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QUANTILE BASED RELEVATION TRANSFORM AND ITS PROPERTIES

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1. INTRODUCTION

Relevation transforms, introduced by Krakowski (1973) have attracted considerable interest of researchers in survival analysis and reliability theory. Let X and Y be two absolutely continuous non-negative random variables, with survival functions $\bar{F}(.)$ and $\bar{G}(.)$ respectively. Consider a component from a population where lifetime has survival function $\bar{F}(x)$. This component is replaced at the time of its failure at age x, by another component of the same age x from another population. The lifetime of the second component has survival function $\bar{G}(x)$. Let $X\#Y$ denote the total lifetime of the random variable Y given it exceeds a random time X, (i.e $X\#Y \stackrel{d}{=} \{Y|Y>X\}$). Then the survival function of $X\#Y$,

$$\bar{T}_{X\#Y}(x)=\bar{F}\#\bar{G}(x)=\bar{F}(x)-\bar{G}(x)\int_0^x \frac{1}{\bar{G}(t)}d\bar{F}(t), \tag{1}$$

is called the relevation transform of X and Y. The probability density function (p.d.f.) of $X\#Y$ is obtained as

$$t_{X\#Y}(x)=T'_{X\#Y}(x)=g(x)\int_0^x \frac{f(t)}{\bar{G}(t)}dt. \tag{2}$$

Grosswald *et al.* (1980) provided two characterizations of the exponential distribution based on relevation transform. The concept of dependent relevation transform and

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its importance in reliability analysis are given in Johnson and Kotz (1981). Baxter (1982) discussed certain reliability applications of the relevation transform. Shanthikumar and Baxter (1985) provided closure properties of certain ageing concepts in the context of relevation transforms. Improved versions of the results in Grosswald *et al.* (1980) are given by Lau and Rao (1990). Chukova *et al.* (1993) established characterizations of the class of distributions with almost lack of memory property based on the relevation transform.

All these theoretical results and their applications are based on the distribution function. It is well known that any probability distribution can also be specified in terms of its quantile function. For a non-negative random variable X with distribution function $F(x)$, the quantile function $Q(u)$ is defined by

$$Q(u) = F^{-1}(x) = \inf\{x : F(x) \geq u\}, \ 0 \leq u \leq 1. \tag{3}$$

The derivative of $Q(u)$ is the quantile density function denoted by $q(u)$. When $f(x)$ is the probability density function (p.d.f.) of X, then we get

$$q(u) f(Q(u)) = 1. \tag{4}$$

One of the basic concepts employed for modelling and analysis of lifetime data is the hazard rate. In quantile set up, Nair and Sankaran (2009) defined hazard quantile function, which is equivalent to the hazard rate, $h(x) = \frac{f(X)}{\bar{F}(X)}$. The hazard quantile function $H(u)$ is defined as

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{5}$$

Thus $H(u)$ can be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of the unit until $100(1-u)$ % point of the distribution. Note that $H(u)$ uniquely determines the distribution through the identity

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{6}$$

Another important concept useful in the context of lifetime studies is the mean residual life. For a non-negative random variable X, the mean residual life $m(x)$ is defined as

$$m(x) = E(X - x | X > x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t) dt. \tag{7}$$

Nair and Sankaran (2009) defined the mean residual quantile function, which is the quantile version of the mean residual life $m(x)$ as

$$M(u) = m(Q(u)) = \frac{1}{1-u} \int_u^1 (Q(p) - Q(u)) dp. \tag{8}$$

$M(u)$ is interpreted as the mean remaining life of a unit beyond the $100(1-u)\%$ of the distribution. $M(u)$ uniquely determines $Q(u)$ by

$$Q(u)=\int_0^u \frac{M(p)}{1-p}dp - M(u)+\mu, \tag{9}$$

where $\mu = E(X)$. For a detailed and recent study on quantile functions, its properties, and its usefulness in the identification of models, we refer to Lai and Xie (2006), Nair and Sankaran (2009), Sankaran and Nair (2009), Nair and Vineshkumar (2011), Nair *et al.* (2013) and the references therein. Throughout this paper, the terms increasing and decreasing are used in a wide sense, that is, a function g is increasing (decreasing) if $g(x) \leq (\geq)\, g(y)$ for all $x \leq y$. Whenever we use a derivative, an expectation, or a conditional random variable, we are tacitly assuming that it exists.

The rest of the article is organized as follows. In Section 2, we introduce the quantile version of the relevation transform and study its basic properties. The quantile based relevation transform in the context of proportional hazards model and equilibrium distribution are discussed in Section 3 and Section 4 respectively. Finally, Section 5 summaries major conclusions of the study.

2. Quantile based relevation transform

To introduce the quantile based relevation transform between X and Y, we denote $Q_X(u)$ and $Q_Y(u)$ as the quantile functions corresponding to the distribution functions $F(.)$ and $G(.)$ respectively. From (1), by taking $x = Q_X(u)$, we define quantile based relevation transform as

$$\begin{aligned} T_{X\#Y}(Q_X(u)) &= u-\bar{G}(Q_X(u))\int_0^u \frac{1}{\bar{G}(Q_X(p))}dp \\ &= u-(1-Q_Y^{-1}(Q_X(u)))\int_0^u \frac{1}{(1-Q_Y^{-1}(Q_X(p)))}dp. \end{aligned} \tag{10}$$

Denote $T^*_{X\#Y}(u) = T_{X\#Y}(Q_X(u))$ and $Q_1(u) = Q_Y^{-1}(Q_X(u))$, (10) becomes

$$T^*_{X\#Y}(u) = u-(1-Q_1(u))\int_0^u \frac{1}{(1-Q_1(p))}dp. \tag{11}$$

If $Q(u)$ is a quantile function and $K(x)$ is a non-decreasing function of x, then $K(Q(u))$ is again a quantile function (Gilchrist, 2000). Now, since $T(.)$ and $Q_Y^{-1}(.)$ are non-decreasing functions, $T_{X\#Y}(Q_X(u))$ and $Q_Y^{-1}(Q_X(p))$ represent the quantile functions of $F(T^{-1}(x))$ and $F(G^{-1}(x))$. We call $T^*_{X\#Y}(u)$ as the relevation quantile function (RQF). Note that, in general, the quantile based relevation transform is not symmetric, namely $T^*_{X\#Y}(u) \neq T^*_{Y\#X}(u)$. We can interpret $T^*_{X\#Y}(u)$ as the probability that the total lifetime is less than or equal to $100u\%$ point of X, given it exceeds a random time X.

From (10), we have

$$\begin{aligned} T^*_{X\#Y}(u) &= T_{X\#Y}(Q_X(u)) \\ \Rightarrow Q_{X\#Y}(T^*_{X\#Y}(u)) &= Q_X(u) \\ \Rightarrow \qquad Q_{X\#Y}(u) &= Q_X(T^{*^{-1}}_{X\#Y}(u)). \end{aligned} \tag{12}$$

Thus, we can compute the quantile function of the relevation random variable $X\#Y$ from the relevation quantile function $T^*_{X\#Y}(u)$ using the identity (12).

PROPOSITION 1. *Let X and Y be two random variables with survival functions $\bar{F}(x)$ and $\bar{G}(x)$ with quantile functions $Q_X(u)$ and $Q_Y(u)$ respectively. Then $T^*_{X\#Y}(u) \leq u$ for all $u \in (0,1)$.*

PROOF. Denote $T^*_{X\#Y}(u) = u - S(u)$, where

$$S(u) = (1 - Q_1(u)) \int_0^u \frac{1}{(1 - Q_1(p))} dp \geq 0 \text{ for all } u \in (0,1). \tag{13}$$

Since $Q_1(u) = Q_Y^{-1}(Q_X(u)) = F_Y(Q_X(u))$, we have $Q_1(u) \in (0,1)$ for all $u \in (0,1)$. This implies, $S(u) \geq 0$ for all $u \in (0,1)$. From this, we get $T^*_{X\#Y}(u) \leq u$ for all $u \in (0,1)$. □

REMARK 2. *From Proposition 1, we have*

$$\begin{aligned} &T^*_{X\#Y}(u) \leq u \textit{ for all } u \in (0,1) \\ \Leftrightarrow\ &T(Q_X(u)) \leq u \textit{ for all } u \in (0,1) \\ \Leftrightarrow\ &Q_X(u) \leq Q_{X\#Y}(u) \textit{ for all } u \in (0,1). \end{aligned} \tag{14}$$

There are many situations in practice where we need to compare the characteristics of two distributions. Stochastic orders are used for the comparison of lifetime distributions. We shall consider the following stochastic orders. Their basic properties and interrelations can be seen in Barlow and Proschan (1975) and Shaked and Shanthikumar (2007).

If X and Y are lifetime random variables with absolutely continuous distribution functions $F(x)$ and $G(x)$ respectively. Let $f(x)$ and $g(x)$ are the corresponding probability density functions. Then we have the following:

(i) X is smaller than Y in the usual stochastic order denoted by $X \leq_{st} Y$ if and only if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

(ii) X is smaller than Y in hazard rate order, denoted by $X \leq_{hr} Y$, if and only if $\frac{\bar{G}(x)}{\bar{F}(x)}$ is increasing in x.

Nair *et al.* (2013) provided equivalent definitions of the above two orders in terms of the quantile functions as follows

(i) $X \leq_{st} Y$ if and only if $Q_X(u) \leq Q_Y(u)$ for all $0 < u < 1$.

(ii) $X \leq_{hr} Y$, if and only if $H_X(u) \geq H_{Y^*}(u)$, for all $0 < u < 1$, where $H_X(u) = h_X(Q_X(u))$ and $H_{Y^*}(u) = h_Y(Q_X(u))$.

Now from (14), since $Q_X(u) \leq Q_{X\#Y}(u)$ for all $u \in (0,1)$, we get $X \leq_{st} X\#Y$. Psarrakos and Di Crescenzo (2018) showed that $X \leq_{hr} X\#Y$. From Nair *et al.* (2013), we have $X \leq_{hr} Y$, if and only if $\frac{\bar{F}_Y(Q_X(1-u))}{u}$ is decreasing in u. This implies

$$\frac{1 - T_{X\#Y}(Q_X(1-u)}{u} = \frac{1 - T^*_{X\#Y}(1-u)}{u}, \text{ is decreasing in } u.$$

In the next proposition, we establish the relation between hazard quantile functions of the random variable $X\#Y$ and X.

PROPOSITION 3. *Let $H_{X\#Y}(u)$ and $H_X(u)$ be the hazard quantile functions corresponding to the random variables $X\#Y$ and X. Then*

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{1}{q_X(u)} \frac{d}{du}(-\log(1 - T^*_{X\#Y}(u))), \tag{15}$$

or equivalently,

$$\frac{H_{X\#Y}(T^*_{X\#Y}(u))}{H_X(u)} = (1-u)\frac{d}{du}(-\log(1 - T^*_{X\#Y}(u))). \tag{16}$$

PROOF. From (12), we have,

$$Q_{X\#Y}(T^*_{X\#Y}(u)) = Q_X(u).$$

Differentiating both sides with respect to u, we get,

$$\begin{aligned}
& q_{X\#Y}(T^*_{X\#Y}(u))(T^*_{X\#Y}(u))' = q_X(u). \\
\Rightarrow \quad & \frac{1}{q_{X\#Y}(T^*_{X\#Y}(u))(T^*_{X\#Y}(u))'} = \frac{1}{q_X(u)} \\
\Rightarrow \quad & \frac{H_{X\#Y}(T^*_{X\#Y}(u))}{(T^*_{X\#Y}(u))'} = \frac{1}{(1 - T^*_{X\#Y}(u))q_X(u)} \\
\Rightarrow \quad & H_{X\#Y}(T^*_{X\#Y}(u))q_X(u) = \frac{(T^*_{X\#Y}(u))'}{(1 - T^*_{X\#Y}(u))}.
\end{aligned} \tag{17}$$

From (17), we have,

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{1}{q_X(u)} \frac{d}{du}(-\log(1 - T^*_{X\#Y}(u))). \tag{18}$$

Since $q_X(u) = \frac{1}{(1-u)H_X(u)}$, (16) follows directly from (18), which completes the proof. □

PROPOSITION 4. *Suppose X and Y be two random variables with same support D and $Q_{Exp}(u)$ be the quantile function of the unit exponential distribution. Then*

$$H_{X\#Y}(T^*_{X\#Y}(u)) = \frac{H_X(u)}{H_Z(u)}, \tag{19}$$

where $H_Z(u)$ is the hazard quantile function corresponding to the quantile function $Q_Z(u) = Q_{Exp}(T^*_{X\#Y}(u))$.

PROOF. Since X and Y have the same support, D, we have, $T^*_{X\#Y}(0) = 0$ and $T^*_{X\#Y}(1) = 1$. From Gilchrist (2000), we have, if $Q(u)$ is a quantile function and $K(u)$ is a non-decreasing function of u satisfying the boundary conditions $K(0) = 0$ and $K(1) = 1$, then $Q(K(u))$ is again a quantile function of a random variable with the same support. This gives

$$Q_Z(u) = Q_{Exp}(T^*_{X\#Y}(u)) = -\log(1 - T^*_{X\#Y}(u)) \tag{20}$$

is a quantile function with support $Q_Z(0)$, $Q_Z(1) = (0, \infty)$.

From (20), we have

$$H_Z(u) = \left((1-u)\frac{d}{du}(-\log(1 - T^*_{X\#Y}(u)))\right)^{-1} \tag{21}$$

is the hazard quantile function of $Q_Z(u)$. Now the result (19) follows from (16) and (21), which completes the proof. □

EXAMPLE 5. *Suppose X follows uniform distribution with quantile function $Q_X(u) = \theta u$ and Y follows the exponential distribution with quantile function $Q_Y(u) = -\frac{1}{\lambda}\log(1-u)$. Then $Q_1(u) = Q_Y^{-1}(Q_X(u)) = 1 - \exp(-\lambda\theta u)$, and hence*

$$T^*_{X\#Y}(u) = u - \frac{1}{\lambda\theta}(1 - \exp(-\lambda\theta u)). \tag{22}$$

The identity (12) *is useful for generating random observations of the relevation random variable $X\#Y$. Since $T^*_{X\#Y}(u)$ given in* (22) *is not directly invertible, we generate the random sample of $X\#Y$ by first carrying out the numerical inversion of* (22) *and then using the relation $Q_{X\#Y}(u) = Q_X(T^{*^{-1}}_{X\#Y}(u))$.*

Relevation quantile function is not unique. There exist different distribution pairs with same relevation quantile function. We illustrate this with the following example.

EXAMPLE 6. *Let X, Y, W, and Z be four random variables with quantile functions,* *respectively by*

$$Q_X(u) = -\frac{1}{\lambda_1}\log(1-u);\ \lambda_1 > 0,\ \ [\textit{exponential distribution}(\lambda_1)],$$

$$Q_Y(u) = -\frac{1}{\lambda_2}\log(1-u);\ \lambda_2 > 0,\ \ [\textit{exponential distribution}(\lambda_2)],$$

$$Q_W(u) = (1-u)^{-\frac{1}{\lambda_1}} - 1;\ \lambda_1 > 0,\ \ [\textit{Pareto-II distribution}(\lambda_1)],$$

and

$$Q_Z(u) = (1-u)^{-\frac{1}{\lambda_2}} - 1;\ \lambda_2 > 0,\ \ [\textit{Pareto-II distribution}(\lambda_2)].$$

Now we obtain

$$Q_Y^{-1}(Q_X(u)) = Q_Z^{-1}(Q_W(u)) = 1-(1-u)^{\frac{\lambda_2}{\lambda_1}}.$$

This gives

$$T^*_{X\#Y}(u) = T^*_{W\#Z}(u) = \frac{\lambda_1\big((1-u)^{\lambda_2/\lambda_1}-1\big)+u\,\lambda_2}{\lambda_2-\lambda_1}.$$

Note that $Q_Y^{-1}(Q_X(u))$ *is the quantile function of the rescaled beta distribution and* $T^*_{X\#Y}(u)$ *is the linear combination of the quantile functions of the rescaled beta and the uniform distributions.*

EXAMPLE 7. *Suppose X follows Govindarajalu distribution with quantile function,* $Q_X(u) = \sigma((\beta+1)u^\beta - \beta u^{\beta+1})$ *and* Y *is uniform over the interval* $(0,1)$. *In this case,* $Q_1(u) = \beta u^{\beta+1} - (\beta+1)u^\beta + 1$, *then*

$$T^*_{X\#Y}(u) = \frac{(\beta(u-1)-1)\left(\frac{\beta u}{\beta+1}\right)^\beta B_{\frac{u\beta}{\beta+1}}[1-\beta,0]}{\beta} + u. \tag{23}$$

EXAMPLE 8. *Let X follows uniform* $(0,\theta_1)$ *and Y follows uniform* $(0,\theta_2)$, *with* $\theta_2 \geq \theta_1$. *Then* $Q_1(u) = \left(\frac{\theta_1}{\theta_2}\right)u$ *and*

$$T^*_{X\#Y}(u) = u - \left(1-\frac{\theta_1 u}{\theta_2}\right)\left(u - \frac{\theta_1 u^2}{2\theta_2}\right). \tag{24}$$

Since $T^*_{X\#Y}(u)$ is a quantile function, we can make use of the models obtained in Examples 5, 6 and 7 for modeling various types of lifetime data sets.

3. Proportional hazards relevation transform

In reliability theory, proportional hazards model (PHM) plays a vital role in the comparison of the lifetime of two components. The random variables X and Y satisfy PHM if

$$h_Y(x) = \theta h_X(x), \;\; \theta > 0, \tag{25}$$

where $h_Y(x)$ and $h_X(x)$ are the hazard rate functions of X and Y. An equivalent representation of (25) is

$$\bar{G}(x) = (\bar{F}(x))^{\theta}, \;\; \theta > 0. \tag{26}$$

For more details on PHM, one could refer to Lawless (2003) and Kalbfleisch and Prentice (2011). When Y is the PHM of X with survival functions as in (26), we call the transformation given in (1) as the proportional hazards relevation transform (PHRT).

When X is the PHM of Y, denote

$$\begin{aligned} T^*_{PH}(u) = T_{X\#Y}(Q_X(u)) &= u - (1-u)^{\theta} \int_0^u \frac{1}{(1-p)^{\theta}} dp \\ &= \frac{1-u\theta}{1-\theta} - \frac{(1-u)^{\theta}}{1-\theta}, \;\; u \in (0,1). \end{aligned} \tag{27}$$

We call $T^*_{PH}(u)$ as the proportional hazards relevation quantile function (PHRQF).

When $\theta = 1$,

$$T^*_{PH}(u) = T_{X\#X}(Q_X(u)) = u + (1-u)\log(1-u), \;\; u \in (0,1). \tag{28}$$

Proposition 9. *Let X and Y be two independent random variables. Then Y is the PHM of X if and only if $T^*_{PH}(u)$ satisfies the relation*

$$T^*_{PH}(u) = Q_A(u) - Q_B(u), \tag{29}$$

where

(i) $Q_A(u)$ and $Q_B(u)$ are the quantile functions of uniform $(0, \frac{\theta}{\theta-1})$ and rescaled beta $(0, \frac{1}{\theta-1})$ respectively, when $\theta > 1$, and

(ii) $Q_A(u)$ is rescaled beta $(0, \frac{1}{1-\theta})$ and $Q_B(u)$ is uniform $(0, \frac{\theta}{1-\theta})$, when $\theta < 1$.

Proof. From (27), we have

$$T^*_{PH}(u) = \frac{\theta u}{\theta - 1} - \frac{1}{\theta - 1}\left(1 - (1-u)^{\theta}\right). \tag{30}$$

This can be written as

$$T^*_{PH}(u) = \begin{cases} \frac{\theta u}{\theta-1} - \frac{1}{\theta-1}\left(1-(1-u)^{\theta}\right) & \text{if } \theta > 1 \\ \frac{1}{1-\theta}\left(1-(1-u)^{\theta}\right) - \frac{\theta u}{1-\theta} & \text{if } \theta < 1, \end{cases} \tag{31}$$

which completes the proof for the 'if' part of the proposition. Conversely, assume that $T^*_{PH}(u)$ has the form (29), now for $\theta > 1$, from (11), we have

$$T^*_{PH}(u) = u - \vartheta(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{\theta u}{\theta - 1} - \frac{1}{\theta - 1}\left(1-(1-u)^{\theta}\right),$$

where $\vartheta(u) = 1 - Q_Y^{-1}(Q_X(u))$.

This implies

$$\vartheta(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{((1-u)^{\theta} - (1-u))}{1-\theta}. \tag{32}$$

Differentiating both sides with respect to u, we get

$$\vartheta'(u)\int_0^u \frac{1}{\vartheta(p)}dp = \frac{\theta}{1-\theta}(1-(1-u)^{\theta-1}). \tag{33}$$

Dividing (33) by (32), we obtain

$$\frac{\vartheta'(u)}{\vartheta(u)} = \frac{\theta(1-(1-u)^{\theta-1})}{(u-1)(1-(1-u)^{\theta-1})} = \frac{-\theta}{1-u}, \tag{34}$$

which implies

$$\frac{d}{du}\log(\vartheta(u)) = \frac{-\theta}{1-u}. \tag{35}$$

On integration (35) reduces to

$$\log(\vartheta(u)) = \log(1-u)^{\theta}.$$

This gives $\vartheta(u)) = (1-u)^{\theta}$. Now from (11), we have

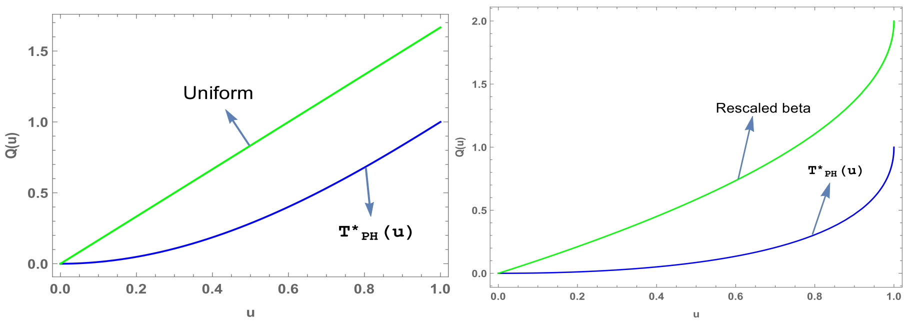
$$\vartheta(u) = 1 - Q_Y^{-1}(Q_X(u)) = (1-u)^{\theta},$$

which gives

$$Q_X(u) = Q_Y(1-(1-u)^{\theta}), \text{ or equivalently } \bar{G}(x) = (\bar{F}(x))^{\theta}. \tag{36}$$

Thus, Y is the PHM of X. Proof for the case $\theta < 1$ is similar and hence the details are omitted. □

REMARK 10. *From Proposition 9, we can see that $T^*_{PH}(u)$ lies below uniform $(0, \frac{\theta}{\theta-1})$ quantile function when $\theta > 1$ and it lies below rescaled beta $(0, \frac{1}{1-\theta})$ quantile function when $\theta < 1$. We illustrate this for two particular cases of θ such as 0.5 and 2.5 in Figure 1.*



(a) $\theta = 2.5$ (b) $\theta = 0.5$

Figure 1 –: (a) Uniform $(0, \frac{\theta}{\theta-1})$ with $T^*_{PH}(u)$, and (b) rescaled beta $(0, \frac{1}{1-\theta})$ with $T^*_{PH}(u)$.

Since $T^*_{PH}(u)$ is a unit support quantile function, we can use $T^*_{PH}(u)$ with an additional scale parameter for modelling lifetime datasets. Thus, consider the model

$$Q^*(u) = \begin{cases} \sigma\left(\frac{1-u\theta}{1-\theta} - \frac{(1-u)^\theta}{1-\theta}\right) & \text{if } \theta \neq 1 \\ u + (1-u)\log(1-u) & \text{if } \theta = 1. \end{cases} \tag{37}$$

The hazard quantile function has the form

$$H^*(u) = \begin{cases} \frac{1-\theta}{\theta\sigma((1-u)^\theta + u - 1)} & \text{if } \theta \neq 1 \\ (\sigma(u-1)\log(1-u))^{-1} & \text{if } \theta = 1. \end{cases} \tag{38}$$

Note that, when $\theta = 1$, $q^*(u) = \frac{d}{du}Q^*(u) = -\sigma\log(1-u)$, which is the quantile function of an exponential distribution with mean σ. Thus $q^*(u)$ is non decreasing when $\theta = 1$. $H^*(u)$ is bathtub shaped for all choices of the parameters. Change point of $H^*(u)$ is $u_0 = 1 - \left(\frac{1}{\theta}\right)^{\frac{1}{\theta-1}}$, when $\theta \neq 1$ and for $\theta = 1$, change point $u_0 = 0.63$ (see Figure 2).

We illustrate the utility of the above model with the aid of a real data set studied by Zimmer *et al.* (1998). The data consists of time to first failure of 20 electric carts. There are different methods for the estimation of parameters of the quantile function. The method of percentiles, method of L-moments, method of minimum absolute deviation, method of least squares and method of maximum likelihood are commonly used techniques. With small and moderate samples the method of L-moments is more efficient than other method of estimations. For more details, one could refer to Hosking (1990) and Hosking and Wallis (1997). We estimate the parameters using the method of

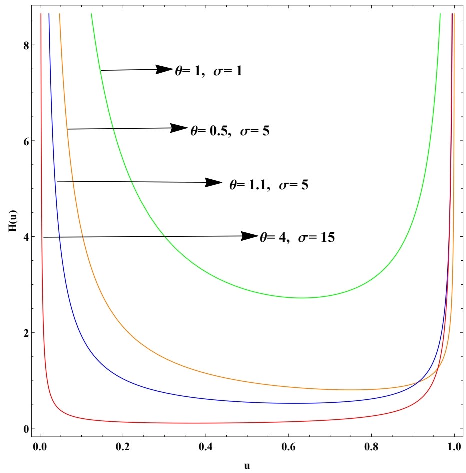


Figure 2 –: Hazard quantile function for different choices of parameters.

L-moments. The first and second L-moments are given by

$$L_1 = \int_0^1 Q(u)\,du = \frac{\theta\left(\frac{1}{\theta+1} - \frac{1}{2}\right)\sigma}{1-\theta}, \tag{39}$$

and

$$L_2 = \int_0^1 (2u-1)Q(u)\,du = \frac{\theta\left(\frac{1}{\theta^2+3\theta+2} - \frac{1}{6}\right)\sigma}{1-\theta}. \tag{40}$$

We equate sample L—moments to corresponding population L-moments. Let $x_1, x_2, ..., x_n$ be a random sample of size n from the population with quantile function (37). To define the L- moments of a sample, we first take the order statistics of the sample: $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ and define the sample L-moments, $l_l, l_2, ..., l_n$ by

$$l_r = \sum_{k=0}^{r-1} p_{r,k}\, b_k, \tag{41}$$

where

$$b_k = \left(\frac{1}{n}\right) \sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-k)}{(n-1)(n-2)...(n-k)} x_{(i)}, \quad \text{for } k = 0, 1, 2, ...n-1. \tag{42}$$

In particular, the first two sample L—moments are given by

$$\begin{aligned}
l_1 &= \left(\frac{1}{n}\right)\sum_{i=1}^{n} x_{(i)}, \\
l_2 &= \left(\frac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1}-\binom{n-i}{1}\right)x_{(i)},
\end{aligned} \tag{43}$$

where $x_{(i)}$ is the i-th order statistic. For estimating the parameters θ and σ, we equate the above two sample L—moments to corresponding population L-moments given in (39) and (40). The parameters are obtained by solving the equations

$$l_r = L_r; \quad r = 1,2. \tag{44}$$

The estimates of the parameters are obtained as, $\hat{\theta} = 2.004$, $\hat{\sigma} = 43.988$. Midhu *et al.* (2013) fitted the data using the class of distributions with linear mean residual quantile function. The quantile function of the model proposed by Midhu *et al.* (2013) is

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \ \mu > 0, -\mu < c < \mu. \tag{45}$$

Mudholkar and Srivastava (1993) analyzed this data using an exponentiated-Weibull model with distribution function

$$F(x) = (1-\exp(-(\lambda x)^{\gamma})), \ \lambda > 0, \gamma > 0. \tag{46}$$

To compare the performance of the aforementioned models with the proposed quantile function, we use the Akaike information criterion (AIC)(Akaike, 1974). The AIC is defined by

$$AIC = 2k - 2\log(\hat{L}),$$

where $\hat{L}$ is the estimate of likelihood function. The AIC measure for the proposed model is 151.5, whereas the AIC's for the models proposed by Midhu *et al.* (2013) and Mudholkar and Srivastava (1993) are 153.45 and 748.41 respectively. On the basis of AIC values, our model gives better fit.

4. Relevation transform with equilibrium distribution

The equilibrium distribution is a widely accepted tool in the context of analysis of ageing phenomena. The equilibrium distribution associated with the random variable X is defined by the survival function

$$\bar{F}_E(x) = \frac{1}{\mu_X}\int_x^{\infty} \bar{F}(t)dt. \tag{47}$$

Setting $x = Q_X(u)$, from Nair *et al.* (2013), we have

$$F_E(Q_X(u)) = \frac{1}{\mu_X}\int_0^u (1-p)q_X(p)dp, \tag{48}$$

where the integral

$$\phi_X(u) = \frac{1}{\mu_X}\int_0^u (1-p)q_X(p)dp \tag{49}$$

is called the scaled total time on test transform (TTT) of the random variable X. For various properties and applications of $\phi_X(u)$, one could refer to Nair *et al.* (2013).

From (48) and (49), we have

$$Q_X(u) = Q_E(\phi_X(u)), \tag{50}$$

where $Q_E(.)$ is the quantile function corresponding to the equilibrium distribution of X.

PROPOSITION 11. *Let X and Y be two non-negative random variables. Then Y is the equilibrium random variable of X if and only if*

$$T^*_{X\#Y}(u) = u - (1-\phi_X(u))\int_0^u \frac{dp}{(1-\phi_X(p))}. \tag{51}$$

PROOF. Assume Y is the equilibrium random variable of X. From (10), we have

$$T(Q_X(u)) = u - (1-Q_Y^{-1}(Q_X(u)))\int_0^u \frac{1}{(1-Q_Y^{-1}(Q_X(p)))}dp. \tag{52}$$

Since Y is the equilibrium random variable of X, we have

$$Q_X(u) = Q_Y(\phi_X(u)). \tag{53}$$

Now using (53) in (52), we get

$$\begin{aligned} T^*_{X\#Y}(u) &= u - \left(1-Q_Y^{-1}(Q_Y(\phi_X(u)))\right)\int_0^u \frac{1}{\left(1-Q_Y^{-1}(Q_Y(\phi_X(p)))\right)}dp, \\ &= u - (1-\phi_X(u))\int_0^u \frac{dp}{(1-\phi_X(p))}. \end{aligned} \tag{54}$$

Conversely, assume (51) is true. Now from (11), we have

$$u - (1-Q_1(u))\int_0^u \frac{1}{(1-Q_1(p))}dp = u - (1-\phi_X(u))\int_0^u \frac{dp}{(1-\phi_X(p))}.$$

Taking derivative on both sides with respect u, and simplifying, we get

$$\begin{aligned} & Q_1(u) = \phi_X(u) \\ \Leftrightarrow \quad & Q_Y^{-1}(Q_X(u)) = \phi_X(u) \\ \Leftrightarrow \quad & Q_X(u) = Q_Y(\phi_X(u)). \end{aligned}$$

Thus Y is the equilibrium random variable of X, which completes the proof. □

COROLLARY 12. *Suppose Y is the equilibrium random variable of X. Then $T^*_{X\#Y}(u)$ uniquely determines $\phi_X(u)$ through the identity*

$$\phi_X(u) = 1 - \exp\left(\int_0^u \frac{(T^*_{X\#Y}(p))'}{T^*_{X\#Y}(p) - p} dp\right). \tag{55}$$

PROOF. From (51), we have

$$(1 - \phi_X(u)) \int_0^u \frac{dp}{(1 - \phi_X(p))} = u - T^*_{X\#Y}(u). \tag{56}$$

Differentiating both sides with respect to u, we get

$$\begin{aligned} & 1 - \phi'_X(u) \int_0^u \frac{dp}{(1 - \phi_X(p))} = 1 - (T^*_{X\#Y}(u))' \\ \Leftrightarrow \quad & \phi'_X(u) \int_0^u \frac{dp}{(1 - \phi_X(p))} = (T^*_{X\#Y}(u))'. \end{aligned} \tag{57}$$

From (56), we have $\int_0^u \frac{dp}{(1-\phi_X(p))} = \frac{u - T^*_{X\#Y}(u)}{(1-\phi_X(u))}$. Inserting this in (57), we obtain

$$\begin{aligned} & \frac{\phi'_X(u)}{1 - \phi_X(u)} = \frac{(T^*_{X\#Y}(u))'}{u - T^*_{X\#Y}(u)}, \\ \text{or} \quad & \frac{d}{du}(\log(1 - \phi_X(u))) = \frac{(T^*_{X\#Y}(u))'}{T^*_{X\#Y}(u) - u}, \end{aligned}$$

which gives

$$\phi_X(u) = 1 - \exp\left(\int_0^u \frac{(T^*_{X\#Y}(p))'}{T^*_{X\#Y}(p) - p} dp\right).$$

This completes the proof. □

REMARK 13. *From Nair* et al. *(2013), we have $\phi_X(u)$ uniquely determines the distribution through the relation*

$$Q(u) = \int_0^u \frac{\mu_X \phi'_X(p)}{1 - p} dp. \tag{58}$$

Now, from Corollary 12, we have $T^*_{X\#Y}(u)$ uniquely determines the baseline distribution.

EXAMPLE 14. *Let X be distributed as generalized Pareto with quantile function*

$$Q_X(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}} - 1\right], \; b > 0, a > -1. \tag{59}$$

Since $\mu_X = b$, we get

$$\phi_X(u) = \frac{1}{\mu}\int_0^u (1-p)q_X(p)dp = \left[1-(1-u)^{\frac{1}{a+1}}\right]. \tag{60}$$

Hence, the equilibrium random variable Y has its quantile function as, $Q_Y(u) = Q_X(\phi_X^{-1}(u))$. Thus from (59) *and* (60)*, we obtain*

$$Q_Y(u) = \frac{b}{a}\left[(1-u)^{-a} - 1\right]. \tag{61}$$

Using (51)*, we get*

$$T^*_{X\#Y}(u) = \frac{1}{a}\left((a+1)\left(1-(1-u)^{\frac{1}{a+1}}\right) - u\right). \tag{62}$$

From Nair *et al.* (2013), $\phi_X(u)$ and $M_X(u)$ are related through the identity

$$M_X(u) = \frac{1-\phi_X(u)}{1-u}. \tag{63}$$

Inserting (63) in (51), we get

$$T^*_{X\#Y}(u) = u - (1-u)M_X(u)\int_0^u \frac{dp}{(1-p)M_X(p)}. \tag{64}$$

Thus $M_X(u)$ uniquely determines $T^*_{X\#Y}(u)$, when Y corresponds to the equilibrium distribution of X.

EXAMPLE 15. *Suppose X follows linear mean residual quantile function distribution with $M_X(u) = \mu + c\,u$, and $Q_X(u)$ as in* (45) *(Midhu* et al.*, 2013). In this case*

$$T^*_{X\#Y}(u) = u + \frac{(1-u)(cu+\mu)\log\left(\frac{\mu-\mu u}{cu+\mu}\right)}{c+\mu}. \tag{65}$$

In the next proposition, we provide a characterization for the exponential distribution using $T^*_{X\#Y}(u)$, when Y is the equilibrium random variable of X.

PROPOSITION 16. *Let X be a non-negative random variable and Y be the corresponding equilibrium random variable. Then X has exponential distribution if and only if*

$$T^*_{X\#Y}(u) = T^*_{X\#X}(u), \text{ for all } u \in (0,1). \tag{66}$$

PROOF. Assume X follows exponential distribution with quantile function $Q_X(u) = \frac{-1}{\lambda}\log(1-u)$, $\lambda > 0$. We get $\mu_X = \frac{1}{\lambda}$ and $\phi_X(u) = u$. Since $\phi_X(u) = u$, from (50), we have $Q_X(u) = Q_Y(u)$. This implies

$$T^*_{X\#Y}(u) = T^*_{X\#X}(u).$$

Conversely, we have, $T^*_{X\#Y}(u) = T^*_{X\#X}(u)$ for all $u \in (0,1)$. Now from (11), we have

$$T^*_{X\#Y}(u) = u + (1-u)\log(1-u). \tag{67}$$

Now using (55), we get $\phi_X(u) = u$. Thus from (58), the baseline quantile function of X is obtained as $Q_X(u) = -\mu_X \log(1-u)$, which is exponential. This completes the proof. □

In $\bar{T}_{X\#Y}(x)$, given in (1), if we take $\bar{F}(x) = \frac{1}{\mu_X}\int_x^\infty \bar{G}(t)dt$ and $u = G(x)$, we get

$$\begin{aligned} 1 - T_{X\#Y}(Q_Y(u)) &= \frac{\int_0^u (1-p)q_Y(p)dp}{\mu_Y} + (1-u)\int_0^u \frac{1}{\mu_Y}dp, \\ &= 1 - \frac{\int_0^u (1-p)q_Y(p)dp}{\mu_Y} + \frac{(1-u)u}{\mu_Y}, \end{aligned}$$

which implies

$$\begin{aligned} T_{X\#Y}(Q_Y(u)) &= 1 - \phi_Y(u) + \frac{(1-u)u}{\mu_Y}, \\ &= 1 - \frac{(1-u)}{\mu_Y}[M_Y(u) + u], \end{aligned} \tag{68}$$

where $M_Y(u)$ is the mean residual quantile function of Y.

5. CONCLUSION

The present work introduced an alternative approach to relevation transform using quantile functions. Various properties and applications of quantile based relevation transform were discussed. Quantile based relevation transform in the context of proportional hazards and equilibrium models were presented. The PHRQF model is applied to a real lifetime data set. It was proved that $T^*_{X\#Y}(u)$ uniquely determines the distribution of X, when Y is the equilibrium random variable of X. We can develop quantile based analysis of a sequence of random variables constructed based on the relevation transform. The identity (66) can be used to test exponentiality. For this, one has to develop non-parametric estimator of $T^*_{X\#Y}(u)$. The work in these directions will be reported elsewhere.

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SUMMARY

Relevation transform introduced by Krakowski (1973) is extensively studied in the literature. In this paper, we present a quantile based definition of the relevation transform and study its properties in the context of lifetime data analysis. We give important special cases of relevation transform in the context of proportional hazards and equilibrium models in terms of quantile function.

Keywords: Relevation transform; Quantile function; Hazard quantile function; Mean residual quantile function; Proportional hazard model; Equilibrium distribution.