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THE ODD LOG-LOGISTIC POWER SERIES FAMILY OF DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

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1. INTRODUCTION

Since 1997 when Marshall and Olkin proposed a way to add a parameter to the exponential distribution by compounding with the geometric distribution (obtaining what is referred to as the exponential geometric (EG) distribution), many new distributions have been proposed based on compounding lifetime distributions with members of the power series family. The exponential Poisson (EP) and exponential logarithmic (EL) distributions were introduced and studied by Kus (2007) and Tahmasbi and Rezaei (2008), respectively. Chahkandi and Ganjali (2009) proposed the exponential power series (EPS) distribution, which contains as special cases these distributions. Recently, Morais and Barreto-Souza (2011) proposed the Weibull power series (WPS) distribution which contains the EPS distribution as a special case.

The odd log-logistic (OLL) family of distributions was developed by Gleaton and Lynch (2004). The name "odd" originates from the idea of evaluating the odds of death of a patient. The OLL family of distributions have the cumulative distribution function (cdf) and probability density function (pdf) specified by

$$G(x;\gamma,\tau)=\frac{\Pi(x;\tau)^{\gamma}}{\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}} \qquad (1)$$

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and

$$g(x;\gamma,\tau)=\frac{\gamma\pi(x;\tau)\left[\Pi(x;\tau)\overline{\Pi}(x;\tau)\right]^{\gamma-1}}{\left[\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}\right]^{2}}, \tag{2}$$

respectively, where $\pi(x;\tau)$, $\Pi(x;\tau)$ and $\overline{\Pi}(x;\tau)$ are the pdf, cdf and survival function of the baseline distribution. The survival function of the OLL distribution is $\overline{G}(x;\tau)=1-G(x;\tau)$, where τ are parameters of the baseline distribution.

In this paper, we introduce the OLL power series distribution (OLL-PS) obtained by compounding the OLL family and power series distributions. The compounding procedure follows key ideas of Marshall and Olkin (1997), builds a wider and more flexible family of continuous lifetime distributions.

The contents of this paper are organized as follows: Section 2 introduces the OLL-PS family of distributions; Section 3 derives some mathematical properties; estimation of the parameters of the OLL-PS family of distributions by maximum likelihood is investigated in Section 4; Section 5 presents five special cases of the OLL-PS family of distributions; a simulation study is given in Section 6; applications to two real data sets are illustrated in Section 7. The paper is concluded in Section 8.

2. The new distribution

Let $X_1,X_2,\ldots,X_N$ be a random sample following the OLL distribution with cdf and pdf given by (1) and (2), respectively. Let $\overline{G}(x;\varsigma)=1-G(x;\varsigma)$ denote the survival function, where $\varsigma=(\gamma,\tau)$ are the parameters of (1) and (2). Let N follow the zero-truncated power series distribution. A distribution is said to be a power series distribution (Noack, 1950) if its probability mass function (pmf) can be written in the form

$$P(N=n;\theta)=\frac{a_n\theta^n}{A(\theta)} \tag{3}$$

for $n=0,1,2,\ldots$, where a_n depends only on n and not on θ, $A(\theta)=\sum_{n=0}^{\infty}a_n\theta^n$ and $\theta>0$ is such that $A(\theta)$ is finite. In (3), θ is the power parameter of the distribution and $A(\cdot)$ is the series function. Thus, we can define a zero-truncated power series distribution with pmf as follows:

$$P(N=n;\theta)=\frac{a_n\theta^n}{C(\theta)}$$

for $n=1,2,3,\ldots$, where $C(\theta)=A(\theta)-a_0=\sum_{n=1}^{\infty}a_n\theta^n$. For more details about power series distributions, see Johnson *et al.* (2005).

Table 1 in Appendix E shows useful quantities of some power series distributions (truncated at zero) such as Poisson, geometric, logarithmic, negative binomial and binomial distributions.

The OLL-PS family of distributions is defined by the random variable $X = \min\{X_i\}_{i=1}^{N}$. Then

$$
\begin{aligned}
F(x \mid n; \gamma, \tau) &= 1 - \overline{G}(x; \gamma, \tau)^n \\
&= 1 - \sum_{n=1}^{\infty} \left\{ \frac{\overline{\Pi}(x; \tau)^\gamma}{\Pi(x; \tau)^\gamma + \overline{\Pi}(x; \tau)^\gamma} \right\}^n \frac{a_n \theta^n}{C(\theta)}.
\end{aligned}
$$

The marginal cdf of X, if N has truncated at zero power series distribution, is

$$
\begin{aligned}
F(x; \xi) &= \sum_{n=1}^{\infty} F(x \mid n; \gamma, \tau) P(N = n; \theta) \\
&= 1 - \sum_{n=1}^{\infty} \left\{ \frac{\overline{\Pi}(x; \tau)^\gamma}{\Pi(x; \tau)^\gamma + \overline{\Pi}(x; \tau)^\gamma} \right\}^n \frac{a_n \theta^n}{C(\theta)} \\
&= 1 - [C(\theta)]^{-1} \sum_{n=1}^{\infty} a_n \left\{ \frac{\theta \overline{\Pi}(x; \tau)^\gamma}{\Pi(x; \tau)^\gamma + \overline{\Pi}(x; \tau)^\gamma} \right\}^n \\
&= 1 - [C(\theta)]^{-1} C\left(\theta \left\{ \frac{\overline{\Pi}(x; \boldsymbol{\tau})^\gamma}{\Pi(x; \boldsymbol{\tau})^\gamma + \overline{\Pi}(x; \tau)^\gamma} \right\} \right),
\end{aligned}
\tag{4}
$$

where $\xi = (\theta, \gamma, \tau)$ is the parameter vector of the OLL-PS family of distributions. The random variable X following (4) extends some distributions introduced in the literature. The EPS (Chahkandi and Ganjali, 2009) and WPS (Morais and Barreto-Souza, 2011) distributions are obtained by taking exponential and Weibull distributions as the baseline distributions and $\gamma = 1$.

THEOREM 1. *The exponentiated OLL family of distributions with shape parameter c is a limiting special case of the OLL-PS family of distributions when $\theta \to 0^+$, where $c = \min\{n \in \mathbb{N} : a_n > 0\}$.*

PROOF. Let $c = \min\{n \in \mathbb{N} : a_n > 0\}$. Using $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ for $x > 0$, we have that

$$
\begin{aligned}
\lim_{\theta \to 0^+} F(x; \xi) &= 1 - \lim_{\theta \to 0^+} \frac{1}{C(\theta)} C\left(\theta \overline{G}(x; \varsigma) \right) \\
&= 1 - \lim_{\theta \to 0^+} \frac{\sum_{n=c}^{\infty} a_n \left[\theta \overline{G}(x; \varsigma) \right]^n}{\sum_{n=c}^{\infty} a_n \theta^n} \\
&= 1 - \lim_{\theta \to 0^+} \frac{\left[\overline{G}(x; \varsigma) \right]^c + a_c^{-1} \sum_{n=c+1}^{\infty} a_n \left[\theta \overline{G}(x; \varsigma) \right]^n}{1 + a_c^{-1} \sum_{n=c+1}^{\infty} a_n \theta^n}
\end{aligned}
$$

$$= \quad 1-\left[\overline{G}(x;\varsigma)\right]^{c}$$

for $x>0$. For more details see Nadarajah and Kotz (2006). □

3. Some useful properties

3.1. Density, survival and failure rate functions

Let X be an OLL-PS random variable with cdf (4). The corresponding pdf, survival function and failure rate function are

$$\begin{aligned} f(x;\xi) &= \theta[C(\theta)]^{-1} g(x;\varsigma) C'\left(\theta\overline{G}(x;\varsigma)\right) \\ &= \frac{\gamma\theta\pi(x;\tau)\left(\Pi(x;\tau)\overline{\Pi}(x;\tau)\right)^{\gamma-1}}{C(\theta)\left(\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}\right)^{2}} C'\left(\frac{\theta\overline{\Pi}(x;\tau)^{\gamma}}{\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}}\right), \end{aligned} \tag{5}$$

$$\overline{F}(x;\xi)=[C(\theta)]^{-1}C\left(\theta\left\{\frac{\overline{\Pi}(x;\tau)^{\gamma}}{\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}}\right\}\right), \tag{6}$$

and

$$h(x;\xi)=\frac{\gamma\theta\pi(x;\tau)\left(\Pi(x;\tau)\overline{\Pi}(x;\tau)\right)^{\gamma-1}}{\left(\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}\right)^{2}}\left\{\frac{C'\left(\frac{\theta\overline{\Pi}(x;\tau)^{\gamma}}{\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}}\right)}{C\left(\theta\left\{\frac{\overline{\Pi}(x;\tau)^{\gamma}}{\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}}\right\}\right)}\right\},$$

respectively, for $x>0$. Two useful linear representations for (4) and (5) can be derived using the concept of power series. We can prove that the cdf (4) admits the expansion

$$\overline{G}(x)^{n}=\left[\frac{\overline{\Pi}(x)^{\gamma}}{\Pi(x)^{\gamma}+\overline{\Pi}(x)^{\gamma}}\right]^{n}=\frac{\sum_{r=0}^{\infty}\lambda_r\overline{\Pi}(x)^{r}}{\sum_{r=0}^{\infty}\rho_r\overline{\Pi}(x)^{r}}=\sum_{r=0}^{\infty}b_r\overline{\Pi}(x)^{r},$$

where

$$\lambda_r=\sum_{l=r}^{\infty}(-1)^{l+r}\binom{\gamma n}{l}\binom{l}{r},\ \rho_r=h_r(\gamma,n)$$

and

$$b_r=b_r(\gamma,n)=\frac{1}{\rho_0}\left(\rho_r-\frac{1}{\rho_0}\sum_{s=1}^{r}\rho_s b_{r-s}\right)$$

for $r > 1$ and $b_0 = \frac{\lambda_0}{\rho_0}$, see Appendix A. For more details, see Cordeiro *et al.* (2015). Second by using the power series expansion for $C\left(\theta\overline{G}(x)\right)$, we have

$$C\left(\theta\overline{G}(x)\right)=\sum_{n=1}^{\infty}a_n\theta^n\overline{G}(x)^n=\sum_{n=1}^{\infty}a_n\theta^n\sum_{r=0}^{\infty}b_r\overline{\Pi}(x)^r=\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}a_n\theta^n b_r\overline{\Pi}(x)^r.$$

Then, (6) can be expressed as

$$\overline{F}(x)=[C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}a_n\theta^n b_r\overline{\Pi}(x)^r \tag{7}$$

and the pdf of the OLL-PS distribution can be represented as

$$f(x)=[C(\theta)]^{-1}\pi(x)\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}c_{n,r}\theta^n\overline{\Pi}(x)^r, \tag{8}$$

where $c_{n,r}=-(r+1)a_n b_{r+1}$. The OLL-PS pdf is an infinite linear combination of the baseline survival function.

In Appendix B, we introduce and calculate the following

$$\varkappa(a,b,c)=E\left[X^a\pi(X)^{b-1}\overline{\Pi}(X)^c\right]=\int_{-\infty}^{\infty}x^a[\pi(x)]^b\left[\overline{\Pi}(x)\right]^c dx.$$

This can be used to express the moments, moment generating function, mean residual functions, etc.

3.2. *Moments, moment generating and mean residual life time functions*

Moments are the most important measures in statistical analysis especially in applied work. For all $s > 0$, direct integration of (8) shows that

$$\begin{aligned} E[X^s] &= [C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}c_{n,r}\theta^n\int_{-\infty}^{+\infty}x^s\overline{\Pi}(x)^r\,\pi(x)\,dx \\ &= [C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}c_{n,r}\theta^n\varkappa(s,1,r). \end{aligned} \tag{9}$$

The moment generating function of X can be determined by using an expansion of $\exp(tX)$ and (9). It is given by

$$\begin{aligned} M_X(t) &= E\left[e^{tX}\right]=E\left[\sum_{s=0}^{\infty}\frac{(tX)^s}{s!}\right]=\sum_{s=0}^{\infty}\frac{t^s}{s!}E[X^s] \\ &= [C(\theta)]^{-1}\sum_{n=1}^{\infty}\sum_{r,s=0}^{\infty}c_{n,r}\theta^n\frac{t^s}{s!}\varkappa(s,1,r). \end{aligned}$$

Given the survival to time x_0 until the time of failure, the mean residual lifetime is

$$\begin{aligned} m(x_0) &= E[X - x_0 | X > x_0] \\ &= \left[\overline{F}(x_0)\right]^{-1} \int_{x_0}^{\infty} \overline{F}(v) dv \\ &= [C(\theta)]^{-1} \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} a_n \theta^n b_r \int_{x_0}^{\infty} \overline{\Pi}(v)^r \, dv \\ &= [C(\theta)]^{-1} \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} a_n \theta^n b_r \varkappa_{x_0}(0, 0, r), \end{aligned}$$

where the incomplete kappa function $\varkappa_{x_0}(a, b, c)$ is introduced in Appendix B.

3.3. *Random number generation*

By inverting (4), we obtain the quantile function of X, say $x = Q(u) = F^{-1}(u)$, as

$$x = Q(u) = \Pi^{-1}\left(\left[1 + \left\{\frac{u'(\theta)}{1 - u'(\theta)}\right\}^{\frac{1}{\gamma}}\right]^{-1}\right), \tag{10}$$

where $\Pi^{-1}(\cdot)$ denotes the inverse cdf of the baseline distribution, $C^{-1}(\cdot)$ denotes the inverse $C(\cdot)$ function and $u'(\theta) = \theta^{-1} C^{-1}(u C(\theta))$. So, the OLL-PS distribution can be simulated as follows: if U is a uniform $[0, 1]$ random variable, then the previous transformation of U has the OLL-PS distribution.

3.4. *Entropies*

Many information measures are suggested in the literature. Two most widely used uncertainty measures are Rényi entropy (Rényi, 1961) and Shannon entropy (Shannon, 1948). The Shannon entropy of a random variable X is given by $I_S(X) = -E[\log(f(X))]$. By (5),

$$\begin{aligned} I_S(X) = \; & \log(C(\theta)) - \log(\gamma) - \log(\theta) - E[\log(\pi(X))] \\ & + 2E\left[\log\left(\Pi(X)^\gamma + \overline{\Pi}(X)^\gamma\right)\right] \\ & -(\gamma - 1)\left\{E[\log\{\Pi(X)\}] + E\left[\log\left\{\overline{\Pi}(X)\right\}\right]\right\} \\ & -E\left[\log\left\{C'\left(\frac{\theta \overline{\Pi}(X)^\gamma}{\Pi(X)^\gamma + \overline{\Pi}(X)^\gamma}\right)\right\}\right]. \end{aligned}$$

We define and calculate

$$B(a_1, a_2, a_3, a_4; a; \theta) = \int_0^1 \frac{u^{a_1}(1-u)^{a_2}}{(u^a + (1-u)^a)^{a_3}} \left\{C'\left(\frac{\theta(1-u)^a}{u^a + (1-u)^a}\right)\right\}^{a_4} du$$

$$
\begin{aligned}
&= \int_0^1 \frac{u^{a_1}(1-u)^{a_2}}{(u^a+(1-u)^a)^{a_3}} \sum_{n=0}^{\infty} c_{a_4,n} \frac{\theta^n (1-u)^{na}}{(u^a+(1-u)^a)^n} du \\
&= \sum_{n=0}^{\infty} c_{a_4,n} \theta^n \int_0^1 \frac{u^{a_1}(1-u)^{a_2+na}}{(u^a+(1-u)^a)^{a_3+n}} du \\
&= \sum_{n=0}^{\infty} c_{a_4,n} \theta^n A(a_1, a_2+na, a_3+n; a),
\end{aligned}
$$

where $A(a_1,a_2,a_3;a)$ is as defined by Cordeiro *et al.* (2015). After some calculus, it is possible to derive the Shannon entropy for a member of the OLL-PS family.
For X a random variable with pdf (5),

$$E[\log\{\Pi(x)\}] = \frac{\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma+t-1, \gamma-1, 2, 1; \gamma; \theta),$$

$$E\left[\log\left\{\overline{\Pi}(x)\right\}\right] = \frac{\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma+t-1, 2, 1; \gamma; \theta),$$

$$E\left[\log\left\{\Pi^{\gamma}(x)+\overline{\Pi}^{\gamma}(x)\right\}\right] = \frac{\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma-1, 2+t, 1; \gamma; \theta)$$

and

$$E\left[\log\left\{C'\left(\frac{\theta\overline{\Pi}(x)^{\gamma}}{\Pi(x)^{\gamma}+\overline{\Pi}(x)^{\gamma}}\right)\right\}\right] = \frac{\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma-1, 2, 1+t; \gamma; \theta).$$

Hence, the Shannon entropy of X is

$$
\begin{aligned}
I_S(X) = {} & \log(C(\theta)) - \log(\gamma) - \log(\theta) - E[\log(\pi(X))] \\
& - \frac{\theta\gamma(\gamma-1)}{C(\theta)} \frac{\partial}{\partial t} B(\gamma+t-1, \gamma-1, 1, 1; \gamma; \theta) \\
& - \frac{\theta\gamma(\gamma-1)}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma+t-1, 1, 1; \gamma; \theta) \\
& + \frac{2\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma-1, 1+t, 1; \gamma; \theta) \\
& - \frac{\theta\gamma}{C(\theta)} \frac{\partial}{\partial t} B(\gamma-1, \gamma-1, 2, 1+t; \gamma; \theta).
\end{aligned}
$$

The Rényi entropy of a random variable with pdf $f(x)$ is defined by

$$I_R(X) = \frac{1}{1-\eta} \log\left\{\int_0^{\infty} f(x)^{\eta} dx\right\}$$

for $\eta > 0$ and $\eta \neq 1$. The Shannon entropy is a special case of the Rényi entropy.

Now, we derive expressions for the Rényi entropy for the OLL-PS family. Since $C(\theta) = \sum_{n=1}^{\infty} a_n \theta^n$ and $\theta > 0$, $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$ is a power series too. By (20),

$$\left\{C'(\theta)\right\}^{\eta} = \sum_{n=0}^{\infty} c_{\eta,n} \theta^n,$$

where $c_{\eta,n} = (n a_1)^{-1} \sum_{m=1}^{n} (n+1)[m(n+1)-n] a_{m+1} c_{\eta,n-m}$. Using the power series for the ratio of two power series, we have

$$\begin{aligned}
\frac{u^{a_1}(1-u)^{a_2}}{(u^a + (1-u)^a)^{a_3}} &= \sum_{i=0}^{\infty} (-1)^i \binom{a_2}{i} \frac{u^{a_1+i}}{(u^a + (1-u)^a)^{a_3}} \\
&= \sum_{i=0}^{\infty} (-1)^i \binom{a_2}{i} \frac{\sum_{k=1}^{\infty} \delta_{1,k} u^k}{\sum_{k=1}^{\infty} \delta_{2,k} u^k} \\
&= \sum_{i,k=0}^{\infty} (-1)^i \binom{a_2}{i} \delta_{3,k} u^k, \qquad (11)
\end{aligned}$$

where $\delta_{1,k} = a_k(a_1 + i)$, $\delta_{2,k} = h_k(a, a_3)$ and

$$\delta_{3,k} = \frac{1}{\delta_{2,0}} \left(\delta_{1,k} - \frac{1}{\delta_{2,0}} \sum_{r=1}^{k} \delta_{2,r} \delta_{3,k-r} \right).$$

For more details, see Appendix A. Hence, by using (5) and (11), we have

$$\begin{aligned}
[f(x)]^{\eta} &= \left[\frac{\gamma\theta}{C(\theta)}\right]^{\eta} \frac{\pi(x)^{\eta} \left(\Pi(x)\overline{\Pi}(x)\right)^{\eta(\gamma-1)}}{\left(\Pi(x)^{\gamma} + \overline{\Pi}(x)^{\gamma}\right)^{2\eta}} \sum_{n=0}^{\infty} c_{\eta,n} \left[\frac{\theta \overline{\Pi}(x)^{\gamma}}{\Pi(x)^{\gamma} + \overline{\Pi}(x)^{\gamma}}\right]^{n} \\
&= \left[\frac{\gamma\theta}{C(\theta)}\right]^{\eta} \pi(x)^{\eta} \sum_{n=0}^{\infty} c_{\eta,n} \frac{\theta^n \Pi(x)^{\eta(\gamma-1)} \overline{\Pi}(x)^{\gamma(\eta+n)-\eta}}{\left(\Pi(x)^{\gamma} + \overline{\Pi}(x)^{\gamma}\right)^{2\eta+n}} \\
&= \left[\frac{\gamma\theta}{C(\theta)}\right]^{\eta} \pi(x)^{\eta} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^i c_{\eta,n} \theta^n \binom{\eta(\gamma-1)}{i} \delta_{3,k} \left[\overline{\Pi}(x)\right]^{k}. \qquad (12)
\end{aligned}$$

By using (12) and direct integration, we obtain

$$I_R(X) = \frac{1}{1-\eta} \log\left\{\int_0^{\infty} f^{\eta}(x)\, dx\right\} dx$$

$$= \frac{\eta}{1-\eta}\left[\log\{\gamma\}+\log\{\theta\}-\log\{C(\theta)\}\right] + \frac{1}{1-\eta}\log\left\{\sum_{n,i,k=0}^{\infty} w_{n,i,k}\,\varkappa(0,\eta,k)\right\},$$

where $w_{n,i,k}=(-1)^i c_{\eta,n}\theta^n\binom{\eta(\gamma-1)}{i}\delta_{3,k}$ and $\varkappa(0,\eta,k)=\int_0^\infty \pi(x)^\eta\,\overline{\Pi}(x)^k dx$.

3.5. *Order statistics*

Let $X_1,X_2,\ldots,X_m$ be independent OLL-PS random variables. Let $X_{i:m}$ denote the ith order statistic. The pdf of $X_{i:m}$, say $f_{i:m}(x)$, is

$$\begin{aligned} f_{i:m}(x) &= Kf(x)F(x)^{i-1}\{1-F(x)\}^{m-i} \\ &= Kf(x)\sum_{j=1}^{i-1}(-1)^j\binom{i-1}{j}\left(\overline{F}(x)\right)^{m+j-i}, \end{aligned}$$

where $K=m!/[(i-1)!(m-i)!]$. An explicit expression for this is derived by the following theorem.

THEOREM 2. *The pdf of $X_{i:m}$ can be expressed as*

$$f_{i:m}(x)=\frac{K\gamma\theta^{m-i+1}\pi(x)}{[C(\theta)]^{m-i+1}}\sum_{j=1}^{i-1}\sum_{n=1}^{\infty}\sum_{r,k=0}^{\infty} c_{i,j,n,r}\,\delta_{3,k}\left[\overline{\Pi}(x)\right]^k. \tag{13}$$

For the proof see Appendix C.

Equation (13) is the main result of this section. It reveals that the pdf of the OLL-PS order statistic is a linear combination of survival functions. So, some mathematical properties of the OLL-PS order statistic such as ordinary, incomplete and factorial moments, moment generating function, mean deviations, etc can be easily found. For example, the sth moment of the ith order statistic $X_{i:m}$ can be expressed as

$$E\left[X_{i:m}^s\right]=\frac{K\gamma\theta^{m-i+1}}{[C(\theta)]^{m-i+1}}\sum_{j=1}^{i-1}\sum_{n=1}^{\infty}\sum_{r,k=0}^{\infty} c_{i,j,n,r}\,\delta_{3,k}\,\varkappa(s,1,k).$$

4. MAXIMUM LIKELIHOOD ESTIMATION

Suppose $X_1,X_2,\ldots,X_n$ is a random sample with observed values $x_1,x_2,\ldots,x_n$ from the OLL-PS family of distributions with unknown parameters $\xi=(\theta,\gamma,\tau)$. Furthermore, let $g(x,\varsigma)$ and $\overline{G}(x,\varsigma)$ denote the pdf and survival function of the OLL-PS family of distributions with unknown parameters $\varsigma = (\gamma, \boldsymbol{\tau})$. The log-likelihood function of ξ is

$$\begin{aligned}\ell(\boldsymbol{\xi}|\mathbf{x}) &= n\log\gamma + n\log\theta - n\log[C(\theta)] + \sum_{i=1}^{n}\log[\pi(x_i;\boldsymbol{\tau})] \qquad (14)\\ &+(\gamma-1)\sum_{i=1}^{n}\log[\Pi(x_i;\boldsymbol{\tau})]\\ &+(\gamma-1)\sum_{i=1}^{n}\log\left[\overline{\Pi}(x_i;\boldsymbol{\tau})\right] - 2\sum_{i=1}^{n}\log\left[\Pi(x_i;\boldsymbol{\tau})^{\gamma} + \overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}\right]\\ &+\sum_{i=1}^{n}\log\left[C'\left(\frac{\theta\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\right)\right]. \qquad (15)\end{aligned}$$

The log-likelihood function can be maximized by solving the nonlinear likelihood equations obtained by differentiating (14). The components of the score function $U_n(\xi) = (\partial\ell/\partial\theta, \partial\ell/\partial\gamma, \partial\ell/\partial\boldsymbol{\tau})$ are given by

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^{n}\frac{\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\left\{\frac{C'\left(\frac{\theta\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\right)}{C''\left(\frac{\theta\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\right)}\right\},$$

$$\begin{aligned}\frac{\partial\ell}{\partial\gamma} &= \frac{n}{\gamma} + \sum_{i=1}^{n}\log[\Pi(x_i;\boldsymbol{\tau})] + \sum_{i=1}^{n}\log\left[\overline{\Pi}(x_i;\boldsymbol{\tau})\right]\\ &-2\sum_{i=1}^{n}\frac{\Pi(x_i;\boldsymbol{\tau})^{\gamma}\log[\Pi(x_i;\boldsymbol{\tau})] + \overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}\log\left[\overline{\Pi}(x_i;\boldsymbol{\tau})\right]}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\\ &+\theta\sum_{i=1}^{n}\frac{\Pi(x_i;\boldsymbol{\tau})^{\gamma}\,\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}\left\{\log\left[\overline{\Pi}(x_i;\boldsymbol{\tau})\right] - \log[\Pi(x_i;\boldsymbol{\tau})]\right\}}{\left[\Pi(x;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x;\boldsymbol{\tau})^{\gamma}\right]^2}\\ &\cdot\left\{\frac{C'\left(\frac{\theta\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\right)}{C''\left(\frac{\theta\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}{\Pi(x_i;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma}}\right)}\right\},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial\boldsymbol{\tau}} &= \sum_{i=1}^{n}\frac{\pi_{\boldsymbol{\tau}}(x_i;\boldsymbol{\tau})}{\pi(x_i;\boldsymbol{\tau})} + (\gamma-1)\sum_{i=1}^{n}\frac{\Pi_{\boldsymbol{\tau}}(x_i;\boldsymbol{\tau})}{\Pi(x_i;\boldsymbol{\tau})} + (\gamma-1)\sum_{i=1}^{n}\frac{\overline{\Pi}_{\boldsymbol{\tau}}(x_i;\boldsymbol{\tau})}{\overline{\Pi}(x_i;\boldsymbol{\tau})}\\ &-2\gamma\sum_{i=1}^{n}\frac{\Pi_{\boldsymbol{\tau}}(x_i;\boldsymbol{\tau})\Pi(x_i;\boldsymbol{\tau})^{\gamma-1} + \overline{\Pi}_{\boldsymbol{\tau}}(x_i;\boldsymbol{\tau})\overline{\Pi}(x_i;\boldsymbol{\tau})^{\gamma-1}}{\Pi(x;\boldsymbol{\tau})^{\gamma}+\overline{\Pi}(x;\boldsymbol{\tau})^{\gamma}}\end{aligned}$$

$$+\gamma\theta\sum_{i=1}^{n}\frac{\overline{\Pi}(x_i;\tau)^{\gamma-1}\Pi^{\gamma}(x_i;\tau)\left[\overline{\Pi}_{\tau}(x_i;\tau)\Pi(x_i;\tau)-\Pi_{\tau}(x_i;\tau)\overline{\Pi}(x_i;\tau)\right]}{\left[\Pi(x;\tau)^{\gamma}+\overline{\Pi}(x;\tau)^{\gamma}\right]^2}$$

$$\cdot\left\{\frac{C'\left(\frac{\theta\overline{\Pi}(x_i;\tau)^{\gamma}}{\Pi(x_i;\tau)^{\gamma}+\overline{\Pi}(x_i;\tau)^{\gamma}}\right)}{C''\left(\frac{\theta\overline{\Pi}(x_i;\tau)^{\gamma}}{\Pi(x_i;\tau)^{\gamma}+\overline{\Pi}(x_i;\tau)^{\gamma}}\right)}\right\},$$

where $C'(\cdot)$ and $C''(\cdot)$ are the first and second derivatives of $C(\cdot)$, respectively. Furthermore,

$$\pi_{\tau}(x;\tau)=\frac{d}{d\tau}\pi(x;\tau),\ \Pi_{\tau}(x;\tau)=\frac{d}{d\tau}\Pi(x;\tau),\ \overline{\Pi}_{\tau}(x;\tau)=\frac{d}{d\tau}\overline{\Pi}(x;\tau).$$

The maximum likelihood estimate (MLE) of ξ say $\widehat{\xi}$ should satisfy the following equation $U_n(\xi)=0$. The solution of this nonlinear system of equations has no closed form. To solve this equation, it is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function. In the data applications section, the MLEs were obtained by directly maximizing (14) with respect to the parameters. The optim routine in R was used for maximization.

Asymptotic properties of the MLE are needed for interval estimation and tests of hypotheses. Under certain regularity conditions (Lehmann and Casella, 1998), $\sqrt{n}\left(\widehat{\xi}-\xi\right)$ approaches $N\left(0,K(\xi)^{-1}\right)$ in distribution as $n\to\infty$, where $K(\xi)=\lim_{n\to\infty}n^{-1}I_n(\xi)$ and $I_n(\xi)$ denotes the observed information matrix.

5. Some special cases

In this section, we study some special cases of the OLL-PS distribution. To illustrate the flexibility of the distributions, plots of the pdf and failure rate function for some selected values of the parameters are presented.

5.1. *Odd exponential power series (OEPS) distribution*

The pdf and cdf of the exponential distribution are given by $\Pi(x;\beta)=1-e^{-\beta x}$ and $\pi(x;\beta)=\beta e^{-\beta x}$, respectively. Inserting these into (5) gives the OEPS pdf

$$f(x;\beta,\gamma,\theta)=\frac{\gamma\theta\beta e^{-\beta x}\left[\left(1-e^{-\beta x}\right)e^{-\beta x}\right]^{\gamma-1}}{C(\theta)\left[\left(1-e^{-\beta x}\right)^{\gamma}+e^{-\beta\gamma x}\right]^2}C'\left(\theta\left\{\left(e^{\beta x}-1\right)^{\gamma}+1\right\}^{-1}\right)$$

for $x>0$, $\beta>0$ and $\gamma>0$.

5.2. Odd Lindley power series (OLPS) distribution

Consider the Lindley distribution with cdf and pdf given by

$$\Pi(x;\beta)=1-\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}$$

and

$$\pi(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)e^{-\beta x},$$

respectively. Inserting these into (5) gives the OLPS pdf

$$f(x;\beta,\gamma,\theta) = \frac{\gamma\theta\beta^2(1+x)e^{-\beta x}\left[\left(1-\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}\right)\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}\right]^{\gamma-1}}{(\beta+1)C(\theta)\left[\left(1-\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}\right)^{\gamma}+\left(\frac{\beta+1+\beta x}{\beta+1}e^{-\beta x}\right)^{\gamma}\right]^2} \cdot C'\left(\theta\left\{1+\left(\frac{\beta+1}{\beta+1+\beta x}e^{\beta x}-1\right)^{\gamma}\right\}^{-1}\right)$$

for $x>0$, $\beta>0$ and $\gamma>0$.

5.3. Odd Weibull power series (OWPS) distribution

The OWPS distribution is defined from (5) by taking $\Pi(x;\alpha,\beta)=1—e^{-\beta x^{\alpha}}$ and $\pi(x;\alpha,\beta)=\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}$. Its pdf is

$$f(x;\alpha,\beta,\gamma,\theta)=\frac{\gamma\theta\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}\left[\left(1-e^{-\beta x^{\alpha}}\right)e^{-\beta x^{\alpha}}\right]^{\gamma-1}}{C(\theta)\left[\left(1-e^{-\beta x^{\alpha}}\right)^{\gamma}+e^{-\beta\gamma x^{\alpha}}\right]^2} \cdot C'\left(\theta\left\{\left(e^{\beta x^{\alpha}}-1\right)^{\gamma}+1\right\}^{-1}\right)$$

for $x>0$, $\beta>0$ and $\alpha\gamma>0$. Figures 1 and 2 in Appendix D display the pdf and failure rate function of the OWPS distribution for selected parameter values.

5.4. Odd Lomax power series (OLxPS) distribution

The cdf and pdf of the Lomax distribution are $\Pi(x;\beta)=1-[1+\beta x]^{-\alpha}$ and $\pi(x;\beta)=\alpha\beta[1+\beta x]^{-\alpha-1}$, respectively. Inserting these into (5) gives the OLxPS pdf

$$f(x;\alpha,\beta,\gamma,\theta)=\frac{\gamma\theta\alpha\beta[1+\beta x]^{-\alpha-1}\left\{[1+\beta x]^{-\alpha}-[1+\beta x]^{-2\alpha}\right\}^{\gamma-1}}{C(\theta)\left[\left(1-[1+\beta x]^{-\alpha}\right)^{\gamma}+[1+\beta x]^{-\alpha\gamma}\right]^2} \cdot C'\left(\theta\{1+([1+\beta x]^{\alpha}-1)^{\gamma}\}^{-1}\right)$$

for $x>0$, $\alpha>0$, $\beta>0$ and $\gamma>0$.

5.5. *Odd log-logistic power series (OLLPS) distribution*

Consider the log-logistic distribution distribution with cdf and pdf (for $x > 0$) given by $\Pi(x;\alpha,\beta) = \left[1+(\beta x)^{-\alpha}\right]^{-1}$ and $\pi(x;\alpha,\beta) = \alpha\beta(\beta x)^{\alpha-1}[1+(\beta x)^{\alpha}]^{-2}$, respectively. The OLLPS pdf is

$$f(x;\alpha,\beta,\gamma,\theta) = \frac{\alpha\beta\gamma\theta(\beta x)^{\alpha\gamma-1}}{C(\theta)[1+(\beta x)^{\alpha\gamma}]^2} C'(\theta[1+(\beta x)^{\alpha\gamma}]) \tag{16}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$. Since the log-logistic distribution is closed under the OLL generalization, (16) is the pdf of the log-logistic distribution compounded with the power series distribution.

6. Simulation study

In this section, we assess the performance of the MLEs of the OWG distribution as the special case of the OLL-PS family with respect to sample size n. Samples of sizes 50, 100, 200 and 500 were generated for different combinations of $\xi = (\alpha,\beta,\gamma,\theta)$ from the OWG distribution by using (10). We repeated the simulation $k = 10000$ times and calculated the MLEs of the parameters. The standard deviation (SD) of the parameter estimates were calculated by inverting the observed information matrices. The biases, mean squared errors (MSEs), coverage probabilities (CP) and coverage lengths (CL) were computed by

$$\text{bias}_\epsilon(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\epsilon}_i - \epsilon),$$

$$\text{MSE}_\epsilon(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\epsilon}_i - \epsilon)^2,$$

$$\text{CP}_\epsilon(n) = \frac{1}{10000}\sum_{i=1}^{10000}\mathbf{1}\left(\widehat{\epsilon}_i - 1.95996 s_{\widehat{\epsilon}_i} < \epsilon < \widehat{\epsilon}_i + 1.95996 s_{\widehat{\epsilon}_i}\right)$$

and

$$\text{CL}_\epsilon(n) = \frac{3.91992}{10000}\sum_{i=1}^{10000} s_{\widehat{\epsilon}_i}$$

for $\epsilon = \alpha, \beta, \gamma, \theta$, where $\mathbf{1}(\cdot)$ denotes the indicator function and $\widehat{\epsilon}_i$ is ith MLE of ϵ with standard error $s_{\widehat{\epsilon}_i}$. The empirical results given in Table 2 in Appendix E indicate that the MLEs perform well for estimating the model parameters. When the sample size increases, the biases and standard deviations of the estimates decrease. Furthermore, i) the MSEs for each parameter decrease to zero and appear reasonably small at $n = 500$; ii) although the coverage probabilities are slightly above or below the nominal level, they reach the nominal level at $n = 500$; iii) the coverage lengths for each parameter decrease to zero and appear reasonably small at $n = 500$.

7. Illustrative real data examples

In this section, we provide illustrations to two real data sets to show the importance of the OLL-PS family. We consider the special cases: odd Weibull Poisson (OWP), odd Weibull geometric (OWG), odd Weibull logarithmic (OWL), odd Weibull negative binomial (OWN) and odd Weibull binomial (OWB) distributions specified by the pdfs

$$f_{OWP}(x;\xi_4) = \frac{\alpha\beta\gamma\theta x^{\alpha-1} e^{\beta x^\alpha}\left(e^{\beta x^\alpha}-1\right)^{\gamma-1}}{(e^\theta - 1)\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}^2} \exp\left(\theta\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}^{-1}\right),$$

$$f_{OWG}(x;\xi_5) = \frac{\alpha\beta\gamma(1-\theta) x^{\alpha-1} e^{\beta x^\alpha}\left(e^{\beta x^\alpha}-1\right)^{\gamma-1}}{\left\{\left(e^{\beta x^\alpha}-1\right)^\gamma + 1-\theta\right\}^2},$$

$$f_{OWL}(x;\xi_6) = \frac{\alpha\beta\gamma\theta x^{\alpha-1} e^{\beta x^\alpha}\left(e^{\beta x^\alpha}-1\right)^{\gamma-1}}{-\log(1-\theta)\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}\left\{1-\theta+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}},$$

$$f_{OWN}(x;\xi_7) = \frac{m\alpha\beta\gamma\theta x^{\alpha-1} e^{\beta x^\alpha}\left(e^{\beta x^\alpha}-1\right)^{\gamma-1}}{\left[(1-\theta)^{-m}-1\right]\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}^2} \cdot\left[1-\theta\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}^{-1}\right]^{-m-1},$$

and

$$f_{OWB}(x;\xi_8) = \frac{m\alpha\beta\gamma\theta x^{\alpha-1} e^{\beta x^\alpha}\left(e^{\beta x^\alpha}-1\right)^{\gamma-1}}{\left[(1+\theta)^m-1\right]\left\{1+\left(e^{\beta x^\alpha}-1\right)^\gamma\right\}^2}\left[\frac{1+\theta+\left(e^{\beta x^\alpha}-1\right)^\gamma}{1+\left(e^{\beta x^\alpha}-1\right)^\gamma}\right]^{m-1},$$

respectively, where $\xi_i = (\alpha, \beta, \gamma, \theta)^\top$ for $i = 4, 5, \ldots, 8$, $\alpha\gamma > 0$ and $\beta > 0$. The θ in the OWP distribution is allowed to take values in $(-\infty, +\infty)$ for more flexibility. Similar extensions may be applied for parameters of other OLL-PS distributions, as can be viewed in Table 1 in Appendix E. For the OWN and OWB distributions, we assume $m = 5$, so every fitted distribution has four parameters. The MLEs of the parameters and the goodness-of-fit statistics were computed and compared with those of the popular odd Weibull (OW) (Cooray, 2006), beta Weibull (BW) (Famoye *et al.*, 2005) and beta generalized exponential (BGE) (Barreto-Souza *et al.*, 2010) distributions specified by the pdfs

$$f_{OW}(x;\xi_1) = \frac{\alpha\beta\gamma x^{\alpha} e^{\beta x^{\alpha}} \left(e^{\beta x^{\alpha}} - 1\right)^{\gamma-1}}{\left\{1 + \left(e^{\beta x^{\alpha}} - 1\right)^{\gamma}\right\}^{-2}},$$
$$x > 0,\ \alpha\gamma > 0,\ \beta > 0,$$

$$f_{BW}(x;\xi_2) = \frac{\alpha\beta x^{\alpha-1}}{B(a,b)} e^{-b\beta x^{\alpha}} \left[1 - e^{-\beta x^{\alpha}}\right]^{a-1},$$
$$x > 0,\ \alpha > 0,\ \beta > 0,\ a > 0,\ b > 0,$$

and

$$f_{BGE}(x;\xi_3) = \frac{\alpha\beta e^{-\beta x}}{B(a,b)} \left(1 - e^{-\beta x}\right)^{a\alpha-1} \left[1 - \left(1 - e^{-\beta x}\right)^{\alpha}\right]^{b-1},$$
$$x > 0,\ \alpha > 0,\ \beta > 0,\ a > 0,\ b > 0,$$

respectively, where $\xi_1 = (\alpha,\beta,\gamma)^{\top}$, $\xi_2 = (\alpha,\beta,a,b)^{\top}$, $\xi_3 = (\alpha,\beta,a,b)^{\top}$ and $B(a,b)$ denotes the beta function.

The first data set consists of the strength of 1.5 cm glass fibres, measured at the National physical laboratory, England (see Smith and Naylor, 1987). The data are: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.00, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.50, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89. The second data are times to death of twenty six psychiatric patients. This data has been studied by Elbatal *et al.* (2015). The data are: 1, 1, 2, 22, 30, 28, 32, 11, 14, 36, 1, 33, 33, 37, 35, 25, 31, 22, 26, 24, 35, 34, 30, 35, 40, 39.

The MLEs, log-likelihood value, the corresponding standard errors, the Kolmogorov Smirnov statistic, its p-value, the AIC value, the AIC_c value and the BIC value are shown in Tables 3 and 4 in Appendix E. For both data sets, we can see that the largest log-likelihood value, the largest p-value, the smallest AIC value, the smallest AIC_c value and the smallest BIC value are obtained for the OLL-PS family. For the first data set, although the results are very close to those obtained by other members of the OLL-PS family, the OWG distribution gives the best fit with respect to all indices. For the second data set too, the OWG distribution gives the best fit. It gives the smallest values for all indices. The histogram of the data sets and plots of the estimated pdfs are displayed in Figures 3 and 4 in Appendix D. Furthermore, estimated quantiles versus observed quantiles for both data sets are shown in Figure 5 and 6 in Appendix D. These figures support good fits of the OLL-PS family of distributions. Plots of the estimated hazard rate functions are displayed in Figure 7 in Appendix D for data sets 1 and 2. The OWG distribution proposes an increasing hazard rate function for data set 1 and a J-shape hazard rate function for data set 2.

8. Concluding remarks

We have proposed a new family of distributions named the OLL-PS family by compounding the OLL family and power series distributions. The family extends some common classes of distributions studied recently. The number of series components are taken to be a power series random variable and the lifetime of each component is taken to be an OLL random variable. Both random variables are assumed to be independent. The OLL-PS distribution contains the OLL, exponential power series and Weibull power series distributions as special cases. The mathematical properties of the OLL-PS distribution derived include: moments, moment generating function, mean residual lifetime, Shannon entropy, and Rényi entropy. We have studied the behaviour of the MLEs by means of a simulation study. Illustrations to two real data sets show that the proposed distribution provides better fits than popular lifetime distributions. A future work is to construct multivariate extensions of the OLL-PS family of distributions.

APPENDIX

A. SOME USEFUL EXPANSIONS

Some power series expansions required for the proofs in Section 3 are as follows.

1. For $a > 0$ real non-integer and $0 \leq y \leq 1$, we have the binomial expansion

$$(1-y)^a = \sum_{k=0}^{\infty} (-1)^k \binom{a}{k} y^k, \tag{17}$$

where $\binom{a}{k} = a\,(a-1)(a-2)\cdots(a-k+1)/k!$.

2. The following expansion holds for any $a > 0$ real non-integer

$$y^a = \sum_{k=0}^{\infty} \alpha_k(a) y^k, \tag{18}$$

where $\alpha_k(a) = \sum_{k=i}^{\infty} (-1)^{k+i} \binom{a}{k}\binom{k}{i}$. The proof follows by writing $y = [1-(1-y)]$ and applying (17) twice.

3. We have

$$y^{\lambda} = \sum_{k=0}^{\infty} (\lambda)_k (y-1)^k / k! = \sum_{k=0}^{\infty} g_k y^k, \tag{19}$$

where λ is a positive integer and

$$g_k = g_k(\lambda) = \sum_{j=0}^{\infty} \frac{(-1)^{j-k}}{j!} \binom{j}{k} (\lambda)_j$$

and $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$.

4. We have

$$\left(\sum_{k=0}^{\infty} a_k y^k \right)^n = \sum_{k=0}^{\infty} c_{n,k} y^k, \tag{20}$$

where the coefficients $c_{n,k}$, $k = 1, 2, \ldots$ are obtained from the recurrence equation

$$c_{n,k} = (k a_0)^{-1} \sum_{m=1}^{k} [m(k+1) - k] a_m c_{n,k-m}$$

and $c_{n,0} = a_0^n$ (Gradshteyn and Ryzhik, 2014).

5. By using (20),

$$\left(\sum_{n=1}^{\infty} a_n y^n\right)^z = \sum_{n=0}^{\infty} d_{z,n} y^{n+z},$$

where

$$d_{z,n} = (na_1)^{-1} \sum_{r=1}^{n} [r(n+1) - n] a_{n+1} d_{z,n-r}.$$

6. We now obtain an expansion for $[x^c + (1-x)^c]^a$. We can write from (17) and (18)

$$[x^c + (1-x)^c] = \sum_{k=0}^{\infty} t_k x^k,$$

where

$$t_k = (-1)^k \left[\binom{c}{k} + \sum_{i=k}^{\infty} (-1)^i \binom{c}{i}\binom{c}{k}\right].$$

Then, using (19), we have

$$[x^c + (1-x)^c]^a = \sum_{j=0}^{\infty} g_j \left(\sum_{k=0}^{\infty} t_k x^k\right)^j,$$

where f_i is defined as before. Finally, using (20), we obtain

$$[x^c + (1-x)^c]^a = \sum_{k=0}^{\infty} h_k x^k,$$

where

$$h_k = h_k(c,a) = \sum_{i=0}^{\infty} g_i m_{i,k},$$

$$m_{i,k} = (k t_0)^{-1} \sum_{j=1}^{k} [j(k+1) - k] t_j m_{i,k-j}$$

and $m_{i,0} = t_0^i$.

B. A USEFUL QUANTITY

In this appendix, we introduce and calculate a very useful quantity $\varkappa(a,b,c)$ defined as

$$\varkappa(a,b,c)=E\left[X^a\pi(X)^{b-1}\overline{\Pi}(X)^c\right]=\int_{-\infty}^{\infty}x^a[\pi(x)]^b\left[\overline{\Pi}(x)\right]^c dx,$$

where a, c are positive real numbers and $b\geq 1$. If $b=1$, $c=0$ and a is a nonnegative integer, then $\varkappa(a,1,0)$ represents the conventional moment about the origin of order a. If $\varkappa(a,1,0)$ exists and $\pi(x)$ is a continuous function, then $\varkappa(a,b,c)$ exists for all $b\geq 1$ and nonnegative c.

If c is a nonnegative integer, then

$$\varkappa(a,1,c)=M(a,0,c)=\sum_{k=0}^{\infty}(-1)^k\binom{c}{k}M(a,k,0),$$

where $M(i,j,l)$ is the (i,j,l)th probability weighted moment (PWM) defined by Greenwood *et al.* (1979) as

$$M(i,j,l)=E\left[X^i\Pi(X)^j\,\overline{\Pi}(X)^l\right]=\int_{-\infty}^{+\infty}x^i\Pi(x)^j\,\overline{\Pi}(x)^l\,\pi(x)dx.$$

In the special case, where a and c are nonnegative integers, $(c+1)\varkappa(a,1,c)$ is the ath moment about the origin of the first order statistic for a sample of size $c+1$. Furthermore, the incomplete kappa function could be defined as

$$\varkappa_{x_0}(a,b,c)=\int_{x_0}^{\infty}x^a[\pi(x)]^b\left[\overline{\Pi}(x)\right]^c dx$$

which arises in mean residual lifetime of reliability models. It is obvious that if $\varkappa(a,b,c)$ exists for a lifetime distribution, then the incomplete kappa function exists for every $x_0>0$.

One can obtain expressions for $\varkappa(a,b,c)$ for some distributions. For others, this quantity can be evaluated numerically. Closed form expressions for $\varkappa$ for exponential, Weibull, Lomax, Lindley and log-logistic distributions are as follows.

1. For the exponential distribution,

$$\varkappa(a,b,c)=\beta^b\int_0^{\infty}x^a e^{-(b+c)\beta x}dx=\beta^{b-a-1}(b+c)^{-a-1}\Gamma(a+1).$$

2. For the Weibull distribution,

$$\varkappa(a,b,c)=\alpha^b\beta^b\int_0^{\infty}x^{b(\alpha-1)+a}e^{-(b+c)\beta x^{\alpha}}dx=\frac{\alpha^{b-1}\Gamma\left(b+\frac{a-b+1}{\alpha}\right)}{(b+c)^b[\beta(b+c)]^{\frac{a-b+1}{\alpha}}}.$$

3. For the Lomax distribution,

$$\begin{aligned} \varkappa(a,b,c) &= \alpha^b \beta^b \int_0^\infty x^a [1+\beta x]^{-\alpha(b+c)-b} dx \\ &= \alpha^b \beta^{b-\alpha-1} B(\alpha(b+c)+b-a-1, a+1). \end{aligned}$$

4. For the Lindley distribution,

$$\varkappa(a,b,c) = \frac{\beta^{2b-a-1}}{(b+c)^{a+1}(\beta+1)^b} \sum_{k=0}^{n+m} c_k \frac{\Gamma(k+a+1)}{[\beta(b+c)]^k},$$

where

$$c_k = c_k(\beta) = \sum_{l=\max\{0,k-m\}}^{n} \binom{b}{l} \binom{c}{k-l} \left(\frac{\beta}{\beta+1}\right)^{k-l}.$$

5. For the log-logistic distribution,

$$\begin{aligned} \varkappa(a,b,c) &= \alpha^b \beta^{\alpha b} \int_0^\infty x^{b(\alpha-1)+a} [1+(\beta x)^\alpha]^{-2b-c} dx \\ &= \alpha^{b-1} \beta^{b-a-1} B\left(b + \frac{a-b+1}{\alpha}, b+c-\frac{a-b+1}{\alpha}\right). \end{aligned}$$

C. Proof of Theorem 2

By using (20), we have

$$\begin{aligned} \left[\overline{F}(x)\right]^{m+j-i} &= \left[\frac{C\left(\theta\overline{G}(x)\right)}{C(\theta)}\right]^{m+j-i} = [C(\theta)]^{i-j-m}\left[\sum_{n=1}^{\infty} a_n \left(\theta\overline{G}(x)\right)^n\right]^{m+j-i} \\ &= [C(\theta)]^{i-j-m} \sum_{n=1}^{\infty} d_{m+j-i,n}\left[\theta\overline{G}(x)\right]^{n+m+j-i}, \end{aligned} \tag{21}$$

where

$$d_{m+j-i,n} = (na_1)^{-1} \sum_{r=1}^{n} [r(n+1)-n] a_{n+1} d_{m+j-i,n-r}.$$

On the other hand $C'(\theta) = \sum_{n=0}^{\infty} b_n \theta^n$, where $b_n = (n+1)a_{n+1}$. By using (21), we have

$$C'\left(\theta\overline{G}(x)\right)\overline{F}(x)^{m+j-i}$$

$$
\begin{aligned}
&= [C(\theta)]^{i-j-m} C'\left(\theta\overline{G}(x)\right)\left\{C\left(\theta\overline{G}(x)\right)\right\}^{m+j-i} \\
&= \left[\theta\overline{G}(x)\right]^{m+j-i} \sum_{n=0}^{\infty} b_n\left[\theta\overline{G}(x)\right]^n \sum_{n=1}^{\infty} d_{m+j-i,n}\left[\theta\overline{G}(x)\right]^n \\
&= \left[\theta\overline{G}(x)\right]^{m+j-i} \sum_{n=1}^{\infty} \phi_{m+j-i,n}\left[\theta\overline{G}(x)\right]^n, \qquad (22)
\end{aligned}
$$

where $b_n = (n+1)a_{n+1}$ and $\phi_{m+j-i,n} = \sum_{l=0}^{\infty} b_l d_{m+j-i,n-l}$. The pdf of ith order statistic can be derived by using (5), (11), (13) and (22) as follows

$$
\begin{aligned}
f_{i:m}(x) &= Kf(x)\sum_{j=1}^{i-1}\binom{i-1}{j}(-1)^j[C(\theta)]^{i-j-m} \\
&\quad \cdot \sum_{n=1}^{\infty} d_{m+j-i,n}\left(\theta\overline{G}(x)\right)^{n+m+j-i} \\
&= K\left\{\frac{\gamma\theta\pi(x)\left(\Pi(x)\overline{\Pi}(x)\right)^{\gamma-1}}{C(\theta)\left(\Pi(x)^\gamma+\overline{\Pi}(x)^\gamma\right)^2}\right\}\sum_{j=1}^{i-1}\binom{i-1}{j}(-1)^j[C(\theta)]^{i-j-m} \\
&\quad \cdot \sum_{n=1}^{\infty}\phi_{m+j-i,n}\left(\frac{\theta\overline{\Pi}(x)^\gamma}{\Pi(x)^\gamma+\overline{\Pi}(x)^\gamma}\right)^{n+m+j-i} \\
&= \gamma\theta^{m-i+1}[C(\theta)]^{i-m-1}K\pi(x) \\
&\quad \cdot \sum_{j=1}^{i-1}\sum_{n=1}^{\infty}\binom{i-1}{j}(-1)^j\theta^{n+j}\phi_{m+j-i,n}[C(\theta)]^{-j} \\
&\quad \cdot \sum_{r,k=0}^{\infty}(-1)^r\binom{\gamma-1}{r}\delta_{3,k}\left[\overline{\Pi}(x)\right]^k \\
&= \frac{\gamma\theta^{m-i+1}K\pi(x)}{[C(\theta)]^{m-i+1}}\sum_{j=1}^{i-1}\sum_{n=1}^{\infty}\sum_{r,k=0}^{\infty} c_{i,j,n,r}\,\delta_{3,k}\left[\overline{\Pi}(x)\right]^k,
\end{aligned}
$$

where $\delta_{3,k}$ is as defined before and

$$
c_{i,j,n,r} = c_{i,j,n,r}(\gamma,\theta) = (-1)^{j+r}\theta^{n+j}\binom{i-1}{j}\binom{\gamma-1}{r}\phi_{m+j-i,n}[C(\theta)]^{-j}.
$$

□

D. Figures

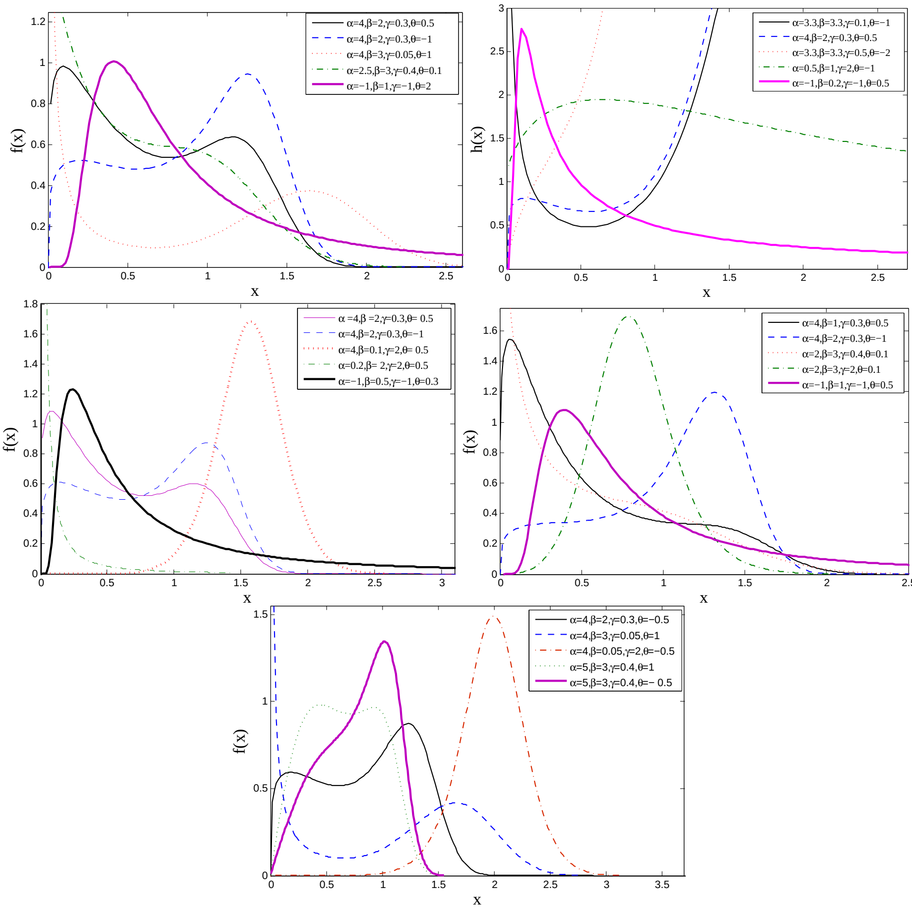

Figure 1 – Plots of the OWPS pdf for some parameter values: odd Weibull Poisson (top left), odd Weibull geometric (top right), odd Weibull logarithmic (middle left), odd Weibull negative binomial (middle right) and odd Weibull binomial (bottom) distributions.

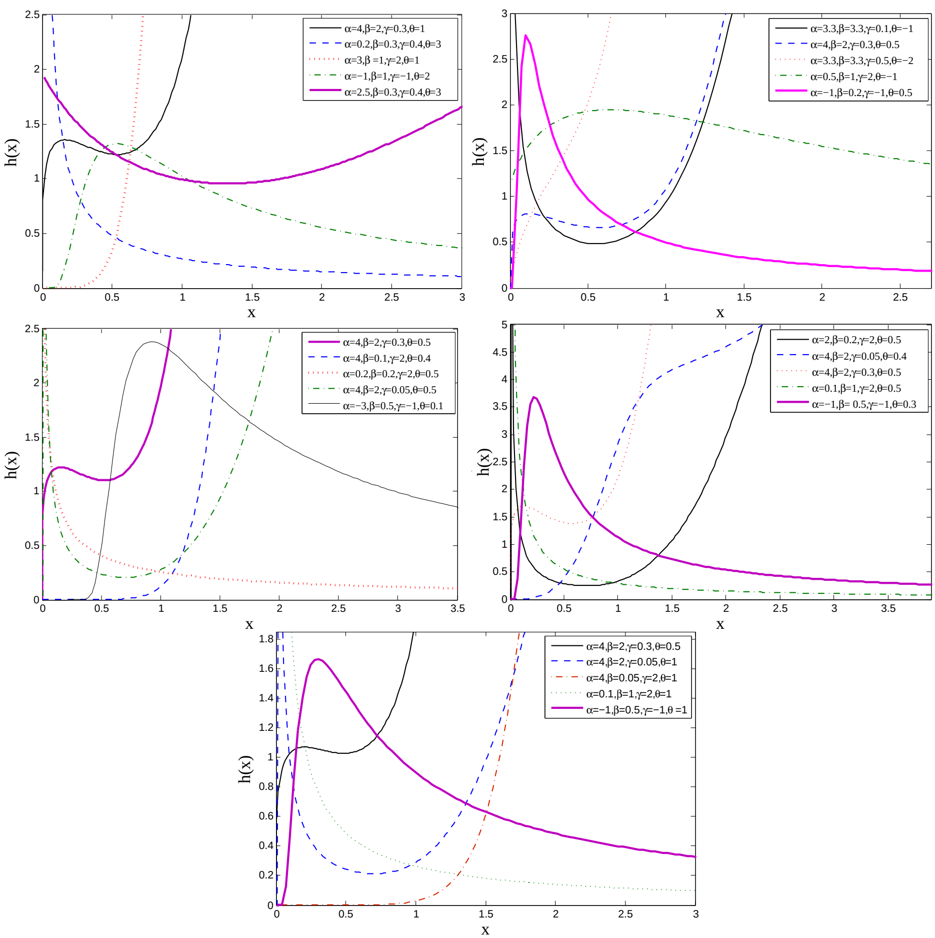

Figure 2 – Plots of the OWPS failure rate function for some parameter values: odd Weibull Poisson (top left), odd Weibull geometric (top right), odd Weibull logarithmic (middle left), odd Weibull negative binomial (middle right) and odd Weibull binomial (bottom) distributions.

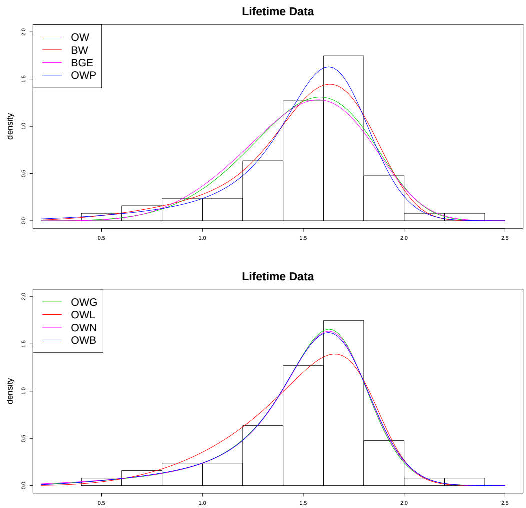

Figure 3 – Estimated pdfs of the OWPS and other competitive distributions for the first data set.

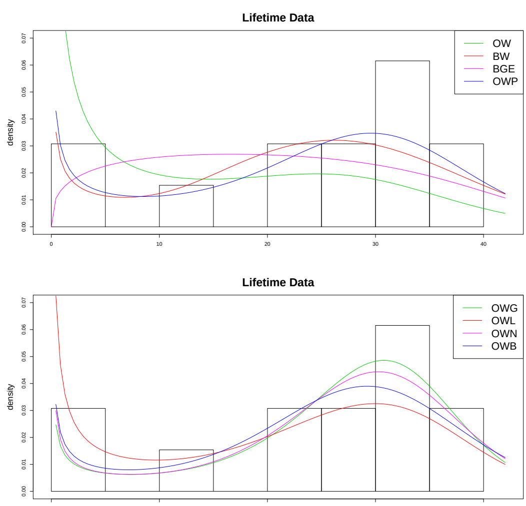

Figure 4 – Estimated pdfs of the OWPS and other competitive distributions for the second data set.

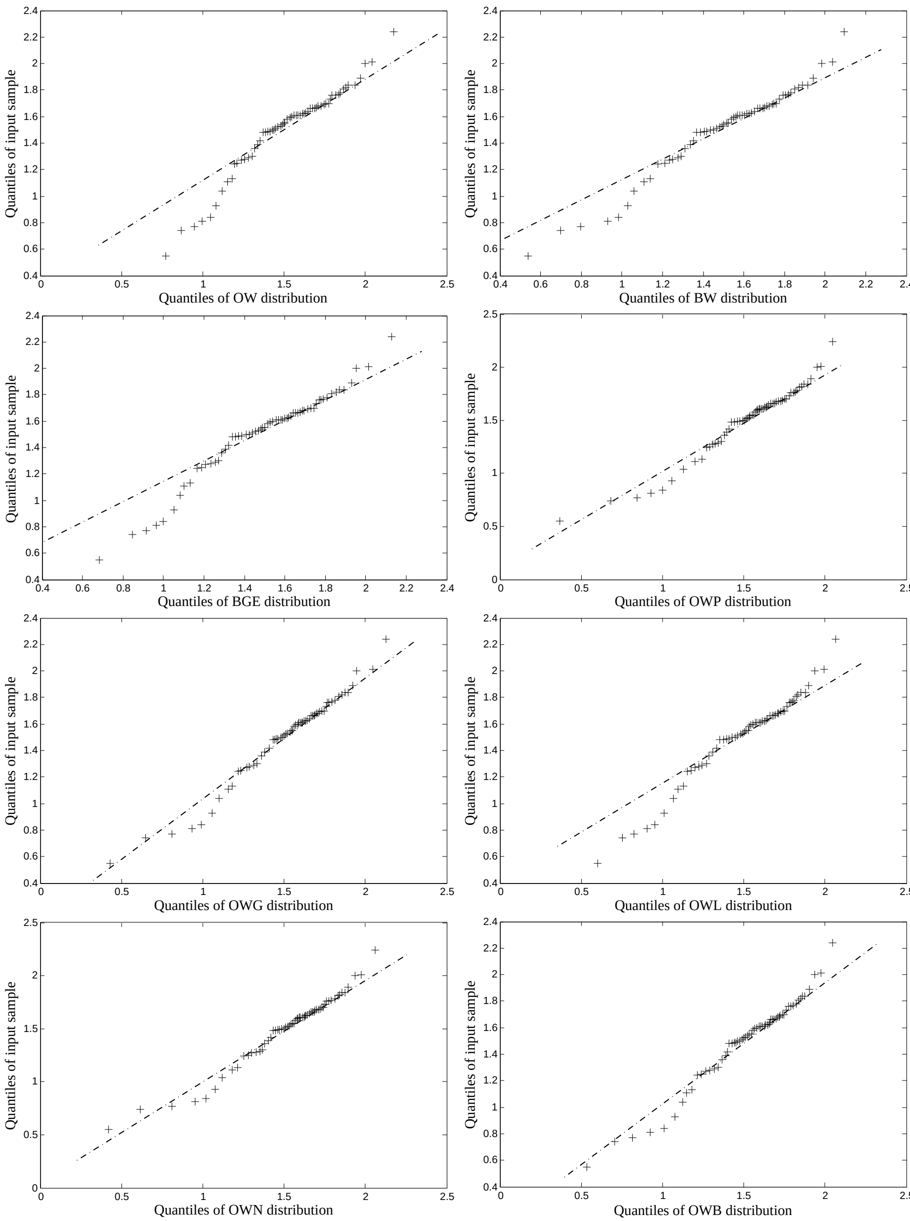

Figure 5 – Quantile-quantile plots for the fitted distributions for the first data set.

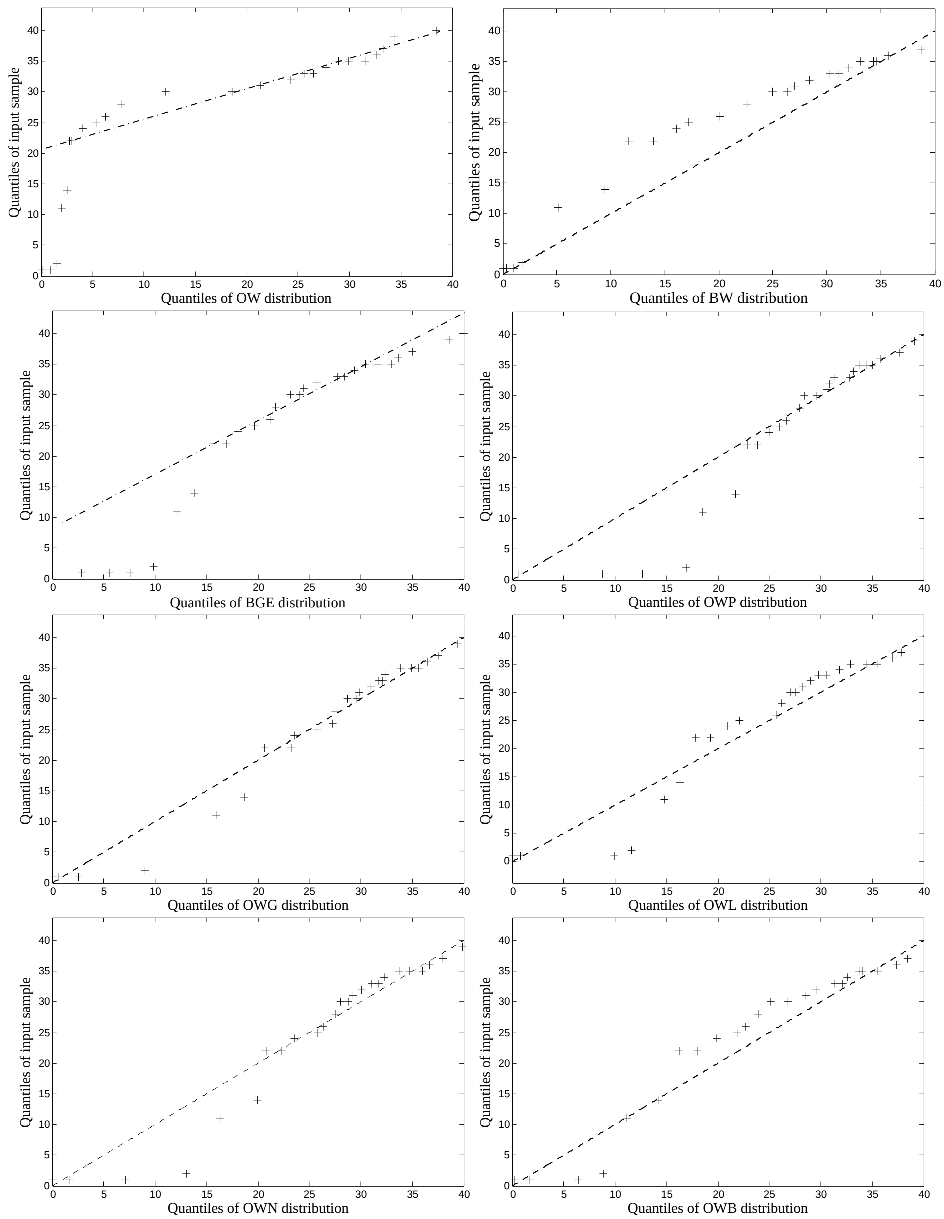

Figure 6 – Quantile-quantile plots for the fitted distributions for the second data set.

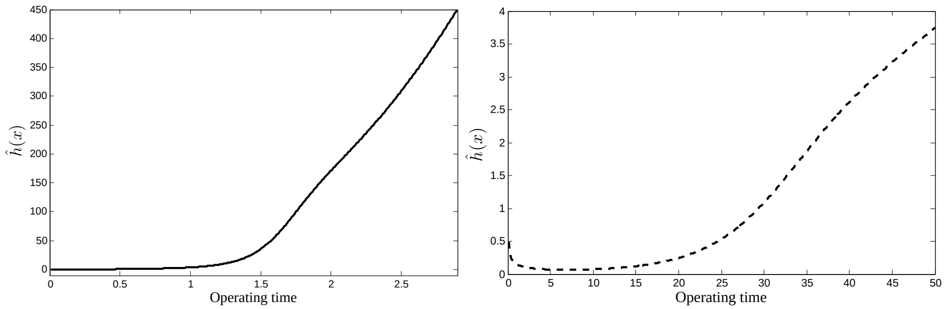

Figure 7 – Plots of the estimated failure rate functions of the OWG distribution for data sets 1 and 2.

E. TABLES

TABLE 1
Members of the power series family.

Distribution	Pdf	θ	The power parameter space in compounded distribution	a_n	$C(\theta)$	$C'(\theta)$
Zero-truncated Poisson	$e^{-\theta}\theta^n/n!(1-e^{-\theta})$	$\theta>0$	$\theta\in(-\infty,+\infty)$	$1/n!$	$e^{\theta}-1$	e^{θ}
Geometric	$(1-\theta)\theta^{n-1}$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	1	$\theta/(1-\theta)$	$(1-\theta)^{-2}$
Logarithmic	$-\theta^n/n\log(1-\theta)$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	$1/n$	-log$(1-\theta)$	$(1-\theta)^{-1}$
Negative binomial	$\binom{n+m-1}{n}(1-\theta)^m\theta^n/1-(1-\theta)^m$	$0<\theta<1$	$\theta\in(-\infty,0)\bigcup(0,1)$	$\binom{n+m-1}{n}$	$(1-\theta)^{-m}-1$	$m(1-\theta)^{-m-1}$
Zero-truncated binomial	$\binom{m}{n}\theta^n/((1+\theta)^m-1)$	$0<\theta<\infty$	$\theta\in(-1,0)\bigcup(0,+\infty)$	$\binom{m}{n}$	$(1+\theta)^m-1$	$m(1+\theta)^{m-1}$

TABLE 2
The mean, bias, MSE, standard error, CP and CL of the MLE estimators from 10000 samples.

Real value	n	ξ	$AV_\xi(n)$	$Bias_\xi(n)$	$MSE_\xi(n)$	$SD(\hat{\xi})$	$CP_\xi(n)$	$CL_\xi(n)$
$\xi = (2, 1, 1.5, 0.7)$	50	α	3.352	1.352	4.862	4.024	0.901	8.233
		β	1.941	0.441	0.320	0.692	0.923	2.735
		γ	1.196	- 0.304	0.483	1.012	0.861	2.965
		θ	0.555	-0.145	0.213	0.954	0.769	3.554
	100	α	2.595	0.595	2.345	2.687	0.969	5.963
		β	0.970	-0.031	0.239	0.602	0.930	1.641
		γ	1.341	-0.169	0.321	0.754	0.936	2.412
		θ	0.561	-0.139	0.154	0.761	0.909	1.689
	200	α	2.286	0.286	0.767	0.823	0.935	3.770
		β	0.960	-0.040	0.102	0.412	0.955	1.379
		γ	1.424	- 0.076	0.123	0.392	0.929	1.322
		θ	0.602	-0.098	0.070	0.494	0.933	0.105
	500	α	2.095	0.095	0.211	0.301	0.954	1.785
		β	0.994	-0.006	0.032	0.052	0.961	0.745
		γ	1.480	-0.020	0.059	0.067	0.939	0.766
		θ	0.683	-0.118	0.0364	0.089	0.951	0.681
$\xi = (0.7, 2, 0.6, 0.5)$	50	α	0.932	0.232	0.252	0.420	0.915	1.897
		β	2.103	0.103	0.555	0.743	0.933	3.065
		γ	0.524	-0.076	0.125	0.303	0.878	1.130
		θ	0.610	0.110	0.099	0.231	0.918	1.294
	100	α	0.794	0.094	0.065	0.123	0.891	1.003
		β	2.016	0.016	0.136	0.369	0.927	1.110
		γ	0.532	-0.068	0.130	0.312	0.946	1.193
		θ	0.584	0.084	0.049	0.145	0.968	0.839
	200	α	0.779	0.079	0.034	0.099	0.941	0.703
		β	2.019	0.019	0.091	0.988	0.973	1.194
		γ	0.572	-0.029	0.011	0.110	0.909	0.451
		θ	0.484	-0.016	0.039	0.135	0.934	0.682
	500	α	0.715	0. 015	0.017	0.088	0.953	0.465
		β	2.003	0.003	0.029	0.460	0.947	0.603
		γ	0.059	-0.001	0.007	0.047	0.932	0.351
		θ	0.492	-0.008	0.008	0.096	0.944	0.373
$\xi = (1.5, 3, 0.5, 0.3)$	50	α	1.931	0.431	0.787	1.032	0.883	3.489
		β	3.287	0.287	0.890	1.251	0.798	2.013
		γ	0.472	-0.028	0.056	0.217	0.899	0.958
		θ	0.368	0.068	0.641	0.281	0.826	1.033
	100	α	1.653	0.153	0.464	0.803	0.868	1.561
		β	3.077	0.077	0.581	0.720	0.837	2.842
		γ	0.474	-0.026	0.022	0.101	0.914	0.564
		θ	0.285	-0.016	0.044	0.367	0.88	0.827
	200	α	1.573	0.073	0.164	0.410	0.934	1.097
		β	3.014	0.014	0.161	0.242	0.969	0.909
		γ	0.492	-0.008	0.027	0.115	0.933	0.631
		θ	0.301	0.001	0.012	0.162	0.927	0.481
	500	α	1.510	0.010	0.012	0.135	0.953	0.364
		β	2.995	-0.005	0.079	0.234	0.946	0.886
		γ	0.496	-0.004	0.006	0.029	0.951	0.215
		θ	0.297	0.003	0.004	0.071	0.934	0.188

TABLE 3
Estimates and goodness-of-fit measures for the first data set.

Model	$\widehat{\xi}$	$-\ell(\widehat{\xi})$	K-S	p-value	AIC	AIC_c	BIC
OW	6.0258, 0.0539, 0.9438	15.187	0.155	0.114	36.374	37.064	42.803
$SE(\widehat{\xi})$	(1.3333, 0.0331, 0.2667)						
BW	7.0138, 0.5533, 0.4498, 0.0499	13.044	0.118	0.387	34.088	35.141	42.661
$SE(\widehat{\xi})$	(0.8896, 0.6459, 0.1810, 0.0464)						
BGE	22.6124, 0.9227, 0.4125, 93.4655	15.599	0.158	0.103	39.198	40.251	47.771
$SE(\widehat{\xi})$	(22.8153, 0.5135, 0.3152, 116.6665)						
OWP	4.3726, 0.6842, 0.3820, -5.0114	11.909	0.097	0.623	31.818	32.871	40.391
$SE(\widehat{\xi})$	(0.0829, 0.5699, 0.1780, 1.6585)						
OWG	3.5469, 0.8086, 0.6918, -14.0978	11.594	0.093	0.681	31.188	32.241	39.761
$SE(\widehat{\xi})$	(1.056, 0.7265, 0.2841, 15.9532)						
OWL	4.7159, 0.2434, 0.8933, -19.5031	13.509	0.121	0.348	35.019	36.072	43.591
$SE(\widehat{\xi})$	(1.0349, 0.1968, 0.2756, 37.2955)						
OWN	4.2334, 0.6283, 0.4739, -1.1735	11.859	0.099	0.613	31.720	32.773	40.292
$SE(\widehat{\xi})$	(0.8850, 0.5539, 0.2199, 0.5532)						
OWB	4.5457, 0.7867, 0.2832, -0.8372	11.725	0.096	0.642	31.450	32.503	40.023
$SE(\widehat{\xi})$	(0.7437, 0.5577, 0.1221, 0.1667)						

TABLE 4
Estimates and goodness-of-fit measures for the second data set.

Model	$\widehat{\xi}$	$-\ell(\widehat{\xi})$	K-S	p-value	AIC	AIC_c	BIC
OW	2.3569, 0.0021, 0.2612	102.166	0.399	0.003	210.332	213.189	214.106
$SE(\widehat{\xi})$	(0.1763, 0.0008, 0.1296)						
BW	2.8460, 0.0021, 0.1813, 0.0285	96.722	0.219	0.278	201.544	204.544	206.476
$SE(\widehat{\xi})$	(0.4389, 0.0004, 0.1482, 0.0433)						
BGE	8.4938, 0.0173, 0.1583, 202.2628	101.603	0.299	0.050	211.206	214.206	216.238
$SE(\widehat{\xi})$	(0.0097, 4.8377, 0.0867, 245.0212)						
OWP	2.2870, 0.0038, 0.1994, -3.0296	94.589	0.169	0.596	197.178	200.178	202.210
$SE(\widehat{\xi})$	(0.1935, 0.0016, 0.1053, 0.9890)						
OWG	2.3142, 0.0048, 0.2028, -8.7458	93.141	0.141	0.803	194.282	197.282	199.314
$SE(\widehat{\xi})$	(0.2026, 0.0021, 0.1107, 5.5701)						
OWL	2.4275, 0.0027, 0.1952, -10.0368	94.488	0.179	0.523	196.976	199.976	202.008
$SE(\widehat{\xi})$	(0.2337, 0.0041, 0.0664, 0.1436)						
OWN	2.3867, 0.0046, 0.1341, -0.9751	93.732	0.145	0.789	195.464	198.464	200.496
$SE(\widehat{\xi})$	(0.2014, 0.0021, 0.0730, 0.3269)						
OWB	2.2758, 0.0064, 0.1271, -0.6981	94.468	0.154	0.707	196.936	199.936	201.968
$SE(\widehat{\xi})$	(0.2337, 0.0041, 0.0664, 0.1436)						

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SUMMARY

A new family of continuous distributions obtained by compounding the odd log-logistic and power series distributions is introduced. The mathematical properties of the proposed family are discussed. The estimation of the parameters is considered by the maximum likelihood method. In order to assess the finite sample performance of maximum likelihood estimators, simulation studies are performed. Finally, the potentiality of the family is illustrated by means of applications to two real data sets.

Keywords: Estimation; Odd log-logistic family of distributions; Power series distribution; Sensitivity analysis.