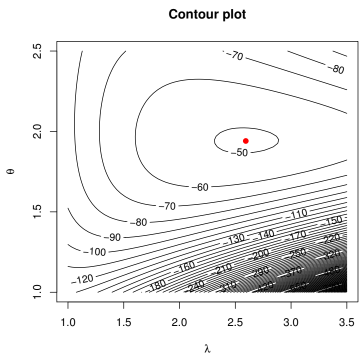

Figure 3 – Contour plot for the real data set 1.

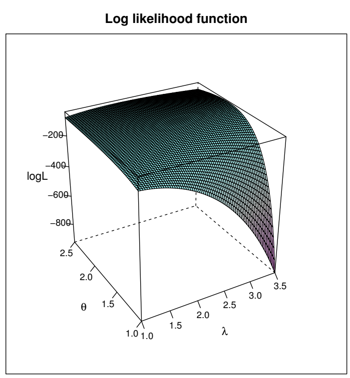

Figure 4 – Log likelihood plot for the real data set 1.

ran to failure at a certain high temperature and current density. The 59 specimens were all tested under the same temperature and current density. The data is as follows:

Real data set 2 (X):
6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923

We have plotted the empirical and theoretical cdf, fitted pdf and the Q-Q plot for the second real data set to confirm the fitting of the above said data to the GB distribution. Plots are presented in Figure 5.

Table 6 gives the values of maximum likelihood estimates of the parameters of the GB distribution, $-\ln L$, AIC, BIC, K-S statistic values and associated $\mathscr{P}$ values. This table shows that the GB distribution gives the good fit to the second real data set, since it has the very small K-S statistic values and the corresponding $p > 0.05$. Also, Figure 5 supports the above findings.

Also, in order to show the accuracy of the obtained MLEs given in Table 6, for the fitted generalized Bilal distribution, we have plotted the contour plot and log-likelihood plots, given in Figure 6 and 7, respectively. Clearly, Figure 6 and 7, supports accuracy of the obtained estimates.

TABLE 6
Summary fit to the real data set 2.

Distribution	Maximum likelihood estimates	$-\ln L$	AIC	BIC	K-S statistics D	*p value*
Generalized Bilal	λ=3.266, θ=7.801	111.470	226.940	231.095	0.076	0.863

By using the real data set 1 and 2; and the expressions obtained in Section 1 and 2, respectively, we have calculated the maximum likelihood, Bayes estimates and Lindley's approximation of the parameters (θ_2, λ_2), (θ_1, λ_1) and $R(t)$. Also, by considering real data set 1 as the stress population (Y) and real data set 2 as the strength population (X), we have calculated the maximum likelihood and Bayes estimates of $\mathscr{P}$ by using the expressions obtained in Section 3. For different values of r and (r,s) results are reported in Table 7.

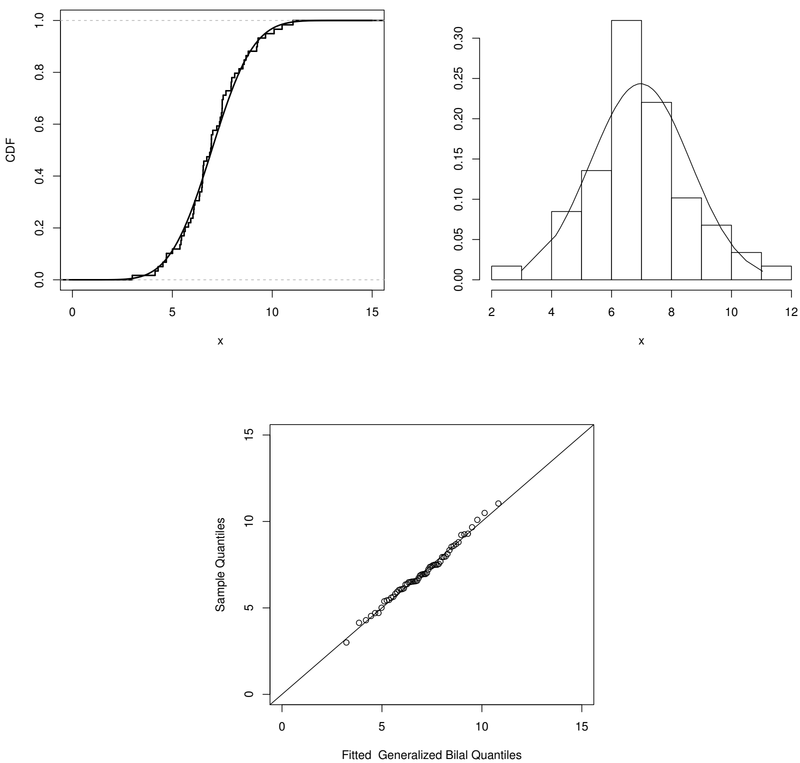

Figure 5 – Empirical and theoretical cdf (upper left), fitted pdf plot (upper right) and Q-Q plot for the real data set 2.

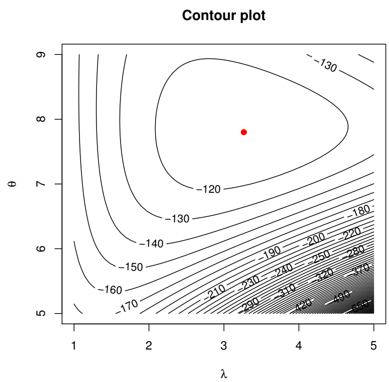

Figure 6 – Contour plot for the real data set 2.

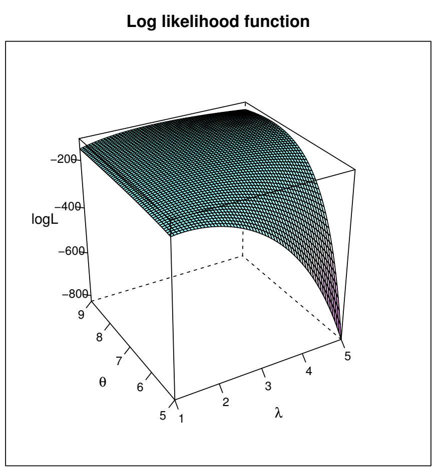

Figure 7 – Log likelihood plot for the real data set 2.

TABLE 7

Estimates based on real data set 1 and real data set 2 (Note: here LY stands for Lindley's approximation).

Estimates of λ, θ and $R(t)$ base on real data set 1 (Y) $(k=3.5, t=1.5)$												
$r\downarrow$		MLE		SELF		LY SELF		LLF $(a{=}1)$	LLF$(a{=}2)$		LY LLF $(a{=}1)$	LY LLF$(a{=}2)$
25	$\tilde{\lambda}_2 \rightarrow$	2.339	$\widehat{\lambda}_{2BS} \rightarrow$	2.407	$\widehat{\lambda}_{2LS} \rightarrow$	2.349	$\widehat{\lambda}_{2LX} \rightarrow$	2.282	2.172	$\widehat{\lambda}_{2LL} \rightarrow$	2.310	2.273
	$\tilde{\theta}_2 \rightarrow$	2.018	$\widehat{\theta}_{2BS} \rightarrow$	2.036	$\widehat{\theta}_{2LS} \rightarrow$	2.023	$\widehat{\theta}_{2LX} \rightarrow$	2.009	1.985	$\widehat{\theta}_{2LL} \rightarrow$	2.017	2.012
	$\tilde{R}(t) \rightarrow$	0.658	$\widehat{R}(t)_{BS} \rightarrow$	0.644	$\widehat{R}(t)_{LS} \rightarrow$	0.656	$\widehat{R}(t)_{LX} \rightarrow$	0.642	0.640	$\widehat{R}(t)_{LL} \rightarrow$	0.6544	0.654
15	$\tilde{\lambda}_2 \rightarrow$	2.400	$\widehat{\lambda}_{2BS} \rightarrow$	2.375	$\widehat{\lambda}_{2LS} \rightarrow$	2.424	$\widehat{\lambda}_{2LX} \rightarrow$	2.241	2.112	$\widehat{\lambda}_{2LL} \rightarrow$	2.381	2.342
	$\tilde{\theta}_2 \rightarrow$	1.971	$\widehat{\theta}_{2BS} \rightarrow$	2.069	$\widehat{\theta}_{2LS} \rightarrow$	1.982	$\widehat{\theta}_{2LX} \rightarrow$	2.023	1.990	$\widehat{\theta}_{2LL} \rightarrow$	1.974	1.966
	$\tilde{R}(t) \rightarrow$	0.641	$\widehat{R}(t)_{BS} \rightarrow$	0.645	$\widehat{R}(t)_{LS} \rightarrow$	0.643	$\widehat{R}(t)_{LX} \rightarrow$	0.643	0.640	$\widehat{R}(t)_{LL} \rightarrow$	0.640	0.640
Estimates of λ, θ and $R(t)$ base on real data set 2 $(X)(k=3.5, t=5.0)$												
30	$\tilde{\lambda}_1 \rightarrow$	3.598	$\widehat{\lambda}_{1BS} \rightarrow$	2.710	$\widehat{\lambda}_{1LS} \rightarrow$	3.607	$\widehat{\lambda}_{1LX} \rightarrow$	2.661	2.613	$\widehat{\lambda}_{1LL} \rightarrow$	3.502	3.422
	$\tilde{\theta}_1 \rightarrow$	7.650	$\widehat{\theta}_{1BS} \rightarrow$	8.664	$\widehat{\theta}_{1LS} \rightarrow$	7.656	$\widehat{\theta}_{1LX} \rightarrow$	8.664	8.664	$\widehat{\theta}_{1LL} \rightarrow$	7.627	7.60
	$\tilde{R}(t) \rightarrow$	0.901	$\widehat{R}(t)_{BS} \rightarrow$	0.890	$\widehat{R}(t)_{LS} \rightarrow$	0.896	$\widehat{R}(t)_{LX} \rightarrow$	0.890	0.890	$\widehat{R}(t)_{LL} \rightarrow$	0.895	0.893
25	$\tilde{\lambda}_1 \rightarrow$	3.751	$\widehat{\lambda}_{1BS} \rightarrow$	2.708	$\widehat{\lambda}_{1LS} \rightarrow$	3.763	$\widehat{\lambda}_{1LX} \rightarrow$	2.658	2.609	$\widehat{\lambda}_{1LL} \rightarrow$	3.648	3.562
	$\tilde{\theta}_1 \rightarrow$	7.548	$\widehat{\theta}_{1BS} \rightarrow$	8.664	$\widehat{\theta}_{1LS} \rightarrow$	7.556	$\widehat{\theta}_{1LX} \rightarrow$	8.664	8.664	$\widehat{\theta}_{1LL} \rightarrow$	7.525	7.496
	$\tilde{R}(t) \rightarrow$	0.903	$\widehat{R}(t)_{BS} \rightarrow$	0.890	$\widehat{R}(t)_{LS} \rightarrow$	0.898	$\widehat{R}(t)_{LX} \rightarrow$	0.890	0.889	$\widehat{R}(t)_{LL} \rightarrow$	0.897	0.896
Estimates of $\mathscr{P} = P(X > Y)$ $(k=4)$												
				(r,s)	$\widetilde{\mathscr{P}}\downarrow$	$\widehat{\mathscr{P}}_{BS}\downarrow$	$\widehat{\mathscr{P}}_{LX}$ $(a=1)\downarrow$		$\widehat{\mathscr{P}}_{LX}$ $(a=2)\downarrow$			
				(30,35)	0.999	0.998	0.998		0.998			
				(30,30)	0.999	0.998	0.998		0.998			
				(25,25)	0.999	0.998	0.998		0.998			

7. Conclusion

The purpose of the paper is manifold. In this paper, we propose the complex expressions for the MLEs and BEs of the parameters λ and θ and reliability functions $R(t)$ and $\mathscr{P}$. The Lindley's approximation's to obtain the Bayes estimators of the parameters and reliability function $R(t)$ are also proposed. These expressions can not be simplified that is why, we obtain the estimates of the parameters λ and θ; and reliability functions $R(t)$ and $\mathscr{P}$, by using the integrate function available in the `R software`. The solution by using integrate is a sub-optimal one and the use of dedicated solutions such as those of the cubature package would be preferred. Accuracy of the procedure and expression obtained are verified by using simulated data. We use MSE to compare the purposed estimators. Also, we present the fitting of the two real data sets. Furthermore, by using the simulated data sets, real data sets and graphical method such as Q-Q plot, contour plot, log-likelihood plot, fitted cdf and pdf plots, accuracy of the estimates and algorithm are proved.

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SUMMARY

We consider here the generalization of the Bilal distribution proposed by Abd-Elrahman (2017) by zeroing in on two measures of reliability, $R(t)$ and $\mathscr{P}$, based on type II censoring. We obtain point estimators namely, λ and θ, of the above said distribution, when both parameters of the distribution are unknown. Maximum likelihood estimators (MLEs), Bayes estimators (BEs) and Lindley's approximation for the Bayes estimators are proposed. By using independent non-informative type of priors for the unknown parameters Bayes estimators are derived. Although the proposed estimators cannot be expressed in closed forms, these can be easily obtained through the use of numerical procedures. The performance of these estimators is studied on the basis of their mean squared error (MSE), computed separately under LINEX loss function (LLF) and squared error loss function (SELF) through Monte-Carlo simulation technique.

Keywords: Maximum likelihood estimators; Bayes estimators; Non-informative prior; Lindley's approximation; Type II censoring scheme; Squared error loss function; LINEX loss function.