STATISTICA, anno LXXVIII, n. 3, 2018

A THREE PARAMETER GENERALIZED LINDLEY DISTRIBUTION: PROPERTIES AND APPLICATION

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1. Introduction

Lindley (1958) proposed the classical one parameter Lindley distribution with scale parameter $\theta > 0$ and the probability density function defined by

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x > 0, \theta > 0. \tag{1}$$

The corresponding cumulative distribution function is given by

$$F(x,\theta) = 1 - \left(\frac{\theta+1+\theta x}{\theta+1}\right)e^{-\theta x}, \quad x > 0, \theta > 0. \tag{2}$$

The probability density function (pdf) of the one-parameter Lindley distribution given in (1) is a two-component mixture of exponential (θ) and gamma (2, θ). Equation (1) can be expressed as

$$f(x,\theta) = p f_1(x) + (1-p) f_2(x), \tag{3}$$

where $f_1(x)$ and $f_2(x)$ are the pdf of the exponential(θ) and gamma($2,\theta$) distribution and p is the mixing proportion. Ghitany *et al.* (2008) studied the properties of the one parameter Lindley distribution and applied it to a waiting time data. Considering some comparison criteria, it was shown that the distribution is a better model than the exponential distribution in modeling lifetime data. But due to the failure rate property of the one parameter Lindley distribution, there are some situations where the distribution fails to provide a good fit in modeling real lifetime data. To address this situation, many researchers have proposed generalized forms of the one parameter Lindley distribution.

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Bhati *et al.* (2015) introduced a new family of distributions with survival function given by

$$\bar{F}(x) = \frac{\theta^2}{\theta+1}\int_0^{-\log G(x)} (1+t)e^{-\theta t}\,dt, \quad x>0, \theta>0. \tag{4}$$

The corresponding density function is given by

$$f(x) = \frac{\theta^2}{\theta+1}([1-\log G(x)]G(x))^{\theta-1}\, g(x), \quad x>0, \theta>0, \tag{5}$$

where $G(x)$ is the cdf of the parent distribution.

Lazri and Zeghdoudi (2016) used the $T-X$ family of distribution framework proposed by Alzaartreh *et al.* (2013) to generate a new family of Lindley distribution called the Lindley-X family of distribution. The cumulative distribution function of the new family is given by

$$F(x) = \frac{\theta^2}{\theta+1}\int_0^{\frac{G(x)}{1-G(x)}} (1+t)e^{-\theta t}\,dt, \quad x>0, \theta>0 \tag{6}$$

and the corresponding density function is given by

$$f(x) = \frac{\theta^2}{\theta+1}\left[\frac{g(x)}{(1-G(x))^2}\left(1+\frac{G(x)}{1-G(x)}\right)\exp\left(-\theta\left[\frac{G(x)}{1-G(x)}\right]\right)\right]. \tag{7}$$

Other generalizations of the Lindley distribution are found in the works of Nadarajah *et al.* (2011); Bakouch *et al.* (2012); Al-Babtain *et al.* (2015); Maya and Irshad (2017) and a host of others.

In this paper, we introduced a new family of Lindley distribution by considering an integral transform of the density function of a random variable which follows the one parameter Lindley distribution with cumulative distribution function defined by

$$\begin{aligned} F(x,\phi) &= \frac{\theta^2}{\theta+1}\int_0^{G(x,\phi)} (1+t)e^{-\theta t}\,dt \\ &= 1-\frac{[1+\theta(1+G(x,\phi))]\,e^{-\theta[G(x,\phi)]}}{(\theta+1)}, \quad x>0, \theta>0, \phi>0. \end{aligned} \tag{8}$$

The density function is given by

$$f(x,\phi) = \frac{\theta^2}{\theta+1}(1+G(x,\phi))e^{-\theta[G(x,\phi)]}g(x,\phi), \quad x>0, \theta>0, \phi>0. \tag{9}$$

In literature, $G(x,\phi)$ could either be the survival function or the cumulative distribution function of any defined probability density function $g(x,\phi)$. Our interest in this paper is to consider $G(x,\phi)\epsilon(0,\infty)$ as a non-negative monotonically increasing function depending on the parameter vector and also differentiable.

The remaining sections of this paper are organized as follows. An account of the mathematical properties of the proposed distribution is given in Sections 2-7. These properties include: the density function, cumulative distribution function, the survival function, the hazard rate function, the quantile function, moments and related measures, Renyi entropy and the distribution of the ordered statistics. An estimation of the parameters of the proposed distribution using maximum likelihood method is presented in Section 8. Finally, Section 9 presents an application of the proposed distribution to two real lifetime data sets alongside with some well-known related lifetime distributions.

2. Density function

Consider a family of distributions whose cumulative distribution function and probability density function is defined by Equations (8) and (9), respectively. Let $G(x,\phi)=\frac{x^\alpha}{\beta}$, then the cumulative distribution function and the density function of the three parameter generalized Lindley distribution (TPGLD) are given by

$$F(x)=1-\frac{(1+\lambda\beta+\lambda x^\alpha)e^{-\lambda x^\alpha}}{1+\lambda\beta}, \quad x>0,\theta,\lambda,\beta>0, \tag{10}$$

and

$$f(x)=\frac{\alpha\lambda^2(\beta+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{1+\lambda\beta}, \quad x>0,\theta,\lambda,\beta>0. \tag{11}$$

The density function in equation (11) which is a two-component mixture of Weibull distribution with shape parameter α and scale parameter λ and a generalized gamma distribution with shape parameters $(2,\alpha)$ and scale parameter (λ) can be expressed as

$$f(x,\lambda)=pf_1(x)+(1-p)f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are pdf of Weibull distribution and generalized gamma distribution, respectively, and $p=\frac{\lambda\beta}{1+\lambda\beta}$ is the mixing proportion.

The graphical representation of the density function of TPGLD for some fixed value of the parameters is shown in Figure 1.

Table 1 shows some of the existing sub-models of the proposed class of distribution.

Remark: For $\alpha=\beta=1$, the TPGLD reduces to the classical one parameter Lindley distribution and also reduce to the Power Lindley distribution when $\beta=1$.

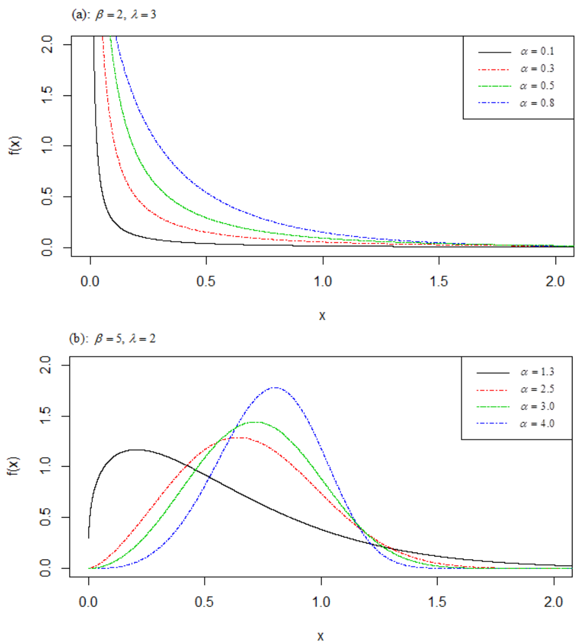

Figure 1 – Density function of the TPGLD for (a):$\beta = 2, \lambda = 3$;   (b):$\beta = 5, \lambda = 2$.

TABLE 1
Distributions and corresponding $G(x;\phi)$.

Distributions	Density function	$G(x;\phi)$	Authors
Lindley	$\frac{\theta^2(1+x)e^{-\theta x}}{(1+\theta)}$	x	Lindley (1958)
Power Lindley	$\frac{\alpha\theta^2(1+x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}}{(1+\theta)}$	x^α	Ghitany *et al.* (2013)
Sushila	$\frac{\theta^2(1+x)e^{-\theta x}}{(1+\theta)}$	$\frac{x}{\beta}$	Shanker *et al.* (2013)
Lindley-Pareto	$\frac{\beta\theta^2 e^\theta x^{2\beta-1}e^{-[\theta(\frac{x}{\alpha})^\beta]}}{\alpha^{2\beta}(1+\theta)}$	$(\frac{x}{\theta})^\beta - 1$	Lazri and Zeghdoudi (2016)
Lindley-Half Logistic	$\frac{\theta^2(1+e^x)e^{[\frac{\theta}{2}(1-e^x)+x]}}{[4(1+\theta)]}$	$\frac{e^x-1}{2}$	Silva *et al.* (2017)
TPGLD	$\frac{\alpha\lambda^2(\beta+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{(1+\lambda\beta)}$	$\frac{x^\alpha}{\beta}$	This paper

3. SURVIVAL AND HAZARD RATE FUNCTION

Let X be a continuous random variable with density function $f(x)$ and cumulative distribution function $F(x)$. The survival (reliability) function and hazard rate (failure rate) function of the three parameter generalized Lindley distribution are defined by

$$s(x) = \frac{(1+\lambda\beta+\lambda x^{\alpha})e^{-\lambda x^{\alpha}}}{1+\lambda\beta}, \quad x>0, \theta, \lambda, \beta > 0, \tag{12}$$

and

$$h(x) = \frac{\alpha\lambda^2(\beta+x^{\alpha})x^{\alpha-1}}{(1+\lambda\beta+\lambda x^{\alpha})}, \quad x>0, \theta, \lambda, \beta > 0. \tag{13}$$

The graph of the hazard rate function of the TPGLD for different value of the parameters is given in Figure 2.

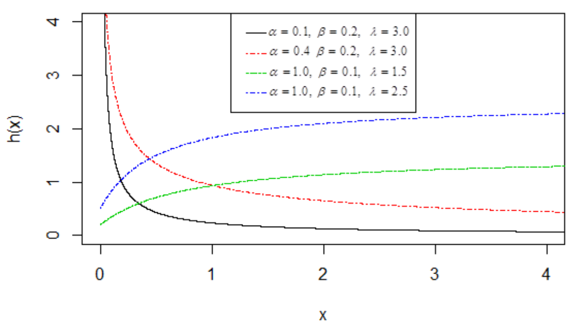

Figure 2 – Hazard rate function of the TPGLD.

Clearly from Figure 2, the TPGLD exhibits both monotone increasing and decreasing failure rate property. It decreases monotonically when $\alpha < 1$ and increases monotonically when $\alpha \geq 1$.

4. QUANTILES

Given the cumulative distribution function $F(X)$ defined by Equation (10), the quantile function of the TPGLD can be obtain as $Q_X(p) = F^{-1}(p)$. The quantile function of the Lindley family of distributions can be expressed in a closed form using the Lambert W function proposed in Jodra (2010).

The p^{th} quantile function is obtained by solving $F(x) = p$, i.e.

$$1 - \frac{(1+\lambda\beta+\lambda x^{\alpha})e^{-\lambda x^{\alpha}}}{(1+\lambda\beta)} = p,$$

$$(1+\lambda\beta+\lambda x^{\alpha})e^{-\lambda x^{\alpha}} = (1+\lambda\beta)(1-p).$$

Multiplying both sides by $e^{-(1+-\lambda\beta)}$, we have

$$(-1-\lambda\beta-\lambda x^{\alpha})e^{-(1+\lambda\beta+\lambda x^{\alpha})} = -(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}.$$

Clearly, we observe that $(-1-\lambda\beta-\lambda x^{\alpha})$ is the Lambert W function of the real argument $-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}$. Thus, we have

$$W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right] = (-1-\lambda\beta-\lambda x^{\alpha}),$$

$$\lambda x^{\alpha} = W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right],$$

$$x = \left(-\beta - \frac{1}{\lambda} - \frac{1}{\lambda}W_{-1}\left[-(1+\lambda\beta)(1-p)e^{-(1+\lambda\beta)}\right]\right)^{\frac{1}{\alpha}}, \tag{14}$$

where $p\epsilon(0,1)$.

The median of the TPGLD can be obtained by substituting $p=\frac{1}{2}$ in Equation (14) which yields

$$\text{Median} = Q_2 = \left(-\beta - \frac{1}{\lambda} - \frac{1}{\lambda}W_{-1}\left[-\frac{1}{2}(1+\lambda\beta)e^{-(1+\lambda\beta)}\right]\right)^{\frac{1}{\alpha}}. \tag{15}$$

5. MOMENTS

Let X be a continuous random variable with density function $f(x)$, then the r^{th} raw moment of X is defined by

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx. \tag{16}$$

Given the pdf in Equation (11), the r^{th} raw moment of the TPGLD is defined by

$$\begin{aligned}\mu_r^{'} = E(X^r) &= \int_0^{\infty} \frac{x^r\alpha\lambda^2(\beta+x^{\alpha})x^{\alpha}-1e^{-\lambda x^{\alpha}}}{1+\lambda\beta}dx\\ &= \frac{\alpha\lambda^2}{1+\lambda\beta}\left[\int_0^{\infty}\beta x^{\alpha+r-1}e^{-\lambda x^{\alpha}}dx + \int_0^{\infty}\beta x^{2\alpha+r-1}e^{-\lambda x^{\alpha}}dx\right].\end{aligned}$$

Using the transformations $y=\lambda x^{\alpha}$, $x=\left(\frac{y}{\lambda}\right)^{\frac{1}{\alpha}}$, $dx=\frac{1}{\alpha\lambda}\left(\frac{y}{\lambda}\right)^{\frac{1}{\alpha}-1}dy$,

$$\int_0^\infty \beta x^{\alpha+r-1}e^{-\lambda x^\alpha}dx = \frac{\beta}{\alpha\lambda}\left(\int_0^\infty \left[\left(\frac{y}{\lambda}\right)^{\frac{1}{\alpha}}\right]^{\alpha+r-1} e^{-y}\left(\frac{y}{\lambda}\right)^{\frac{1}{\alpha}-1}dy\right)$$
$$= \frac{\beta}{\alpha\lambda}\left(\int_0^\infty \frac{y^{\frac{r}{\alpha}}e^{-y}}{\lambda^{\frac{r}{\alpha}}}dy\right)$$
$$= \frac{\beta\Gamma(\frac{r}{\alpha}+1)}{\alpha\lambda^{\frac{r}{\alpha}+1}}.$$

Similarly,

$$\int_0^\infty \beta x^{2\alpha+r-1}e^{-\lambda x^\alpha}dx = \frac{\Gamma(\frac{r}{\alpha}+2)}{\alpha\lambda^{\frac{r}{\alpha}+2}},$$

$$\begin{aligned} \mu_r^{'} &= \frac{\alpha\lambda^2}{1+\lambda\beta}\left[\frac{\beta\Gamma(\frac{r}{\alpha}+1)}{\alpha\lambda^{\frac{r}{\alpha}+1}} + \frac{\Gamma(\frac{r}{\alpha}+2)}{\alpha\lambda^{\frac{r}{\alpha}+2}}\right] \\ &= \frac{r[\alpha(\lambda\beta+1)+r]\Gamma(\frac{r}{\alpha})}{\alpha^2\lambda^{\frac{r}{\alpha}}(1+\lambda\beta)}, \quad r=1,2,3,4,\ldots \end{aligned} \tag{17}$$

From Equation (17), the first four raw moments of the TPGLD can be obtained as follows

$$\mu_1^{'} = \mu = \frac{[\alpha(\lambda\beta+1)+1]\Gamma(\frac{1}{\alpha})}{\alpha^2\lambda^{\frac{1}{\alpha}}(1+\lambda\beta)}, \qquad \mu_2^{'} = \frac{2[\alpha(\lambda\beta+1)+2]\Gamma(\frac{2}{\alpha})}{\alpha^2\lambda^{\frac{2}{\alpha}}(1+\lambda\beta)},$$
$$\mu_3^{'} = \frac{3[\alpha(\lambda\beta+1)+3]\Gamma(\frac{3}{\alpha})}{\alpha^2\lambda^{\frac{3}{\alpha}}(1+\lambda\beta)}, \qquad \mu_4^{'} = \frac{4[\alpha(\lambda\beta+1)+4]\Gamma(\frac{4}{\alpha})}{\alpha^2\lambda^{\frac{4}{\alpha}}(1+\lambda\beta)},$$

so that the variance (σ^2), coefficient of variation (γ), measure of skewness (S_k) and measure of kurtosis (K_s) of the TPGLD can be obtained as

$$\sigma^2 = \mu_2^{'} - \mu^2, \qquad \gamma = \frac{\sigma}{\mu}, \qquad S_k = \frac{\mu_3^{'} - 3\mu_2^{'}\mu + 2\mu^3}{\left(\mu_2^{'} - \mu^2\right)^{\frac{3}{2}}}$$

and

$$K_s = \frac{\mu_4^{'} - 4\mu_3^{'}\mu + 6\mu_2^{'}\mu^2 - 3\mu^4}{\left(\mu_2^{'} - \mu^2\right)^2}.$$

Table 2 shows the theoretical moments of the TPGLD for different values of the parameters. From Table 2, we observed that the TPGLD can be positively skewed and negatively skewed. Also, at some selected values of the parameters, the distribution can be leptokurtic as well as platykurtic.

TABLE 2
Theoretical moments of TPGLD for selected values of the parameters.

α	β	λ	μ_1'	μ_2'	μ_3'	μ_4'	σ^2	γ	S_k	K_s
2	2	3	0.5482	0.3810	0.3107	0.2857	0.0804	2.0070	0.5981	3.1661
		4	0.4677	0.2778	0.1939	0.1528	0.0590	2.3523	0.6097	3.1919
	3	3	0.5372	0.3667	0.2942	0.2667	0.0780	2.0479	0.6134	3.2003
		4	0.4602	0.2692	0.1853	0.1442	0.0575	2.3910	0.6201	3.2164
4	2	3	0.7133	0.5482	0.4464	0.3810	0.0394	1.5424	-0.1105	2.7428
		4	0.6587	0.4677	0.3520	0.2778	0.0338	1.6703	-0.1020	2.7434
	3	3	0.7059	0.5372	0.4334	0.3667	0.0389	1.5586	-0.0994	2.7438
		4	0.6532	0.4602	0.3437	0.2692	0.0334	1.6843	-0.0947	2.7449

6. RENYI ENTROPY

An entropy of a random variable X is a measure of variation of uncertainty associated with the random variable X. Renyi (1961) defined the Renyi entropy of X with density function $f(x)$ as

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[\int f^{\gamma}(x)dx\right], \qquad \gamma > 0, \gamma \neq 1. \tag{18}$$

Using Equation (18), the Renyi entropy of the TPGLD is defined by

$$\begin{aligned}\tau_R(\gamma) &= \frac{1}{1-\gamma}\log\left[\int_0^\infty \frac{(\alpha\lambda^2)^\gamma(\beta+x^\alpha)^\gamma x^{\gamma(\alpha-1)}e^{-\gamma\lambda x^\alpha}}{(1+\lambda\beta)^\gamma}dx\right] \\ &= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{1+\lambda\beta}\right)^\gamma\int_0^\infty (\beta+x^\alpha)^\gamma\, x^{\alpha\gamma-\gamma}e^{-\gamma\lambda x^\alpha}dx\right].\end{aligned} \tag{19}$$

From the series expansion

$$(a+b)^n = \sum_{j=0}^{\infty}\binom{n}{j}a^{n-j}b^j,$$

$$= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{1+\lambda\beta}\right)^\gamma\sum_{j=0}^{\infty}\binom{n}{j}\beta^{\gamma-j}\int_0^\infty x^{\alpha j+\alpha\gamma-\gamma}e^{-\gamma\lambda x^\alpha}dx\right],$$

but

$$\int_0^\infty x^{\alpha j+\alpha\gamma-\gamma}e^{-\gamma\lambda x^\alpha}dx = \frac{\Gamma\left(j+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}\right)}{\alpha(\gamma\lambda)^{j+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}}},$$

so that

$$\tau_R(\gamma) = \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha\lambda^2}{1+\lambda\beta}\right)^{\gamma}\sum_{j=0}^{\infty}\binom{n}{j}\beta^{\gamma-j}\frac{\Gamma\left(j+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}\right)}{\alpha(\gamma\lambda)^{j+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}}}\right]$$
$$= \frac{1}{1-\gamma}\log\left[\left(\frac{\alpha^{\gamma-1}\lambda^{\gamma\left(1+\frac{1}{\alpha}\right)-\frac{1}{\alpha}}}{(1+\lambda\beta)^{\gamma}\gamma^{\gamma(1-\frac{1}{\alpha})+\frac{1}{\alpha}}}\right)\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\beta^{\gamma-j}}{(\gamma\lambda)^j}\Gamma\left(j+\gamma-\frac{\gamma}{\alpha}+\frac{1}{\alpha}\right)\right]. \tag{20}$$

7. The distribution of the ordered statistics

Suppose that $Y_{1:n} < Y_{2:n} < \cdots < Y_{n:n}$ is the order statistics of a random sample generated from TPGLD, then the probability density function of the k^{th} order statistics, say $X = Y_{n:n}$ is given by

$$g_k(x) = \frac{n!}{(n-k)!(k-1)!}[G(x)]^{k-1}[1-G(x)]^{n-k}\, g(x), \tag{21}$$

where

$$g(x) = \frac{\alpha\lambda^2}{(1+\lambda\beta)}(\beta+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}, \qquad G(x) = 1-\frac{(1+\lambda\beta+\lambda x^{\alpha})}{(1+\lambda\beta)}e^{-\lambda x^{\alpha}},$$

$$g_k(x) = \frac{n!}{(n-k)!(k-1)!}\sum_{j=0}^{\infty}\binom{n-k}{j}(-1)^j[G(x)]^{j+k-1}g(x)$$

$$g_k(x) = \frac{\alpha\lambda^2 n!}{(n-k)!(k-1)!(1+\lambda\beta)}\sum_{j=0}^{\infty}\binom{n-k}{j}(-1)^j(\beta+x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}} \times\left[1-\frac{(1+\lambda\beta+\lambda x^{\alpha})}{(1+\lambda\beta)}e^{-\lambda x^{\alpha}}\right]^{j+k-1}. \tag{22}$$

Using the series expression

$$\left[1-\frac{(1+\lambda\beta+\lambda x^{\alpha})}{(1+\lambda\beta)}e^{-\lambda x^{\alpha}}\right]^{j+k-1} = \sum_{m=0}^{\infty}\binom{j+k-1}{m}(-1)^m\frac{(1+\lambda\beta+\lambda x^{\alpha})}{(1+\lambda\beta)^m}e^{-m\lambda x^{\alpha}},$$

Equation (22) becomes

$$= \frac{\alpha\lambda^{2+p}\beta^{p+1-q}n!}{(n-k)!(k-1)!(1+\lambda\beta)^{m+1}}\sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{p=0}^{m}\sum_{q=0}^{p+1}\binom{n-k}{j}\binom{j+k-1}{m}\binom{m}{p} \times\binom{p+1}{q}(-1)^{j+m}x^{\alpha(1+q)-1}e^{-\lambda(m+1)x^{\alpha}}. \tag{23}$$

The s^{th} moment of the k^{th} order statistics from the TPGLD is defined by

$$E(X_k^s) = \int_0^\infty x^s g_k(x) dx \tag{24}$$

$$= \frac{\alpha\lambda^{2+p}\beta^{p+1-q} n!}{(n-k)!(k-1)!(1+\lambda\beta)^{m+1}} \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{p=0}^{m}\sum_{q=0}^{p+1} \binom{n-k}{j}\binom{j+k-1}{m}\binom{m}{p}$$
$$\times \binom{p+1}{q}(-1)^{j+m}\int_0^\infty x^{s+\alpha(1+q)-1} e^{-\lambda(m+1)x^\alpha} dx \tag{25}$$

but

$$\int_0^\infty x^{s+\alpha(1+q)-1} e^{-\lambda(m+1)x^\alpha} dx = \frac{\Gamma\left(\frac{s}{\alpha}+q+1\right)}{\alpha\left(\lambda(m+1)\right)^{\frac{s}{\alpha}+q+1}}.$$

Thus

$$E(X_k^s) = \frac{\alpha\lambda^{2+p}\beta^{p+1-q} n!}{(n-k)!(k-1)!(1+\lambda\beta)^{m+1}} \sum_{j=0}^{\infty}\sum_{m=0}^{j}\sum_{p=0}^{m}\sum_{q=0}^{p+1} \binom{n-k}{j}\binom{j+k-1}{m}$$
$$\times \binom{m}{p}\binom{p+1}{q}(-1)^{j+m}\frac{\Gamma\left(\frac{s}{\alpha}+q+1\right)}{\alpha\left(\lambda(m+1)\right)^{\frac{s}{\alpha}+q+1}}. \tag{26}$$

8. Maximum likelihood estimation

Let $(x_1, x_2, \cdots, x_n)$ be random samples from the TPGLD, then the log-likelihood function is defined as

$$\ell(x,\phi) = \sum_{i=1}^{n} \log\left[\frac{\alpha\lambda^2(\beta+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{1+\lambda\beta}\right], \qquad \phi = (\alpha,\beta,\lambda)^T, \tag{27}$$

$$= n\log\alpha + 2n\log\lambda + \sum_{i=1}^{n}\log(\beta+x_i^\alpha) + (\alpha-1)\sum_{i=1}^{n}\log x_i - \lambda\sum_{i=1}^{n}(x_i^\alpha) - n\log(1+\lambda\beta). \tag{28}$$

On differentiating the log-likelihood function with respect to the parameters, we obtain the score function as

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha} \log x_i}{(\beta + x_i^{\alpha})} + \sum_{i=1}^{n} \log x_i - \lambda \sum_{i=1}^{n} x_i^{\alpha} \log x_i,$$
$$\frac{\partial \ell}{\partial \lambda} = \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i^{\alpha} - \frac{n\beta}{(1+\lambda\beta)}.$$

The maximum likelihood estimator $\hat{\phi}$ of ϕ can be obtained by solving the system of non-linear equation $\frac{\partial \ell}{\partial \phi} = 0$. This non-linear equation can be solved using the Newton Raphson iterative scheme given by

$$\hat{\phi} = \phi_k - H^{-1}(\phi_k) U(\phi_k), \tag{29}$$

where $U(\phi_k)$ is the score function and $H(\phi_k)$ is the Hessian matrix, which is the second derivative of the log-likelihood function.

A closed form expression of the Fisher information matrix is defined by

$$I(\phi_k) = -E[H(\phi_k)] = -E \begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & \frac{\partial^2 \ell}{\partial \lambda \partial \beta} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix}$$

The elements of the observed information matrix of the TPGLD are available upon request from the authors.

9. Data analysis

In this section, we fit the proposed distribution to two real data sets alongside with some well-known lifetime distributions with the following density functions.

(i) Exponentiated Power Lindley distribution (EPLD) reported in Warahena-Liyanage and Pararai (2014):

$$f(x) = \frac{\alpha \lambda^2 \beta (1 + x^{\alpha}) x^{\alpha-1} e^{-\lambda x^{\alpha}}}{1+\lambda} \left[1 - \left(1 + \frac{\lambda x^{\alpha}}{1+\lambda} \right) e^{-\lambda x^{\alpha}} \right]^{\beta-1}, \quad x > 0, \alpha, \lambda, \beta > 0.$$

(ii) Exponentiated Lindley geometric distribution (ELGD) reported in Wang (2013):

$$f(x) = \frac{\alpha \beta^2 (1-\lambda)(1+x) e^{-\beta x} \left[1 - \left(1 + \frac{\beta x}{1+\beta} \right) e^{-\beta x} \right]^{\alpha-1}}{(1+\beta) \left[1 - \lambda + \lambda \left[1 - \left(1 + \frac{\beta x}{1+\beta} \right) e^{-\beta x} \right]^{\alpha} \right]^2}, \quad x > 0, \alpha, \beta > 0, 0 < \lambda < 1.$$

(iii) Power Lindley distribution (PLD) reported in Ghitany *et al.* (2013):

$$f(x) = \frac{\alpha\lambda^2(1+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}}{1+\lambda}, \qquad x > 0, \alpha, \lambda > 0.$$

(iv) Lindley-exponential distribution (LED) reported in Bhati *et al.* (2015):

$$f(x) = \frac{\alpha\lambda^2 e^{-\alpha x}(1-e^{-\alpha x})^{\lambda-1}(1-log(1-e^{-\alpha x}))}{1+\lambda}, \qquad x > 0, \alpha, \lambda > 0.$$

(v) Lindley distribution reported in Lindley (1958):

$$f(x) = \frac{\lambda^2(1+x)e^{-\lambda}}{1+\lambda}, \qquad x > 0, \lambda > 0.$$

The comparison criteria considered in this work includes, the estimates of the parameters of the distribution,$-2\log(L)$, Akaike information criterion $[AIC = 2k - 2\log(L)]$, Bayesian information criterion $[BIC = k\log(n) - 2\log(L)]$, Anderson Darling test statistic (A^*) and Crammer-von Mises test statistic (W^*), where n is the number of observations, k is the number of estimated parameters and L is the value of the likelihood function evaluated at the parameter estimates.

Data Set 1: This data set consists of 72 exceedances of flood peaks (in m^3/s) of the Wheaton river near Carcross in Yukon Territory, Canada for the years 1958-1984. This data was first used by Choulakian and Stephens (2001) to examine the applicability of the generalized Pareto distribution and also was reported in Akinsete *et al.* (2008). This data set is given in Table 3.

TABLE 3
Exceedances of Wheaton river flood data.

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	27.0	14.4	1.7
37.6	0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1
0.6	9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0
3.6	5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7
64.0	1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5

Plots of the density and the cumulative distribution fit for the Wheaton river flood data is shown in Figure 3.

Data Set 2: This dataset consists of the strengths of 1.5 cm glass fibres measured by the National Physical Laboratory in England reported in Bera (2015). The data was originally used by Smith and Naylor (1987). Table 5 presents the data set.

TABLE 4
Summary statistics for the data set 1 (standard error in parenthesis).

Models	Estimates	-2 log L	AIC	BIC	A^*	W^*
TPGLD	α=0.870(0.110) β=16.515(35.714) λ=0.154(0.083)	502.730	508.730	515.560	0.808	0.141
EPLD	α=0.730(0.236) β=0.916(0.599) λ=0.300(0.281)	504.425	510.425	517.255	0.858	0.149
ELGD	α=0.559(0.121) β=0.095(0.024) λ=0.281(0.466)	505.089	511.089	517.919	0.842	0.141
PLD	α=0.700(0.057) λ=0.339(0.056)	504.444	508.444	512.997	0.877	0.153
LED	α=0.062(0.012) λ=1.121(0.141)	503.073	507.073	511.626	0.845	0.154
Lindley	α=0.150(0.013)	528.424	530.424	532.700	7.421	0.818

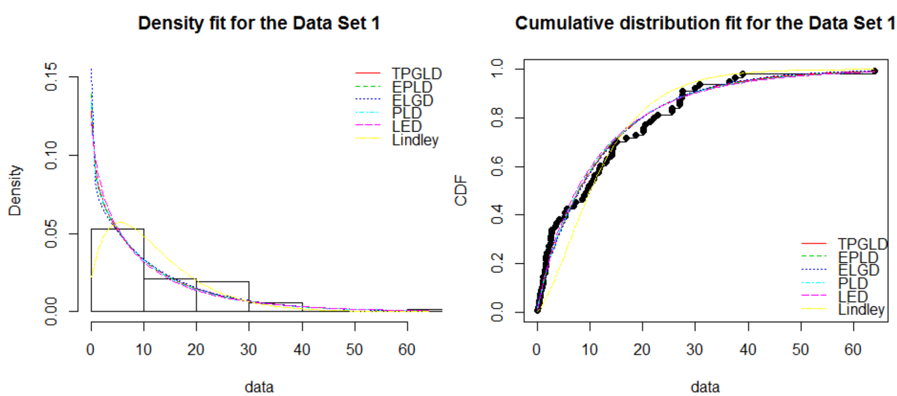

Figure 3 – Density and cumulative distribution fit for the Wheaton river flood data.

TABLE 5
Strength of glass fibers data.

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68	1.73	1.81	2.00
0.74	1.04	1.27	1.39	1.49	1.53	1.59	1.61	1.66	1.68	1.76	1.82	2.01
0.77	1.11	1.28	1.42	1.50	1.54	1.60	1.62	1.66	1.69	1.76	1.84	2.24
0.81	1.13	1.29	1.48	1.50	1.55	1.61	1.62	1.66	1.70	1.77	1.84	0.84
1.24	1.30	1.48	1.51	1.55	1.61	1.63	1.67	1.70	1.78	1.89		

TABLE 6
Summary statistics for the data set 2 (standard error in parenthesis).

Models	Estimates	-2 log L	AIC	BIC	A^*	W^*
TPGLD	α=4.944(0.658) β=3.429(4.401) λ=0.156(0.071)	28.421	34.421	40.851	0.999	0.166
EGLD	α=9.521(20.587) β=6.217(0.628) λ=-653.220(1341.380)	31.663	37.663	44.093	1.280	0.171
PLD	α=4.458(0.387) λ=0.222(0.047)	29.380	33.380	37.666	1.119	0.190
LED	α=2.612(0.239) λ=32.308(9.558)	62.816	66.816	71.102	4.341	0.799
Lindley	α=0.996(0.095)	162.557	164.557	166.700	16.245	3.332

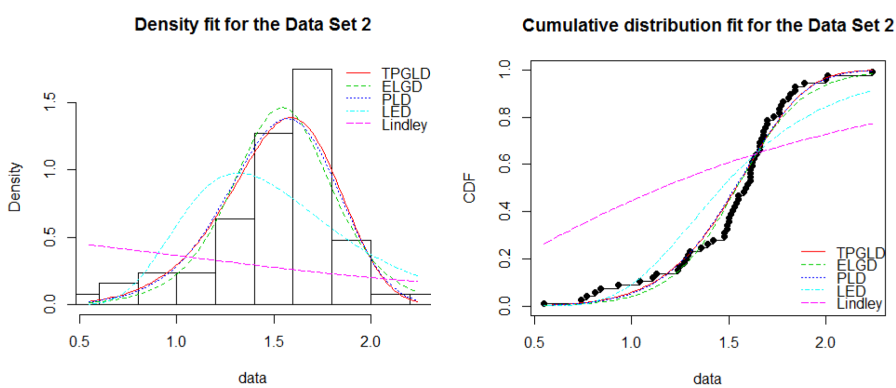

Figure 4 – Density and cumulative distribution fit for the strengths of glass fiber data.

Plots of the density and the cumulative distribution fit for the strengths of glass fiber data are shown in Figure 4. When comparing lifetime distributions, the distribution with the smallest $-2\log L$, AIC, BIC, A^* and W^* is considered to be the best model in fitting a given data set. However, when the number of parameters of a distribution is small, the likelihood of selecting such a distribution as the best model will be increased in terms of $-2\log L$, AIC and BIC. In this case, one resort to measures of goodness of fit test statistics such as Anderson Darling test, Crammer-von Mises test and Komolgorov-Smirnov test statistics to validate the superiority of a model for a given data set. Consequently, Tables 4 and 6 show that the TPGLD has the least value of $-2\log L$, AIC, BIC, A^* and W^*, which indicates that the TPGLD demonstrates superiority over the EPLD, ELGD, PLD, LED and the classical one parameter Lindley distribution in modeling the lifetime data sets under study. This claim was further supported by inspecting the density and cumulative distribution fit of the distributions for the real lifetime data set.

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SUMMARY

In this paper, we introduced a new class of lifetime distribution and considered the mathematical properties of one of the sub models called a three parameter generalized Lindley distribution (TPGLD). The new class of distributions generalizes some of the Lindley family of distribution such as the power Lindley distribution, the Sushila distribution, the Lindley-Pareto distribution, the Lindley-half logistic distribution and the classical Lindley distribution. An application of the TPGLD to two real lifetime data sets reveals its superiority over the exponentiated power Lindley distribution, the exponentiated Lindley geometric distribution, the power Lindley distribution, the Lindley-exponential distribution and the classical one parameter Lindley distribution in modeling the lifetime data sets under study.

Keywords: Lindley distribution; Power Lindley distribution; Hazard rate; Moments.