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BIVARIATE QUANTILE FUNCTIONS AND THEIR APPLICATIONS TO RELIABILITY MODELLING

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1. Introduction

Quantile functions are alternatives to distribution functions in specifying probability distributions and can be successfully employed in all forms of statistical analysis. Although one can be mathematically derived from the other, quantile functions, which are simple in form and very much flexible, can be generated without reference to the distribution function. This gives a variety of quantile functions to model random phenomenon in addition to distribution functions. For the relative advantages of quantile function, its adaptability to modelling and methods of generating quantile functions we refer to Gilchrist (2000) and Nair *et al.* (2013). There has been a spurt in interest in recent times to utilize quantile functions and concepts derived from it in modelling and analysis of lifetime data, and in information theory, see for example Nair and Sankaran (2009), Nair and Vineshkumar (2010, 2011), Franco-Pereira *et al.* (2012), Soni and Dewan (2012), Nair *et al.* (2013), Lin *et al.* (2016), Kumar and Rani (2018), Kayal and Tripathy (2018), Sadeghi *et al.* (2019) and their references.

Some attempts have been made in the literature to extend the concept of quantile functions to higher dimensions, as can be seen from the works of Chen and Welsh (2002), Serfling (2002), Belzunce *et al.* (2007) and Cai (2010). However, none of these approaches appear to have been utilized in the context of reliability analysis. Moreover, extenstion of univariate quantiles to represent bivariate life distributions does not seem to have been discussed in the literature. The present work is, therefore, an attempt to discuss a definition of bivariate quantile function appropriate to analyze bivariate lifetime data and to derive some basic results in this connection. This is motivated by extending to the bivariate case the advantages of the univariate quantile-based approach namely, the benefits of alternative methodology, new simple and flexible models, certain new results

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that are difficult to find through the distribution function approach, more robust estimation procedures and sometimes better insight into the data generating mechanism. In the process, it is conceived that for a nonnegative random vector (X_1, X_2) , the bivariate quantile function is a pair that transforms the unit square $[0,1]^2$, to points in the plane $x_1 - x_2$, where (x_1, x_2) stands for the realization of (X_1, X_2) . By doing so we ensure that the bivariate results are consistent with quantile-based reliability concepts and results in the univariate case, and also the basic notions in the existing distribution function approach. Based on the proposed bivariate quantile function, we define the bivariate hazard and mean residual quantile functions, and establish some of their properties. Some new flexible quantile functions are suggested and a real data set is modeled with one of them.

The paper is organized into seven sections. In Section 2, some basic definitions and results in univariate case needed for sequel are presented. The definition of bivariate quantile function and some examples form the material in Section 3. This is followed in Section 4 by the discussions on the concepts of bivariate hazard and mean residual quantile functions. Section 5 is devoted to the derivation of new models using special forms bivariate hazard and mean residual quantile functions. In Section 6, the application of the results to real data set is considered, and the work is concluded in Section 7.

2. Basic results in the univariate case

Let X be a nonnegative random variable with absolutely continuous distribution function $F(x)$, quantile function

$$Q(u) = \inf\{x \,|F(x) \geq u\}, \quad 0 \leq u \leq 1$$

and the quantile density function $q(u) = \frac{dQ(u)}{du}$. If $f(x)$ is the probability density function of X we have the relationship $f(Q(u)) = [q(u)]^{-1}$. Analogous to the hazard rate of the random variable X, defined as

$$h(x) = \frac{f(x)}{1 - F(x)},$$

in the quantile function approach we have the hazard quantile function

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1},$$

which determines $Q(u)$ uniquely as

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}.$$

Similarly, corresponding to the mean residual function

$$m(x) = \frac{1}{1-F(x)} \int_x^\infty (t-x)f(t)dt,$$

the mean residual quantile function is defined as

$$M(u) = m(Q(u)) = (1-u)^{-1} \int_u^1 (1-p) q(p) dp.$$

Further

$$[H(u)]^{-1} = M(u) - (1-u) \frac{dM(u)}{du}$$

and

$$Q(u) = \int_0^u \frac{M(p) - (1-p)\frac{dM(p)}{dp}}{1-p} dp.$$

Let X and Y be two lifetimes with hazard quantile functions $H_X(.)$ and $H_Y(.)$, and mean residual quantile functions $M_X(.)$ and $M_Y(.)$, respectively. When X and Y are to be compared, we say that X is smaller than Y

1. in hazard quantile function order, written as $X \leq_{HQ} Y$ if $H_X(u) \geq H_Y(u)$ for $0 \leq u \leq 1$, and
2. in mean residual quantile function order , denoted by $X \leq_{MRQ} Y$ if $M_X(u) \leq M_Y(u)$ for $0 \leq u \leq 1$.

Mathematically 1 and 2 are equivalent respective to the dispersive order and excess wealth order, respectively, although the two have different interpretations and reliability properties. For details see Vineshkumar *et al.* (2015). It may be noted that the hazard (mean residual) quantile orders mentioned above are different from the usual hazard rate (mean residual life) orders. For detailed study of the concepts and definitions given in this section we refer to Nair *et al.* (2013) and Vineshkumar *et al.* (2015).

3. BIVARIATE QUANTILE FUNCTIONS

Let (X_1, X_2) be a nonnegative random vector with absolutely continuous distribution function $F(x_1, x_2)$, survival function $\bar{F}(x_1, x_2)$ and probability density function $f(x_1, x_2)$. We denote by $F_i(x_i)$ ($\bar{F}_i(x_i)$) the marginal distribution (survival) function of X_i, $i = 1, 2$. The quantile function of X_i is

$$Q_i(u_i) = \inf\{x_i \,|F_i(x_i) \geq u_i\}, 0 \leq u_i \leq 1, i = 1, 2. \tag{1}$$

In defining bivariate quantile functions we seek transformations of points $[0, 1]^2$ in the unit square representing the $u_1 - u_2$ plane to points in the $x_1 - x_2$ plane. Our approach, which is slightly different from the existing ones, consists in choosing the quantile functions of $P(X_1 > x_1)$ and $P(X_2 > x_2 | X_1 > x_1)$ that makes up the joint distribution function through

$$\bar{F}(x_1, x_2) = P(X_1 > x_1) P(X_2 > x_2 | X_1 > x_1).$$

The advantage of this approach is that the terms on the right are more convenient to define various reliability functions.

DEFINITION 1. *The bivariate quantile function of* (X_1, X_2) *is defined as the pair* $(Q_1(u_1), Q_{21}(u_2 \mid u_1))$*, where* $Q_1(u)$ *is as given in (1) and*

$$Q_{21}(u_2|u_1) = \inf\{x_2 | P(X_2 \le x_2 | X_1 > Q_1(u_1)) \ge u_2\}. \tag{2}$$

EXAMPLE 2. *For the Gumbel's bivariate exponential distribution*

$$\bar{F}(x_1, x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2],\ x_1, x_2 > 0;\ \lambda_1, \lambda_2 > 0,\ 0 \le \theta \le \lambda_1 \lambda_2$$

$F_1(x_1) = 1 - \exp[-\lambda_1 x_1]$ *gives* $Q_1(u_1) = \frac{-\log(1-u_1)}{\lambda_1}$*. Also*

$$P(X_2 > x_2 | X_1 > x_1) = e^{-(\lambda_2 + \theta x_1)x_2}$$

has quantile function

$$Q_{21}(u_2|u_1) = \frac{-\lambda_1 \log(1-u_2)}{\lambda_1 \lambda_2 - \theta \log(1-u_1)}.$$

EXAMPLE 3. *Consider the bivariate Pareto distribution with survival function*

$$\bar{F}(x_1, x_2) = (1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-p},\ x_1, x_2 > 0,\ a_1, a_2 > 0,\ 0 \le b \le (p-1)a_1 a_2.$$

The marginal distribution function of X_1 *is*

$$F_1(x_1) = 1 - (1 + a_1 x_1)^{-p},$$

which yields

$$Q_1(u_1) = \frac{1}{a_1}(1-u_1)^{\frac{-1}{p}} - 1.$$

From the conditional distribution

$$P[X_2 > x_2 | X_1 > x_1] = \frac{(1 + a_1 x_1 + a_2 x_2 + b x_1 x_2)^{-p}}{(1 + a_1 x_1)^{-p}},$$

we can find the quantile function $Q_{21}(u_2 \mid u_1)$ *as*

$$Q_{21}(u_2|u_1) = \frac{a_1 (1-u_1)^{-\frac{1}{p}} \left((1-u_2)^{-\frac{1}{p}} - 1\right)}{a_1 a_2 + b\left((1-u_1)^{-\frac{1}{p}} - 1\right)}.$$

More examples are given in Table 1.

If needed the distribution function of (X_1, X_2) can be recovered from the quantile functions Q_1 and Q_{21} using

$$F_1(x_1) = \inf[u_1 \,|\, Q_1(u_1) \geq x_1]$$

and

$$P(X_2 \leq x_2 \,|\, X_1 > x_1) = \inf[u_2 \,|\, Q_{21}(u_2 \,|\, u_1) > x_2].$$

By way of illustration, from Example 2, since Q_1 and Q_{21} are strictly increasing, from $Q_1(u_1)$, $F_1(x_1) = 1 - \exp[-\lambda_1 x_1]$ and $P(X_2 \leq x_2 \,|\, X_1 > x_1) = 1 - \exp(-\lambda_2 x_2 - \theta x_1 x_2)$ after setting $Q_1(u_1) = x_1$ and $Q_{21}(u_2 \,|\, u_1) = x_2$. The joint survival function now follows.

4. Basic reliability functions

4.1. *Bivariate hazard quantile function*

Based on the definitions given in Section 2, the bivariate hazard quantile function of (X_1, X_2) is defined as the pair $(H_1(u_1), H_{21}(u_1, u_2))$, where

$$H_1(u_1) = [(1-u_1)q_1(u_1)]^{-1}, \;\; H_{21}(u_1, u_2) = [(1-u_2)q_{21}(u_2 \,|\, u_1)]^{-1}, \tag{3}$$

in which $q_1(u_1) = \frac{dQ_1(u_1)}{du_1}$ and $q_{21}(u_2 \,|\, u_1) = \frac{dQ_{21}(u_2|u_1)}{du_2}$ are the quantile density functions of Q_1 and Q_{21}, respectively. It is further observed that the distribution of (X_1, X_2) is uniquely determined by (H_1, H_{21}) through the equations

$$Q_1(u_1) = \int_0^{u_1} \frac{dp}{(1-p)H_1(p)} \tag{4}$$

and

$$Q_{21}(u_2 \,|\, u_1) = \int_0^{u_2} \frac{dp}{(1-p)H_{21}(u_1, p)}. \tag{5}$$

In the case of Gumbel's bivariate exponential distribution considered above $H_1(u_1) = \lambda_1$ and $H_{21}(u_1, u_2) = \lambda_2 - \frac{\theta}{\lambda_1}\log(1-u_1)$ and it is easy to recover Q_1 and Q_{21} from (4) and (5) as obtained in Example 2. See Table 2 for expressions of $(H_1(u_1), H_{21}(u_1, u_2))$ of distributions in Table 1.

Remark 4. *The quantile function corresponding to $\bar{F}(x_1, x_2)$ can also be defined as $Q_{12}(u_1 \mid u_2)$ and $Q_2(u_2)$, the quantile functions of $P(X_1 > x_1 \,|\, X_2 > x_2)$ and $\bar{F}_2(x_2)$, respectively. In this case, the hazard quantile function is the vector $(H_{12}(u_1, u_2), H_2(u_2))$, with*

$$H_{12}(u_1, u_2) = [(1-u_1)q_{12}(u_1 \,|\, u_2)]^{-1}, \; H_2(u_2) = [(1-u_2)q_2(u_2)]^{-1}, \tag{6}$$

where q_{12} and q_2 are the quantile density functions of Q_{12} and Q_2, respectively.

The hazard function (H_1, H_{21}) can be employed to define the ageing concepts, the increasing (decreasing) hazard quantile function, IHQ (DHQ). We say that (X_1, X_2) is IHQ (DHQ) according as H_1 is increasing (decreasing) in u_1 and H_{21} is increasing (decreasing) in u_2 for all u_1. Since u_1 increases with x_1 and u_2 increases with x_2 , the bivariate IHR (DHR) criterion with respect to the vector hazard rate of Johnson and Kotz (1975), defined as $(a_1(x_1, x_2), a_2(x_1, x_2))$, where $a_i(x_1, x_2) = \frac{-\partial \log \bar{F}(x_1, x_2)}{\partial x_i}$, $i = 1,2$, implies IHQ (DHQ).

There are occasions when one has to compare hazard rates of two devices, for example same devise produced by two manufactures under different processes. If (X_1,X_2) and (Y_1,Y_2) are lifetimes with $(a_1(x_1,x_2),a_2(x_1,x_2))$ and $(b_1(x_1, x_2), b_2(x_1, x_2))$ as respective hazard rates, then Hu *et al.* (2003) proposed that (X_1,X_2) has lesser bivariate hazard rate than (Y_1,Y_2), denoted by $(X_1, X_2) \leq_{whr} (Y_1, Y_2)$ whenever

$$a_i(x_1, x_2) \geq b_i(x_1, x_2),\ i = 1,\ 2;\ x_1,\ x_2 > 0.$$

With reference to the bivariate hazard quantile functions (H_1,H_{21}) and (K_1,K_{21}) of (X_1,X_2) and (Y_1,Y_2), we say that (X_1,X_2) has lesser hazard quantile function than (Y_1,Y_2), written as $(X_1, X_2) \leq_{HQ} (Y_1, Y_2)$ if $H_1(u_1) \geq K_1(u_1)$ for all u_1 and $H_{21}(u_1, u_2) \geq K_{21}(u_1, u_2)$ for $0 \leq u_1, u_2 \leq 1$. Since the univariate ordering $a_i(x_1, 0) \geq b_i(x_1, 0)$ neither implies nor implied by $H_1(u_1) \geq K_1(u_1)$ (Vineshkumar *et al.*, 2015), it follows that the orders $\leq_{whr}$ and $\leq_{HQ}$ are not equivalent and latter defines a different stochastic order. Some additional properties of $\leq_{HQ}$ are given below. If $\bar{G}(x_1, x_2)$ denotes the survival function of (Y_1,Y_2), then (X_1,X_2) is smaller than (Y_1,Y_2) in upper orthant order, $(X_1, X_2) \leq_{uo} (Y_1, Y_2)$ (see Shaked and Shanthikumar, 2007) if $\bar{F}(x_1, x_2) \leq \bar{G}(x_1, x_2)$ for $x_1,x_2 > 0$.

THEOREM 5. *If $X_i, i = 1,2$ have the same lower end of their supports then*

$$(X_1, X_2) \leq_{HQ} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{uo} (Y_1, Y_2)$$

PROOF. Since $(X_1,X_2) \leq_{HQ} (Y_1,Y_2)$, from Vineshkumar *et al.* (2015)

$$H_1(u_1) \geq K_1(u_1) \Rightarrow X_1 \leq_{st} Y_1 \Leftrightarrow \bar{F}_1(x_1) \leq \bar{G}_1(x_1),$$

where $\bar{G}_1$ is the survival function of Y_1. Also

$$H_{21}(u_1, u_2) \geq K_{21}(u_1, u_2) \Rightarrow P[X_2 > x_2 | X_1 > x_1] \leq P[Y_2 > x_2 | Y_1 > x_1].$$

Hence

$$\begin{aligned}(X_1,X_2) \leq_{HQ} (Y_1,Y_2) &\Rightarrow H_1(u_1)H_{21}(u_1,u_2) \geq K_1(u_1)K_{21}(u_1,u_2)\\ &\Rightarrow \bar{F}(x_1, x_2) \leq \bar{G}(x_1, x_2) \Rightarrow (X_1, X_2) \leq_{uo} (Y_1, Y_2).\end{aligned}$$

□

EXAMPLE 6. *Let (X_1,X_2) and (Y_1,Y_2) be random vectors with bivariate quantile functions*

$$\left((1-u_1)^{-\frac{1}{2}}-1,\ (1-u_1)^{-\frac{1}{2}}\left((1-u_2)^{-\frac{1}{2}}-1\right)\right)$$

and

$$\left(\left(\frac{u_1}{1-u_1}\right)^{\frac{1}{2}},\left(\frac{u_2}{(1-u_1)(1-u_2)}\right)^{\frac{1}{2}}\right),$$

respectively with corresponding hazard quantile functions

$$(H_1(u_1),H_{21}(u_1,u_2))=\left(2(1-u_1)^{\frac{1}{2}},\ 2((1-u_1)(1-u_2))^{\frac{1}{2}}\right)$$

and

$$(K_1(u_1),K_{21}(u_1,u_2))=\left(2u_1\left(\frac{1-u_1}{u_1}\right)^{\frac{1}{2}},\ 2u_2\left(\frac{(1-u_1)(1-u_2)}{u_2}\right)^{\frac{1}{2}}\right).$$

It is easy to show that $H_1(u_1)>K_1(u_1)$ for all u_1 and $H_{21}(u_1,u_2)>K_{21}(u_1,u_2)$ for all u_1 and u_2. Therefore, $(X_1,\ X_2)\leq_{HQ}(Y_1,\ Y_2)$. The survival functions of (X_1,X_2) and (Y_1,Y_2) are

$$\bar{F}(x_1,x_2)=(1+x_1+x_2)^{-2},\ x_1,x_2>0$$

and

$$\bar{G}(x_1,x_2)=\left(1+x_1^2+x_2^2\right)^{-1},\ x_1,x_2>0.$$

Since $(1+x_1+x_2)^2>1+x_1^2+x_2^2$ for all $x_1,x_2>0$, $\bar{F}(x_1,x_2)<\bar{G}(x_1,x_2)$ for all $x_1,x_2>0$, which implies $(X_1,\ X_2)\leq_{uo}(Y_1,\ Y_2)$. This illustrates Theorem 5.

Another implication is with the reversed hazard quantile function order. The bivariate reversed hazard quantile function of (X_1,X_2) is defined as the pair $(R_1(u_1),\ R_{21}(u_1,\ u_2))$, where

$$R_1(u_1)=[u_1q_1(u_1)]^{-1}$$

and

$$R_{21}(u_1,\ u_2)=[u_2q_{21}(u_2|u_1)]^{-1}.$$

With similar definitions, let $(T_1(u_1),\ T_{21}(u_1,\ u_2))$ be the corresponding function of (Y_1,Y_2). Then we say that (X_1,X_2) is smaller than (Y_1,Y_2) in reversed hazard quantile function order, written as $(X_1,\ X_2)\leq_{RHQ}(Y_1,\ Y_2)$ if $R_1(u_1)\leq T_1(u_1)$ and $R_{21}(u_1,\ u_2)\leq T_{21}(u_1,\ u_2)$ for all $0<u_1,u_2<1$. It is easy to see that

$$(X_1,\ X_2)\leq_{RHQ}(Y_1,\ Y_2)\Leftrightarrow(X_1,\ X_2)\geq_{HQ}(Y_1,\ Y_2),$$

a result which does not hold between usual bivariate hazard and reversed hazard rates.

TABLE 1

Bivariate quantile function of some lifetime distributions.

Model	$\bar{F}(x_1, x_2)$	$(Q_1(u_1), Q_{21}(u_2 \mid u_1))$
1	$\exp\left[-(x_1^m + x_2^m)^{\frac{1}{m}}\right], x_1, x_2 > 0, m > 0$	$\left(-\log(1-u_1), \{[-\log(1-u_1)(1-u_2)]^m - [-\log(1-u_1)]^m\}^{\frac{1}{m}}\right)$
2	$(1-b_1x_1-b_2x_2+cx_1x_2)^p, 0 < x_1 < \frac{1}{b_1}, 0 \le x_2 \le \frac{1-bx_1}{b_2-cx_1}, b_1, b_2 > 0$	$\left(\frac{1}{b_1}(1-(1-u_1))^{\frac{1}{p}}, \frac{b_1(1-(1-u_2))^{\frac{1}{p}}(1-u_1)^{\frac{1}{p}}}{b_1b_2+b_1c\left(1-(1-u_1)^{\frac{1}{p}}\right)}\right)$
3	$(1+a_1x_1+a_2x_2)^{-p}, x_1, x_2 > 0, a_1, a_2, p > 0$	$\left(\frac{(1-u_1)^{-\frac{1}{p}}-1}{a_1}, \frac{(1-u_1)^{-\frac{1}{p}}\left((1-u_2)^{-\frac{1}{p}}-1\right)}{a_2}\right)$
4	$\left(e^{\lambda_1x_1}+e^{\lambda_2x_2}-1\right)^{-1}, x_1, x_2 > 0; \lambda_1, \lambda_2 > 0$	$\left(-\frac{1}{\lambda_1}\log(1-u_1), \frac{1}{\lambda_2}\log\left(\frac{1-u_1+u_1u_2}{(1-u_1)(1-u_2)}\right)\right)$
5	$\frac{\beta}{\beta+x_1^k+x_2^k}, x_1, x_2 > 0, \beta > 0, k > 0$	$\left(\beta\left(\frac{u_1}{1-u_1}\right)^{\frac{1}{k}}, \left(\frac{\beta u_2}{(1-u_1)(1-u_2)}\right)^{\frac{1}{k}}\right)$

TABLE 2
Bivariate hazard quantile functions of lifetime distributions in Table 1.

Model	$(H_1,\ H_{21})$
1	$\left(1,\ [(-\log(1-u_1)(1-u_2))^m-(-\log(1-u_1))^m]^{1-\frac{1}{m}}(-\log(1-u_1)(1-u_2))^{1-m}\right)$
2	$\left(pb_1(1-u_1)^{\frac{-1}{p}},\ p\left(b_2+c(1-(1-u_1)^{\frac{1}{p}}\right)(1-u_1)^{\frac{-1}{p}}(1-u_2)^{\frac{-1}{p}}\right)$
3	$\left(pa_1(1-u_1)^{\frac{1}{p}},\ pa_2(1-u_1)^{\frac{1}{p}}(1-u_2)^{\frac{1}{p}}\right)$
4	$(\lambda_1,\ \lambda_2(1-u_1+u_1u_2))$
5	$\left(ku_1\left(\frac{1-u_1}{\beta u_1}\right)^{\frac{1}{k}},\ ku_2\left(\frac{(1-u_1)(1-u_2)}{\beta u_2}\right)^{\frac{1}{k}}\right)$

4.2. Bivariate mean residual quantile function

The bivariate mean residual quantile function of (X_1, X_2) is defined as the vector $(M_1(u_1),\ M_{21}(u_1,\ u_2))$, where

$$M_1(u_1)=\frac{1}{1-u_1}\int_{u_1}^{1}(1-p)q_1(p)dp \tag{7}$$

and

$$M_{21}(u_1,\ u_2)=\frac{1}{1-u_2}\int_{u_2}^{1}(1-p)q_{21}(p\,|u_1)dp. \tag{8}$$

For example, the Gumbel distribution considered above has

$$M_1(u_1)=\lambda_1^{-1}$$

and

$$M_{21}(u_1,\ u_2)=\left(\lambda_2-\frac{\theta}{\lambda_1}\log(1-u_1)\right)^{-1}.$$

The bivariate hazard quantile function is related to (7) and (8) as

$$[H_1(u_1)]^{-1}=M_1(u_1)-(1-u_1)\frac{dM_1(u_1)}{du_1}$$

and

$$[H_{21}(u_1,\ u_2)]^{-1}=M_{21}(u_1,\ u_2)-(1-u_2)\frac{dM_{21}(u_1,\ u_2)}{du_2},\ \ 0\le u_1\le 1.$$

Further the distribution of (X_1, X_2) is recovered from $(M_1(u_1),\ M_{21}(u_1,\ u_2))$ through the quantile functions

$$Q_1(u_1)=\int_0^{u_1}\frac{M_1(p)-(1-p)\frac{dM_1(p)}{dp}}{1-p}dp$$

and

$$Q_{21}(u_2 | u_1) = \int_0^{u_1} \frac{M_{21}(u_1, p) - (1-p)\frac{dM_{21}(u_1,p))}{dp}}{1-p} dp.$$

As in the case of hazard quantile function, (X_1, X_2) has decreasing (increasing) mean residual life, DMRQ (IMRQ), if $M_1(u_1)$ is decreasing (increasing) in u_1 and $M_{21}(u_1, u_2)$ is decreasing in u_2 for all u_1, and is equivalent to the usual notion of bivariate DMRL (IMRL) using the distribution functions. Recall that the bivariate mean residual life function in the latter case is $(m_1(x_1, x_2), m_2(x_1, x_2))$ where

$$m_i(x_1, x_2) = E(X_i - x_i | X_1 > x_1, X_2 > x_2).$$

Similarly, if $(n_1(x_1, x_2), n_2(x_1, x_2))$ is the mean residual life function of (X_1, X_2), then (X_1, X_2) is smaller than (Y_1, Y_2) in mean residual life order, $(X_1, X_2) \leq_{mrl} (Y_1, Y_2)$, if $m_i(x_1, x_2) \leq n_i(x_1, x_2)$, $x_1, x_2 \geq 0$.

We can also define a stochastic order among (X_1, X_2) and (Y_1, Y_2) by saying that the former is smaller than the latter in mean residual quantile function order denoted by $(X_1, X_2) \leq_{MRQ} (Y_1, Y_2)$ if $M_1(u_1) \leq N_1(u_1)$ for all u_1 and $M_{21}(u_1, u_2) \leq N_{21}(u_1, u_2)$ for $0 \leq u_1, u_2 \leq 1$, where (N_1, N_{21}) is the mean residual quantile function of (Y_1, Y_2). Vineshkumar *et al.* (2015) have shown that $m_1(x_1, 0) \leq n_1(x_1, 0)$ does not imply $M_1(u_1) \leq N_1(u_1)$, and therefore, $(X_1, X_2) \leq_{mrl} (Y_1, Y_2)$ does not imply $(X_1, X_2) \leq_{MRQ} (Y_1, Y_2)$.

THEOREM 7.

$$(X_1, X_2) \leq_{HQ} (Y_1, Y_2) \Rightarrow (X_1, X_2) \leq_{MRQ} (Y_1, Y_2).$$

PROOF. $(X_1, X_2) \leq_{HQ} (Y_1, Y_2)$ implies $H_1(u_1) \geq K_1(u_1)$ and also $H_{21}(u_1, u_2) \geq K_{21}(u_1, u_2)$. Therefore, $(1-u_1)q_1(u_1) \leq (1-u_1)s_1(u_1)$ and $(1-u_2)q_{21}(u_2 | u_1) \leq (1-u_2)s_{21}(u_2 | u_1)$, where s_1 and s_{21} are the quantile density functions of Y_1 and $(Y_2 | Y_1 > x_1)$. Thus,

$$\frac{1}{1-u_1}\int_{u_1}^{1}(1-p)q_1(p)dp \leq \frac{1}{1-u_1}\int_{u_1}^{1}(1-p)s_1(p)dp$$

and

$$\frac{1}{1-u_2}\int_{u_2}^{1}(1-p)q_{21}(p | u_1)dp \leq \frac{1}{1-u_2}\int_{u_2}^{1}(1-p)s_{21}(p | u_1)dp,$$

which implies

$$(X_1, X_2) \leq_{MRQ} (Y_1, Y_2).$$

□

EXAMPLE 8. *For the random vectors defined in Example 6, it is shown that* $(X_1, X_2) \leq_{HQ} (Y_1, Y_2)$. *Now,*

$$M_1(u_1) = (1-u_1)^{-1} \int_{u_1}^{1} (H_1(t))^{-1} dt = (1-u_1)^{-1} \int_{u_1}^{1} \frac{dt}{2(1-t)^{\frac{1}{2}}}$$

$$< (1-u_1)^{-1} \int_{u_1}^{1} \frac{dt}{2(t(1-t))^{\frac{1}{2}}} = N_1(u_1)$$

and

$$M_{21}(u_1, u_2) = (1-u_2)^{-1} \int_{u_2}^{1} (H_{21}(u_1, t))^{-1} dt = (1-u_2)^{-1} \int_{u_2}^{1} \frac{dt}{2((1-u_1)(1-t))^{\frac{1}{2}}}$$

$$< (1-u_2)^{-1} \int_{u_2}^{1} \frac{dt}{2((1-u_1)t(1-t))^{\frac{1}{2}}} = N_{21}(u_1, u_2),$$

which implies that $(X_1, X_2) \leq_{MRQ} (Y_1, Y_2)$. *This verifies Theorem 7.*

The following result is useful from a modelling perspective, as it permits the analyst to commence with a simple functional form for the mean residual quantile function and then to improve it by adding appropriate forms until the desired accuracy is reached without changing the model and the inference procedures associated with it sequentially.

THEOREM 9. *The sum of the mean residual quantile functions of two nonnegative random vectors* (X_1, X_2) *and* (Y_1, Y_2) *is again the mean residual quantile function of a bivariate random vector if and only if its quantile function has components as the sum of the quantile functions of* (X_1, X_2) *and* (Y_1, Y_2).

PROOF. We first observe that the sum of two quantile (quantile density) functions is again a quantile (quantile density) function. Assume first that the quantile functions of (X_1, X_2) and (Y_1, Y_2) are the vectors $(Q_1(u_1),\ Q_{21}(u_2|u_1))$ and $(S_1(u_1), S_{21}(u_2|u_1))$, respectively. Further let, $Q^*(u_1) = Q_1(u_1) + S_1(u_1)$ and $Q^*_{21}(u_2|u_1) = Q_{21}(u_2|u_1) + S_{21}(u_2|u_1)$. Then

$$\frac{1}{1-u_1} \int_{u_1}^{1} (1-p) q_1^*(p) dp = \frac{1}{1-u_1} \int_{u_1}^{1} (1-p) q_1(p) dp + \frac{1}{1-u_1} \int_{u_1}^{1} (1-p) s_1(p) dp \quad (9)$$

$$= M_1(u_1) + N_1(u_1), \ s_1 = \frac{dS_1}{dp}.$$

The left side is a mean residual quantile function since q^* is a quantile density function. By the same argument $M_{21}(u_1, u_2) + N_{21}(u_1, u_2)$ is mean residual quantile function. Conversely if (9) is true for some q^*, then differentiation gives $q^*(u_1) = q_1(u_1) + s_1(u_1)$ and similarly, $q^*_{21}(u_2|u_1) = q_{21}(u_2|u_1) + s_{21}(u_2|u_1)$. This completes the proof. □

EXAMPLE 10. *The mean residal quantile function of the bivariate qantile function given as Model 5 of Table 1 is the vector*

$$\left(\frac{1}{\lambda_1}, \frac{-\log(1-u_1+u_1u_2)}{\lambda_2 u_1(1-u_2)}\right),$$

whose marginal mean residual quantile functions are constants. Therefore the distribution has limited application in analyzing bivarate data with monotonic mean residual functions, which are common in real life situations. In view of Theorem 9, one can improve this mean residual function to expalin more variety of bivariate data by adding one or more mean residual quantile functions. For instance, we obtain a new mean residual quantile function from the above as

$$\left(\tfrac{1}{\lambda_1}+(1-u_1)^{-\frac{1}{p}}, \tfrac{-\log(1-u_1+u_1u_2)}{\lambda_2 u_1(1-u_2)}+((1-u_1)(1-u_2))^{-\frac{1}{p}}\right),\ p>0\ ,$$

which has nondecreasing marginal mean residual quantile functions, and is capable of expalining residual lifes of wide range of data sets. Notice that the quantile function of the added mean residual function is provided as Model 4 of Table 1. We now have a new bivariate quantile function with

$$Q_1(u_1) = \frac{-1}{\lambda_1}\log(1-u_1)+(1-u_1)^{-\frac{1}{p}}-1$$

and

$$Q_{21}(u_2|u_1) = \frac{1}{\lambda_2}\log\left(\frac{1-u_1+u_1u_2}{(1-u_1)(1-u_2)}\right)+(1-u_1)^{-\frac{1}{p}}\left((1-u_2)^{-\frac{1}{p}}-1\right),$$

with $\lambda_1, \lambda_2, p > 0$*. The distributions obtained by this method may have properties different from the individual quantile functions, which are to be further exposed.*

5. NEW QUANTILE FUNCTION MODELS

Most of the quantile function models in the univariate case (except those obtained by inverting standard distribution functions) have the property of assuming simple mathematical form and the ability to subsume many useful distributions either exactly or approximately. This renders such quantile functions to represent a wide variety of data situations, there by contributing to their roles in modelling problems. For details see Nair *et al.* (2013). In the present section we propose some models that are generalizations of univariate models of the above nature.

5.1. *Bivariate linear hazard quantile function model*

We assume that (X_1, X_2) has a bivariate hazard quantile function of the form

$$(H_1(u_1), H_{21}(u_2|u_1)) = (a + b u_1,\ a + b u_1 + c u_2),$$

$a > 0$, $a + b > 0$, $a + c > 0$, $a + b + c > 0$. Notice that the marginal hazard quantile functions of X_1 and X_2 are respectively $H_1(u_1) = a + b u_1$ and $H_2(u_2) = a + c u_2$ giving the quantile functions

$$Q_1(u_1) = \frac{1}{a+b} \log\left(\frac{a + b u_1}{a(1-u_1)}\right) \tag{10}$$

and

$$Q_2(u_2) = \frac{1}{a+c} \log\left(\frac{a + c u_2}{a(1-u_2)}\right).$$

Also

$$Q_{21}(u_2|u_1) = \frac{1}{a + b u_1 + c} \log\left(\frac{a + b u_1 + c u_2}{(a + b u_1)(1-u_2)}\right). \tag{11}$$

From (10) and (11) we find after some algebra, the joint survival function of (X_1, X_2) in closed form as

$$\bar{F}(x_1,\ x_2) = \frac{(a + b u_1 + c)(1-u_1) e^{-(a+bu_1+c)x_2}}{(a + b u_1) + c e^{-(a+bu_1+c)x_2}}, \tag{12}$$

where u_1 is replaced by

$$u_1 = \frac{a\left(1 - e^{-(a+b)x_1}\right)}{a + b e^{-(a+b)x_1}},\ x_1,\ x_2 > 0.$$

Though in closed form, the distribution function is more difficult to work with than the quantile function $(Q_1(u_1), Q_{21}(u_2|u_1))$ obtained in (10) and (11). The marginal distributions of X_1 and X_2 are the linear hazard quantile function distributions discussed in Nair *et al.* (2013) and Midhu *et al.* (2014). There are several special cases of the above bivariate model.

1. When $b = c = 0$ it becomes the bivariate exponential distribution with independent exponential marginals with parameter a .

2. Taking $a = b = c > 0$, $Q_i(u_i) = \frac{1}{2a} \log \frac{1-u_i}{1+u_i}$, the quantile function of half-logistic distribution. Thus (10) and (11) give a bivariate half-logistic distribution with survival function (12) and $u_i = \frac{1-e^{-2ax_i}}{1+e^{-2ax_i}},\ i = 1,\ 2;\ x_i > 0.$

3. If we set $a = \frac{\lambda}{1-p}$, $b = c = \frac{-p\lambda}{1-p}$, $0 < p < 1$; $\lambda > 0$,

$$Q_i(u_i) = \frac{1}{\lambda} \log \frac{1-pu_i}{1-u_i}, \ i = 1, \ 2.$$

and

$$Q_{21}(u_2|u_1) = \frac{1-p}{\lambda(1-p-pu_1)} \log \frac{1-pu_1-pu_2}{(1-pu_1)(1-u_2)},$$

showing that Q_i represents the exponential geometric distribution of Adamidis and Loukas (1998). Thus the distribution is bivariate exponential geometric distribution with parameters λ, p and a with marginals

$$u_i = (1-p)e^{-\lambda x_i}\left(1-pe^{-\lambda x_i}\right)^{-1}, \ x_i > 0.$$

Differentiating $Q_{21}(u_2|u_1)$,

$$q_{21}(u_2|u_1) = \frac{1-p}{\lambda(1-pu_1-pu_2)(1-pu_1)(1-u_2)}.$$

Thus the mean residual quantile function has a closed form with components

$$\left(\frac{1-p}{\lambda p(1-pu_1)} \log \frac{1-pu_1}{1-p}, \ \frac{1-p}{\lambda p(1-pu_1)} \log \frac{1-pu_1-pu_2}{1-p-pu_1}\right),$$

using (7) and (8).

Other reliability aspects, distribution theory and applications will be discussed in a separate work.

5.2. Bivariate linear mean residual quantile function model

Similar to the previous distribution, here we assume a linear form for the bivariate mean residual quantile function and propose the bivariate distribution corresponding to it. Our assumption is

$$(M_1(u_1), \ M_{21}(u_2|u_1)) = (a_1 + b_1u_1, \ a_2 + b_2u_1 + cu_2 + du_1u_2). \tag{13}$$

Direct calculations from the preceding formulas yield

$$Q_1(u_1) = -(a_1 + b_1)\log(1-u_1) - 2b_1u_1, \tag{14}$$

$0 \leq u_1 \leq 1$, $a_1 > 0$, $(a_1 + b_1) > 0$ and

$$Q_{21}(u_2|u_1) = -(a_2 + c + (b_2 + d)u_1)\log(1-u_2) - 2(c + du_1)u_2, \tag{15}$$

$0 \le u_2 \le 1$, $a_2 + c > 0$, $a_2 > 0$, $a_2 + b_2 \ge c + d$. The quantile function of X_2 is obtained from (15) as

$$Q_2(u_2) = -(a_2 + c)\log(1 - u_2) - 2cu_2. \tag{16}$$

Equations (14) and (16) represent the linear mean residual quantile distribution studied in Midhu *et al.* (2013). Neither marginals of X_1 and X_2 nor the joint distribution of (X_1, X_2) have a closed form distribution function to study theoretically the reliability aspects. In view of this there are not many special cases of the model that give standard bivariate distributions except exponential and uniform. When $b_1 = c = d = 0$, we have

$$\bar{F}(x_1,\ x_2) = \exp\left(-\frac{x_1}{a_1} - \frac{x_2}{a_2 + b_2(1 - e^{-x_1})}\right),\ \ x_1,\ x_2 > 0,\ a_1,\ a_2,\ b_2 \ge 0,$$

with exponential marginals having means a_1 and a_2 . Further when $b_2 = 0$ the marginals become independent. Bivariate uniform distribution results when $a_1 > 0$, $a_2 > 0$, $a_1 + b_1 = 0$, $a_1 + c = 0$, $b_2 + d = 0$, $b_2 > 0$.

The quantile density functions

$$q_1(u_1) = (a_1 + b_1)(1 - u_1)^{-1} - 2b_1$$

and

$$q_{21}(u_2 | u_1) = \frac{a_2 + c + (b_2 + d)u_1}{1 - u_2} - 2(c + du_1)$$

provide us the hazard quantile functions

$$H_1(u_1) = (a_1 - b_1 + 2b_1 u_1)^{-1}$$

and

$$H_{21}(u_1,\ u_2) = (a_2 - c + (b_2 - d)u_1 + 2u_2(c + du_1))^{-1}.$$

Notice that H_1 is reciprocal linear and H_{21} is reciprocal bilinear, giving simple forms.

Similar considerations can provide other models as well. For example a wider class of distributions which includes Weibull, Pareto, beta, etc. can be generated if we consider $H_1(u_1) = (1 - u_1)^{\alpha - 1}(-\log(1 - u_1))^{\beta}$ and $H_{21}(u_1,\ u_2) = (1 - u_1)^{\alpha} {u_2}^{\theta}(1 - u_2)^{\phi}$, where α, β, θ are such that H_1 and H_{21} are well defined.

6. Modelling lifetime data

In this section we demonstrate the usefulness of the bivariate quantile function approach and various results derived therefrom in the analysis of lifetime data. The data is on the distribution of the survival times of incubation of individuals known to have sexually transmitted diseases who were later determined to have had sex with an individual possessing the disease verified in a clinic after the time of their encounter (Klein and Moeschberger, 1997, p. 146). The observations were recorded in a 42 month period for 25 individuals as the time in months from the first encounter X_1 and the time in months from the first encounter till the confirmation of disease X_2.

We made an attempt to fit the data to bivariate linear mean residual quantile function model discussed in the last section. We consider the estimation of the parameters by the method of L-moments. The merits of this method over the usual method of moments, maximum likelihood estimation, etc. are well documented in Hosking and Wallis (1997). The first two L-moments of X_1 are

$$L_{1,(X_1)} = E(X_1) = \int_0^1 Q_1(u_1)du_1 = a_1$$

and

$$L_{2,(X_1)} = \int_0^1 (2u_1 - 1)Q_1(u_1)du_1 = \frac{1}{6}(3a_1 + b_1).$$

Similarly,

$$L_{1,(X_2)} = a_2,\ L_{2,(X_2)} = \frac{1}{6}(3a_2 + c),$$

$$L_{1,(X_2|X_1)} = a_2 + b_2 u_1$$

and

$$L_{2,(X_2|X_1)} = \frac{1}{6}(3a_2 + c + (3b_2 + d)u_1).$$

In general terms, the method of L-moments consisting of solving for the parameters from the equations

$$L_r = l_r,\ \ r = 1, 2, \ldots$$

where l_r is the r^{th} sample L-moment, which has the formula

$$l_r = \frac{1}{n}\sum_{j=0}^{r-1} p_{rj}\left(\sum_{r=1}^{n} \frac{(r-1)_{(j)}}{(n-1)_{(j)}}\right)$$

with $p_{ij} = \frac{(-1)^{i-1-j}(i+j-1)!}{(j!)^2(i-j-1)!}$ and n, the sample size.

In the case of conditional moments where u_1 is given, we choose u_1 to be $\hat{u}_1$ obtained from $Q_1(\hat{u}_1) = x_{1(1)}$, where $x_{1(r)}$ is the r^{th} order statistic of the $x_{1,i}$ values in the sample $(x_{1,i}, x_{2,i})$, $i = 1, 2, \ldots, n$. The choice of $x_{1(1)}$ is motivated by the fact that we can make use of the maximum number of sample values of X_2 while considering the event $X_1 > x_1$. In this way, the method gives the estimates of the parameters as

$$\hat{a}_1 = l_{1,(X_1)},\ \hat{b}_1 = 6l_{2,(X_1)} - 3l_{1,(X_1)},\ \hat{a}_2 = l_{1,(X_2)},\ \hat{c} = 6l_{2,(X_2)} - 3l_{1,(X_2)}$$

$$\hat{b}_2 = \frac{1}{\hat{u}_1}\left(l_{1,(X_2|X_1)} - l_{1,(X_2)}\right),\ \hat{d} = \frac{1}{\hat{u}_1}\left(6(l_{2,(X_2|X_1)} - l_{2,(X_2)}) - 3(l_{1,(X_2|X_1)} - l_{1,(X_2)})\right)$$

It is straightforward to find the estimates of a_1, b_1, a_2 and c as $\hat{a}_1 = 15.64$, $\hat{b}_1 = -13.3$, $\hat{a}_2 = 1.132$, $\hat{c} = -1.063$. Since $x_{(1)} = 2$, to find $\hat{u}_1$ we solve

$$-(\hat{a}_1 + \hat{b}_1)\log(1 - \hat{u}_1) - 2\hat{b}_1\hat{u}_1 = 2$$

to find $\hat{u}_1 = 0.06890$. From this $\hat{b}_2 = -0.08344$, $\hat{d} = 0.829881$.

The goodness of fit for the proposed distributions of X_1 and X_2 were ascertained by Q-Q plots, which have been obtained by plotting the points $\left(x_{1(r)},\ Q_1(u_r)\right)$ (Figure 1) and $\left(x_{2(r)},\ Q_{21}(u_r|\hat{u}_1)\right)$ (Figure 2), where $u_r = \frac{r-0.5}{n}$, $r = 1, 2, ..., 25$. The value $\hat{u}_1$ can be replaced by any $\hat{u}_i$, which satisfies $Q_1(\hat{u}_i) = x_{1(i)}$ to draw Figure 2. The figures indicate that the model explains the data satisfactorily.

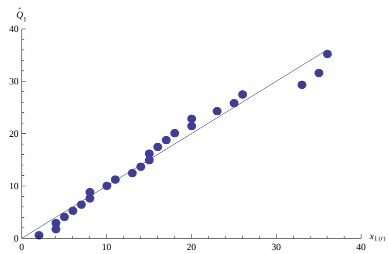

Figure 1 – Q-Q Plot obtained by plotting $(x_{1(r)}, Q_1(u_r))$.

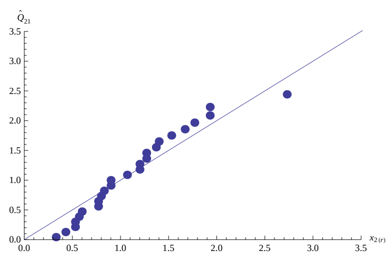

Figure 2 – Q-Q Plot obtained by plotting $(x_{2(r)}, Q_{21}(u_r)|\hat{u}_1)$.

7. Conclusion

In the present work we have suggested an alternative methodology for analysing bivariate lifetime data through quantile functions. The basic reliability functions are defined in terms of bivariate quantile functions and their properties are studied. We have illustrated how new quantile functions can be generated and used in real life data.

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SUMMARY

In this paper we propose a new definition of bivariate quantile function suited for reliability modelling and illustrate its applications. The bivariate hazard and mean residual quantile functions are defined and their properties are studied. Examples of generating new quantile functions and application of the results to model data are provided.

Keywords: Bivariate quantile functions; Hazard and mean residual quantile functions; Bivariate linear hazard (mean residual) quantile function distribution.