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STATISTICAL INFERENCE FOR THE GOMPERTZ DISTRIBUTION BASED ON PROGRESSIVE TYPE-II CENSORED DATA WITH BINOMIAL REMOVALS

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1. Introduction

Censored sampling arises in a life-testing experiment whenever the experimenter does not observe the failure times of all units placed on a life-test. In medical or industrial applications, researchers have to treat the censored data because they usually do not have sufficient time to observe the lifetime of all subjects in the study. Furthermore, subjects/items may fail by cause other than the ones under study. There are numerous schemes of censoring. Here we consider a progressively type-II right-censored sampling. The ordered failure times arising from a progressively type-II right-censored sampling are called progressively type-II right-censored order statistics. The method allows to save time and cost to the experimenter and is useful when the items being tested are very expensive.

Suppose that $X_1, \ldots, X_n$ is an independent and identically distributed (i.i.d.) random lifetimes of n items. A progressive type-II right-censored sample may be obtained as follows: at the time of the first failure, noted as $X_{1:m:n}$, R_1 units are randomly removed from $n-1$ surviving units. Similarly at the time of the second failure, noted as $X_{2:m:n}$, R_2 units from the $n-R_1-2$ units are randomly removed. This process continuous until, at the time of the mth observed failure, the remaining $n-m-R_1-R_2-\ldots-R_{m-1}$ units are all removed from the experiment. Viveros and Balakrishnan (1994) derived explicit expressions for the best linear unbiased estimates (BLUE) of the parameters of both one and two parameter exponential distributions based on progressively type-II right-censored samples. Balakrishnan and Sandhu (1996), Aggarwala and Balakrishnan (1996) and Aggarwala and Balakrishnan (1998) discussed several mathematical results as well as efficient inference procedures using progressive type-II censoring scheme. Singh

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et al. (2015) discussed the one- and two sample prediction problems based on type-I hybrid censored sample for Lindley distribution. Chacko and Mohan (2017) considered the Bayesian estimation of parameters of a Kumaraswamy-exponential distribution under progressive type-II censoring. Sharma (2018) developed estimation procedure for sample prediction problems based on type-II hybrid censored sample for Weibull distribution. Note that, in usual progressive type-II censoring, the scheme $R_1, R_2, \ldots, R_m$ are all pre-fixed. However, in some practical situations, these numbers may occur at random. For example, in some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In such cases, the pattern of removal at each failure is random. This leads to progressive censoring with random removals. In this paper we assume that the random removal R_i follows a binomial distribution with parameter p. It means that each unit leaves with equal probability p and the probability of R_i units leaving after the i^{th} failure occurs is

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1} \tag{1}$$

and, for $i = 2, 3, \ldots, m-1$,

$$P\left(R_i = r_i | R_{i-1} = r_{i-1}, \ldots, R_1 = r_1\right) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=1}^{i-1} r_j}, \tag{2}$$

where $0 \leq r_i \leq n-m-\sum_{j=1}^{i-1} r_j$. Furthermore, we assume that R_i is independent of X_i for all i. The schematic representation of the progressive type-II censoring with binomial removals is illustrated given below.

Process	The number in life testing	Failures	Binomial removals	Remains
1	n	1	$R_1 \sim B(n-m, p)$	$n-1-R_1$
2	$n-1-R_1$	1	$R_2 \sim B(n-m-R_1, p)$	$n-2-R_1-R_2$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$m-1$	$n-(m-2)-\sum_{j=1}^{m-2} R_j$	1	$R_{m-1} \sim B\left(n-m-\sum_{j=1}^{m-2} R_j, p\right)$	$n-(m-1)-\sum_{j=1}^{m-1} R_j$
m	$n-(m-1)-\sum_{j=1}^{m-1} R_j$	1	$R_m = n-m-\sum_{j=1}^{m-1} R_j$	0

The joint distribution of $X = (X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n})$ and $R = (R_1, R_2, ..., R_m)$ is obtained as

$$f_{X,R}(x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}, r | p) = f_X(x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}) P(R | p), \tag{3}$$

where

$$f_X(x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}) = C \prod_{i=1}^{m} f_{X_{i:m:n}}(x_{i:m:n}) \left\{1 - F_{X_{i:m:n}}(x_{i:m:n})\right\}^{r_i},$$

$r = (r_1, r_2, \ldots, r_m)$, C is a constant defined as

$$C = n(n-r_1-1)(n-r_1-r_2-2)\ldots(n-r_1-r_2)\ldots(n-r_1-r_2-\ldots r_{m-1}-m+1), \quad (4)$$

and $P(R|p)$ is the joint probability distribution of $R = (R_1, R_2, \ldots, R_m)$ defined as

$$\begin{aligned} P(R|\,p) = P(R_m = r_m | R_{m-1} = r_{m-1}, \ldots, R_1 = r_1) \times \ldots \\ \times P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1). \end{aligned}$$

Therefore from (1) and (2), we have

$$P(R|\,p) = \frac{(n-m)!}{\left(n-m-\sum_{j=1}^{m-1} r_j\right)! \prod_{j=1}^{m-1} r_j} \, p^{\sum_{j=1}^{m-1} r_j} (1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}. \quad (5)$$

Several authors considered statistical inference on different lifetime distributions under progressive censoring with random removals. Yuen and Tse (1996) considered parametric estimation for Weibull distribution under progressive censoring with random removals. Tse *et al.* (2000) considered statistical analysis for Weibull distributed lifetime data under type-II progressive censoring with binomial removals. Wu and Chang (2003) discussed inference in the Pareto distribution based on progressive type-II censoring with random removals. Wu *et al.* (2004) derived the estimated expected test time for Pareto distribution under progressive censoring data. Amin (2008) discussed Bayesian inference procedures of the Pareto distribution under progressive censoring with binomial removals. Al-Zahrani (2012) derived the maximum likelihood estimators of the generalized Pareto distribution under progressive censoring with binomial removals. Azimi and Yaghmaei (2013) considered the Bayesian estimation based on Rayleigh distribution under progressive type-II censored data with binomial removals. Feroze and El-Batal (2013) derived the maximum likelihood estimators of the Kumaraswamy distribution under progressive type-II censored data with random removals. Azimi *et al.* (2014) discussed statistical inference procedures of the Pareto distribution using progressive type-II censoring data with binomial removals.

In this paper, we consider progressive type-II censored sample taken from a Gompertz distribution with probability density function (pdf) given by

$$f(x|\alpha,\beta) = \alpha\beta e^{\beta x} e^{-\alpha\left(e^{\beta x}-1\right)}, \; x > 0 \quad (6)$$

and cumulative distribution function (cdf) given by

$$F(x|\alpha,\beta) = 1 - e^{-\alpha\left(e^{\beta x}-1\right)}, \; x > 0. \quad (7)$$

The Gompertz model was formulated by Gompertz (1825) to fit mortality tables. It has been used as a growth model, especially in epidemiological and biomedical studies, and also can be used to fit tumor growth. Pollard and Valkovics (1992) studied the Gompertz distribution and its applications. The parameters of the Gompertz distribution have been estimated by Chen (1997). The pdf of the Gompertz distribution is unimodal. It has positive skewness and an increasing hazard rate function. Through out the paper we use the notation Gomp(α, β) to denote Gompertz distribution with pdf defined in (6). Shanubhogue and Jain (2013) derived the minimum variance unbiased estimators of the Gompertz distribution under progressive type-II censored data with binomial removals. Mohan and Chacko (2016) discussed the Bayesian estimation of parameters of Gompertz distribution under progressive type-II censoring.

Progressive type-II censored order statistics have found applications in many epidemiological studies and can be effectively used for estimating parameters of variable of interest. For example, consider the study of survival times of breast cancer patients undergo treatment for a particular group. Suppose n patients diagonalised breast cancer are considered initially for the study. After observing the survival time of the first patient, R_1 patients are randomly withdraw from the study due to any of the reasons such as death due to cause other than cancer, patient lost to follow-up etc. At the time of the second failure R_2 of the remaining patients are randomly withdraw from the study due any of the reasons mentioned above. Finally at the time of observing the survival time of m^{th} patient all the remaining survival patients are censored or removed from the study. Once the data on progressive type-II censoring is available under the assumption that withdrawal after each failure follows a binomial distribution with probability p then the procedure suggested in this paper can be used to estimate the parameters of the variable X when X follows a Gompertz distribution. An advantage of the proposed method is that we can also get an estimate for the removal probability p together with the estimates for the variable of interest.

This paper is organised as follows. In Section 2, maximum likelihood estimates of α, β and p are obtained. The asymptotic variance-covariance matrix of the estimates is obtained in this section. In Section 3, Bayes estimates for α, β and p are obtained for different loss functions such as squared error, LINEX and general entropy. In Section 4, a simulation study is performed for analysing the properties of different estimators developed in this paper. In Section 5, we illustrate the estimation procedure using a real data. Finally, in Section 6, we present some concluding remarks.

2. Maximum likelihood estimation

Let $X_{1:m:n}$, $X_{2:m:n}$, ..., $X_{m:m:n}$, $1 \leq m \leq n$ be a progressively type-II censored sample observed from a life test involving n units taken from a Gomp(α, β) distribution and $(R_1, R_2, ..., R_m)$ being the censoring scheme with probability distribution defined in (5).

Then the joint pdf of $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ is given by

$$\begin{aligned} f_{(X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n})}(x_{1:m:n},x_{2:m:n},\ldots,x_{m:m:n}) &= C\alpha^m\beta^m\prod_{i=1}^{m}e^{\beta x_{i:m:n}} \\ &\times \left[e^{-\alpha\left(e^{\beta x_{i:m:n}}-1\right)}\right]^{r_i+1}, \end{aligned} \tag{8}$$

where C is as given in (4).

Then from (3) the likelihood function of α, β and p is given by

$$\begin{aligned} L(\alpha,\beta,p|x,r) &\propto \alpha^m\beta^m\prod_{i=1}^{m}e^{\beta x_{i:m:n}}\left[e^{-\alpha\left(e^{\beta x_{i:m:n}}-1\right)}\right]^{r_i+1} p^{\sum_{j=1}^{m-1} r_j} \\ &\times (1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}, \end{aligned} \tag{9}$$

where $x=(x_{1:m:n},x_{2:m:n},\ldots,x_{m:m:n})$ and $r=(r_1,r_2,\ldots,r_m)$.

Thus the log-likelihood function is given by

$$\begin{aligned} \ln L(\alpha,\beta,p|x,r) = {} & m\log\alpha + m\log\beta + \beta\sum_{i=1}^{m}x_{i:m:n} \\ & -\alpha\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)+\left(\sum_{j=1}^{m-1}r_j\right)\ln p \\ & +\left((m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j\right)\ln(1-p). \end{aligned} \tag{10}$$

Thus we have

$$\frac{\partial \ln L}{\partial \alpha} = \frac{m}{\alpha}-\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right), \tag{11}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{m}{\beta}+\sum_{i=1}^{m}x_{i:m:n}-\alpha\sum_{i=1}^{m}(r_i+1)x_{i:m:n}e^{\beta x_{i:m:n}} \tag{12}$$

and

$$\frac{\partial \ln L}{\partial p} = \frac{\sum_{j=1}^{m-1} r_j}{p}-\frac{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}{1-p}. \tag{13}$$

The maximum likelihood estimators of the parameters α, β and p respectively can then be obtained as the solution of the following normal equations

$$\frac{\partial \ln L}{\partial \alpha}=0, \tag{14}$$

$$\frac{\partial \ln L}{\partial \beta}=0 \tag{15}$$

and

$$\frac{\partial \ln L}{\partial p}=0. \tag{16}$$

Thus from (11) and (14), we have

$$\hat{\alpha}=\frac{m}{\sum_{i=1}^{m}(r_i+1)\left(e^{\hat{\beta}x_{i:m:n}}-1\right)}. \tag{17}$$

On substituting (17) into (15), we get

$$\frac{m}{\beta}+\sum_{i=1}^{m}x_{i:m:n}-\left[\frac{m}{\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)}\right]\sum_{i=1}^{m}(r_i+1)\,x_{i:m:n}e^{\beta x_{i:m:n}}=0. \tag{18}$$

The maximum likelihood estimator $\hat{\beta}$ of β can be obtained as solution of the non-linear equation of the form $g(\beta)=\beta$, where

$$g(\beta)=m\left(\left[\frac{m}{\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)}\right]\sum_{i=1}^{m}(r_i+1)\,x_{i:m:n}e^{\beta x_{i:m:n}}-\sum_{i=1}^{m}x_{i:m:n}\right)^{-1}. \tag{19}$$

Let $\hat{\beta}$ be the ML estimator of β by solving the non linear equation $g(\beta)=\beta$ and then by using equation (17), the ML estimator of α will be given by

$$\hat{\alpha}=\frac{m}{\sum_{i=1}^{m}(r_i+1)\left(e^{\hat{\beta}x_{i:m:n}}-1\right)}. \tag{20}$$

Also from (13) and (16), we have ML estimator of p is given by

$$\hat{p}=\frac{\sum_{j=1}^{m-1}r_j}{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)\,r_j+\sum_{j=1}^{m-1}r_j}. \tag{21}$$

2.1. *Interval estimation*

In this section, we obtain the appropriate confidence intervals of the parameters based on asymptotic distributions of the MLE of the parameters α, β and p. The elements of the Fisher information matrix for the parameters of the Gompertz distribution based on progressive censored samples have been derived explicitly. The Fisher information matrix can be defined as

$$I = -E\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial p} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial p} \\ \frac{\partial^2 \ln L}{\partial p \partial \alpha} & \frac{\partial^2 \ln L}{\partial p \partial \beta} & \frac{\partial^2 \ln L}{\partial p^2} \end{bmatrix}.$$

Unfortunately, the exact mathematical expressions for the above expectations are difficult to obtain. Therefore, we give the approximate (observed) asymptotic distributions of the MLE of the parameters α, β and p, which is obtained by dropping the expectation operator E and it can be written as

$$\begin{aligned} I &= -\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial p} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta \partial p} \\ \frac{\partial^2 \ln L}{\partial p \partial \alpha} & \frac{\partial^2 \ln L}{\partial p \partial \beta} & \frac{\partial^2 \ln L}{\partial p^2} \end{bmatrix} \\ &= \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & I_{\alpha p} \\ I_{\beta\alpha} & I_{\beta\beta} & I_{\beta p} \\ I_{p\alpha} & I_{p\beta} & I_{pp} \end{bmatrix}, \end{aligned}$$

where

$$I_{\alpha\alpha} = \frac{m}{\alpha^2}, I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{m}(r_i + 1)x_{i:m:n}e^{\beta x_{i:m:n}},$$

$$I_{\beta\beta} = \frac{m}{\beta^2} + \alpha\sum_{i=1}^{m}(r_i + 1)x_{i:m:n}^2 e^{\beta x_{i:m:n}},$$

$$I_{pp} = \frac{\sum\_i = 1^{m-1} r_i}{p^2} + \frac{(m-1)(n-m) - \sum\_i = 1^{m-1}(m-i)r_i}{(1-p)^2},$$

and

$$I_{\alpha p} = I_{p\alpha} = I_{\beta p} = I_{p\beta} = 0.$$

Using these results, we can obtain the Fisher information matrix, which can further be used to derive the elements of the approximate variance-covariance matrix. The variance-covariance matrix may be approximated as

$$V=\begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\beta} & 0 \\ V_{\beta\alpha} & V_{\beta\beta} & 0 \\ 0 & 0 & V_{pp} \end{pmatrix}=\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\beta} & 0 \\ I_{\beta\alpha} & I_{\beta\beta} & 0 \\ 0 & 0 & I_{pp} \end{pmatrix}^{-1}.$$

Thus the asymptotic distribution of the ML estimator $(\hat{\alpha},\hat{\beta},\hat{p})$ is given as

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{p} \end{pmatrix}\approx N\left[\begin{pmatrix} \alpha \\ \beta \\ p \end{pmatrix},\begin{pmatrix} V_{\alpha\alpha} & V_{\alpha\beta} & 0 \\ V_{\beta\alpha} & V_{\beta\beta} & 0 \\ 0 & 0 & V_{pp} \end{pmatrix}\right]. \tag{22}$$

That is, the asymptotic distribution of the maximum likelihood can be written as follows (Miller, 1981,see,),

$$\left[(\hat{\alpha}-\alpha),\left(\hat{\beta}-\beta\right),(\hat{p}-p)\right]\sim N_3(0,V). \tag{23}$$

Since V involves the parameters α, β and p, we replace the parameters by the corresponding MLE's in order to obtain an estimate of V, which is denoted by $\hat{V}$. By using (23), approximate $100(1-\vartheta)\%$ confidence intervals for α, β and p are determined, respectively, as

$$\hat{\alpha}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{\alpha\alpha}},\hat{\beta}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{\beta\beta}} \text{ and } \hat{p}\pm Z_{\vartheta/2}\sqrt{\hat{V}_{pp}}, \tag{24}$$

where Z_ϑ is the upper $100\ \vartheta^{th}$ percentile of the standard normal distribution.

3. Bayesian estimation

In this section, we consider the Bayes estimators of the parameters α and β of Gompertz distribution and the binomial removal probability p using progressively type-II censored data with binomial removals. The Bayes estimates are obtained using symmetric as well as asymmetric loss functions. A symmetric loss function is the squared error which is defined as

$$L_1\left(d(\mu),\hat{d}(\mu)\right)=\left(\hat{d}(\mu)-d(\mu)\right)^2, \tag{25}$$

where $\hat{d}(\mu)$ is an estimate of $d(\mu)$. An asymmetric loss function is the LINEX loss function which is defined as

$$L_2\left(d(\mu),\hat{d}(\mu)\right)=e^{h\left(\hat{d}(\mu)-d(\mu)\right)}-h\left(\hat{d}(\mu)-d(\mu)\right)-1,\ h\neq 0. \tag{26}$$

The Bayes estimate of $d(\mu)$ for the loss function L_2 can be obtained as

$$\hat{d}_{LB}(\mu) = -\frac{1}{h}\log\left\{E_{\mu}\left(e^{-h\mu}\middle|\underline{x}\right)\right\}, \tag{27}$$

provided $E_{\mu}(\cdot)$ exists. Another asymmetric loss function is the general entropy loss function given by

$$L_3\left(d(\mu), \hat{d}(\mu)\right) = \left(\frac{\hat{d}(\mu)}{d(\mu)}\right)^{q} - q\log\left(\frac{\hat{d}(\mu)}{d(\mu)}\right) - 1,\ q \neq 0. \tag{28}$$

In this case, Bayes estimate of $d(\mu)$ is obtained as

$$\hat{d}_{EB}(\mu) = \left(E_{\mu}\left(\mu^{-q}\middle|\underline{x}\right)\right)^{-\frac{1}{q}}. \tag{29}$$

We assume that the prior distributions for α and β follow independent gamma distributions of the form

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \alpha > 0,\ a_1 > 0,\ b_1 > 0 \tag{30}$$

and

$$\pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta}, \beta > 0,\ a_2 > 0,\ b_2 > 0. \tag{31}$$

Here a_1, b_1, a_2 and b_2 are chosen to reflect prior knowledge about α and β. Independently from parameters α and β, p has a beta prior distribution with parameters a and b of the form

$$\pi_3(p) \propto p^{a-1}(1-p)^{b-1}, 0 < p < 1,\ a > 0,\ b > 0. \tag{32}$$

Based on the priors $\pi_1(\alpha)$, $\pi_2(\beta)$ and $\pi_3(p)$, the joint prior pdf of (α, β, p) is

$$\begin{aligned} \pi(\alpha, \beta, p) &\propto \pi_1(\alpha)\pi_2(\beta)\pi_3(p) \\ &\propto \alpha^{a_1-1}e^{-b_1\alpha}\beta^{a_2-1}e^{-b_2\beta}p^{a-1}(1-p)^{b-1};\ \alpha, \beta > 0,\ 0 < p < 1. \end{aligned} \tag{33}$$

Thus the posterior density of (α, β, p) is given by

$$\pi^*(\alpha, \beta, p|x, r) = \frac{L(\alpha, \beta, p|x, r)\pi(\alpha, \beta, p)}{\int_0^\infty\int_0^\infty\int_0^1 L(\alpha, \beta, p|x, r)\pi(\alpha, \beta, p)\,dp\,d\beta\,d\alpha}. \tag{34}$$

It is not possible to compute the integral in the denominator of (34) explicitly, therefore we cannot obtain the posterior density in a closed form. Thus it is not possible to obtain the Bayes estimates of α and β explicitly. Hence we propose the MCMC technique to generate sample from the posterior distribution and then compute the Bayes estimates under different loss functions. We use Gibbs sampling procedure to generate sample from the posterior density. The joint posterior density of α, β and p given in (34) can be written as

$$\pi^*(\alpha,\beta,p|x_1,x_2,...,x_m) \propto \alpha^{m+a_1-1}\beta^{m+a_2-1}p^{a-1}(1-p)^{b-1} \times e^{-\alpha\left[b_1+\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)\right]} e^{\beta\left(-b_2+\sum_{i=1}^{m}x_{i:m:n}\right)} \times p^{\sum_{j=1}^{m-1}r_j}(1-p)^{(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j}. \tag{35}$$

Then the conditional density of α, given β, p and the data is given by

$$\pi_1^*(\alpha|\beta,p,x_1,x_2,...,x_m) \propto \alpha^{m+a_1-1} e^{-\alpha\left[b_1+\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)\right]}. \tag{36}$$

Again the conditional density of β, given α, p and the data is given by

$$\pi_2^*(\beta|\alpha,p,x_1,x_2,...,x_m) \propto \beta^{m+a_2-1} e^{\beta\left(-b_2+\sum_{i=1}^{m}x_{i:m:n}\right)} e^{-\alpha\left[\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}\right)\right]}. \tag{37}$$

Similarly, the conditional density of p, given α, β and the data is given by

$$\pi_3^*(p|\alpha,\beta,x_1,x_2,...,x_m) \propto p^{a+\sum_{j=1}^{m-1}r_j-1}(1-p)^{b+(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j-1}. \tag{38}$$

From (36) we can see that the conditional distribution of α given β, p and data follows Gamma$\left(m+a_1, b_1+\sum_{i=1}^{m}(r_i+1)\left(e^{\beta x_{i:m:n}}-1\right)\right)$. Similarly, from (38) we can see that the conditional distribution of p given α, β and data follows

$$\text{Beta}\left(a+\sum_{j=1}^{m-1}r_j, b+(m-1)(n-m)-\sum_{j=1}^{m-1}(m-j)r_j\right).$$

Therefore one can easily generate samples from the posterior distributions of α and p. But it is not possible to generate random variables from the posterior distribution of β given in (37) using standard random number generation methods. Hence we use Metropolis-Hastings (M-H) algorithm to generate sample from (37). Since plot of (37) is similar to a normal plot we take normal proposal density for β for the M-H algorithm. For updating β we have used MCMC method.

By setting initial values $\alpha^{(0)}$, $\beta^{(0)}$ and $p^{(0)}$, let $\alpha^{(j)}$, $\beta^{(j)}$ and $p^{(j)}$, $j=1,...,N$ be the observations generated from (36), (37) and (38) respectively. Then, by taking the first M iterations as burn-in period, the Bayes estimates of α, β and p against different loss functions are as given below.

The Bayes estimates $\hat{\psi}_{SB}$, $\hat{\psi}_{LB}$ and $\hat{\psi}_{EB}$ of ψ under SB, LB and EB functions, respectively, are obtained as follows

$$\hat{\psi}_{SB} = \frac{1}{N-M} \sum_{j=M+1}^{N} \psi^{(j)}, \tag{39}$$

$$\hat{\psi}_{LB} = -\frac{1}{h} \log\left(\frac{1}{N-M} \sum_{j=M+1}^{N} e^{-h\psi^{(j)}} \right), \tag{40}$$

and

$$\hat{\psi}_{EB} = \left\{ \frac{1}{N-M} \sum_{j=M+1}^{N} \left(\psi^{(j)}\right)^{-q} \right\}^{-\frac{1}{q}}, \tag{41}$$

where ψ stands for α, β or p.

4. SIMULATION STUDY

In this section, a simulation is performed to study the behaviour of different estimators for the parameters α, β and p. The performance of all estimators are compared numerically in terms of their bias and MSE values for different combinations of n, m, α, β and p. First we obtain the MLE's of α, β and p using 1000 generated samples. The bias and MSE for the MLE's of α, β and p for $p = 0.25, 0.5$ and 0.75 are given in Table 1. For the simulation studies for Bayes estimators we took hyperparameters for the prior distributions of α, β and p in such a way that mean of the prior distribution is 1. That is $a_1 = 3$, $a_2 = 3$, $b_1 = 3$, $b_2 = 3$, $a = 3$ and $b = 3$. We have obtained the Bayes estimates for α, β and p using MCMC method. For the MCMC method we do the following.

1. For a given n generate m progressive type-II censored sample from Gomp(α,β).
2. Calculate estimators of the parameters α, β and p using MCMC method as describe below.
   (a) Start with initial guess $\left(\alpha^{(0)}, \beta^{(0)}, p^{(0)}\right)$.
   (b) Set $j = 1$.
   (c) Generate $\alpha^{(j)}$ from Gamma$\left(m + a_1, b_1 + \sum_{i=1}^{m} (r_i + 1)\left(e^{\beta^{(j-1)} x_{i:m:n}} - 1\right)\right)$.
   (d) Generate $p^{(j)}$ from Beta$\left(a + \sum_{j=1}^{m-1} r_j, b + (m-1)(n-m) - \sum_{j=1}^{m-1} (m-j)\, r_j\right)$.
   (e) Using Metropolis-Hastings algorithm, generate $\beta^{(j)}$ from (38) with the normal proposal distribution.
   (f) Set $j = j + 1$.

(g) Repeat the steps from (c) to (f) for N=50,000 times.

(h) Calculate the Bayes estimators of the parameters α, β and p under different functions by taking $M = 5000$.

3. Repeat the steps 1 and 2 for 1000 times.

4. Calculate the bias and MSE of all estimators.

The bias and MSE for the Bayes estimates under different loss functions of α for $p = 0.25, 0.5$ and 0.75 are given in Table 2. The bias and MSE for the estimates under different loss functions of β for $p = 0.25, 0.5$ and 0.75 are given in Table 3. The bias and MSE for the estimates under different loss functions of p for different choices of $p = 0.25, 0.5$ and 0.75 are given in Table 4. For evaluating Bayes estimators under LINEX loss function L_2 we take h=1 and entropy loss function L_3 we take q =1.

From the tables, it is observed that MSE's of Bayes estimators and MLE's decrease as n increases. We can also see that the Bayes estimators have smaller MSE than MLE's. From Table 2, one can see that the bias and MSE of Bayes estimators of α under entropy loss functions($\hat{\alpha}_{EB}$) for different choices of $p = 0.25, 0.5$ and 0.75 are smaller than bias and MSE of other estimators of α. From Table 3, one can see that the bias and MSE of Bayes estimators of β under squared error loss functions($\hat{\beta}_{SB}$) for different choices of $p = 0.25, 0.5$ and 0.75 are smaller than bias and MSE of other estimators of β. From Table 4, one can see that the bias and MSE of Bayes estimators of p under entropy loss functions($\hat{\alpha}_{EB}$) for different choices of $p = 0.25$ and 0.5 are smaller than bias and MSE of other estimators of p and under squared error loss function($\hat{p}_{SB}$) for $p = 0.75$ is smaller than bias and MSE of other estimators of p.

5. ILLUSTRATION USING REAL-LIFE DATA

In this section, we consider a real data obtained from the public use files of the National Cancer Institute's Surveillance Epidemiology and End Results (SEER) Program Research Data (1973-2014). The data consists of survival times in years a group of breast cancer patients who were diagonised the disease at the age of 40 and received 10 years of follow-up. Follow-up of these patients continue through the end of 2014. The data set consisting of survival times (in years) for 59 patients are given below:

0.417, 0.583, 0.667, 0.667, 0.667, 0.833, 0.917, 0.917, 0.917, 0.917, 1.167, 1.167, 1.167, 1.250, 1.417, 1.417, 1.500, 1.583, 1.917, 2.000, 2.250, 2.250, 2.333, 2.583, 2.667, 2.667, 3.083, 3.167, 3.250, 3.333, 3.417, 3.500, 4.083, 4.083, 4.167, 4.250, 4.250, 4.250, 4.500, 4.667, 4.667, 4.667, 4.750, 4.917, 5.000, 5.167, 5.167, 5.250, 5.333, 5.583, 5.917, 6.000, 6.000, 6.083, 6.083, 6.417, 6.417, 6.500, 6.750.

To check for the goodness of fit we use the Anderson-Darling test (see, Stephens, 1974). It is a modification of the Kolmogorov-Smirnov (K-S) test and gives more weight to the tails than does the K-S test. The K-S test is distribution free in the sense that the critical values do not depend on the specific distribution being tested. The Anderson-Darling test makes use of the specific distribution in calculating critical values. We have computed the Anderson-Darling statistic for the data using "DistributionFitTest" function in *Mathematica* and is obtained as 0.897136. The corresponding p-value is 0.415185. Since the p-value is quite high, we cannot reject the null hypothesis that the data are coming from the Gompertz distribution.

From the data set we generated progressive type-II censored observation with binomial removals for different values of p (p=0.25, 0.5, 0.75) and m (m=25, 30, 35) and are given in Table 5. We have obtained the MLE's of α, β and p and are given in Table 6 and the Bayes estimates of α, β and p under squared error, LINEX and general entropy loss functions are given in Table 7.

6. Conclusion

In this paper, we have considered progressive type-II censored sample taken from Gomp(α, β) with binomial removals. The maximum likelihood estimators of the parameters α, β and p have been obtained. The Bayes estimates have also been obtained using different loss functions such as squared error, LINEX and general entropy. To evaluate the Bayes estimates MCMC method has been applied. Based on the simulation study it is observed that, the Bayes estimators perform much better than the MLE's. Among the Bayes estimates of α, estimator under entropy loss function possess minimum bias and MSE. Among the Bayes estimates of β, estimator under squared error loss function possess minimum bias and MSE. Also among the Bayes estimates of p, estimator under entropy loss functions possess minimum bias and MSE for small values of p and squared error loss function possess minimum bias and MSE for large value of p. A real data on survival times of breast cancer patients has been used to illustrate the estimation procedures developed in this paper.

Appendix

A. Tables

TABLE 1
Bias and MSE of the MLE's $\hat{\alpha}$, $\hat{\beta}$ and $\hat{p}$ for different choices n, m, α, β and p=0.25, 0.5 and 0.75.

p	n	m	α	β	$\hat{\alpha}$		$\hat{\beta}$		$\hat{p}$	
					Bias	MSE	Bias	MSE	Bias	MSE
0.25	25	15	0.5	1	-0.354	0.167	0.688	0.704	0.015	0.007
			1	1.5	-0.231	0.212	0.403	0.492	0.016	0.007
			1	2	0.146	0.347	0.070	0.294	0.019	0.007
		20	0.5	1	-0.325	0.141	0.503	0.426	0.010	0.004
			1	1.5	-0.140	0.210	0.427	0.491	0.018	0.005
			1	2	0.115	0.230	0.031	0.264	0.006	0.003
	30	20	0.5	1	-0.327	0.137	0.482	0.375	0.022	0.006
			1	1.5	-0.093	0.204	0.468	0.543	0.025	0.007
			1	2	0.149	0.518	0.088	0.311	0.015	0.005
		25	0.5	1	-0.258	0.121	0.447	0.342	0.011	0.005
			1	1.5	-0.256	0.197	0.282	0.307	0.020	0.005
			1	2	0.118	0.292	0.025	0.245	0.023	0.004
0.5	25	15	0.5	1	-0.331	0.159	0.607	0.534	0.001	0.013
			1	1.5	-0.147	0.261	0.301	0.427	0.037	0.015
			1	2	0.221	0.525	0.063	0.299	0.043	0.017
		20	0.5	1	-0.330	0.158	0.557	0.499	0.036	0.010
			1	1.5	-0.176	0.208	0.387	0.406	-0.006	0.008
			1	2	0.117	0.412	0.059	0.298	0.013	0.009
	30	20	0.5	1	-0.312	0.154	0.497	0.433	0.018	0.013
			1	1.5	-0.155	0.166	0.344	0.394	0.050	0.022
			1	2	0.318	0.648	-0.074	0.295	0.037	0.012
		25	0.5	1	-0.339	0.153	0.517	0.372	0.006	0.028
			1	1.5	-0.167	0.165	0.321	0.359	0.009	0.007
			1	2	0.209	0.477	-0.088	0.260	0.024	0.010
0.75	25	15	0.5	1	-0.327	0.163	0.553	0.511	-0.001	0.012
			1	1.5	-0.204	0.267	0.516	0.607	0.017	0.014
			1	2	0.124	0.322	0.029	0.296	0.004	0.012
		20	0.5	1	-0.351	0.157	0.579	0.497	0.010	0.008
			1	1.5	-0.186	0.253	0.405	0.453	-0.008	0.008
			1	2	0.024	0.244	0.101	0.274	0.008	0.007
	30	20	0.5	1	-0.352	0.170	0.567	0.555	0.029	0.023
			1	1.5	-0.125	0.213	0.272	0.276	0.015	0.016
			1	2	0.138	0.357	0.018	0.249	0.023	0.014
		25	0.5	1	-0.261	0.118	0.397	0.276	0.027	0.012
			1	1.5	-0.229	0.172	0.315	0.358	0.031	0.008
			1	2	0.105	0.263	0.062	0.329	0.011	0.010

TABLE 2
Bias and MSE of the Bayes estimates under squared error loss function $\hat{\alpha}_{SB}$, LINEX loss function $\hat{\alpha}_{LB}$ and entropy loss function $\hat{\alpha}_{EB}$ for different choices n, m, α, β and p=0.25, 0.5 and 0.75.

p	n	m	α	β	$\hat{\alpha}_{SB}$		$\hat{\alpha}_{LB}$		$\hat{\alpha}_{EB}$	
					Bias	MSE	Bias	MSE	Bias	MSE
0.25	25	15	0.5	1	0.298	0.118	0.197	0.065	0.098	0.036
			1	1.5	0.250	0.096	0.114	0.040	0.006	0.025
			1	2	0.384	0.178	0.225	0.081	0.154	0.054
		20	0.5	1	0.314	0.117	0.122	0.046	0.125	0.033
			1	1.5	0.201	0.086	0.096	0.034	0.030	0.028
			1	2	0.350	0.158	0.222	0.075	0.142	0.045
	30	20	0.5	1	0.319	0.131	0.225	0.074	0.127	0.038
			1	1.5	0.212	0.096	0.133	0.052	0.056	0.038
			1	2	0.357	0.172	0.228	0.082	0.153	0.053
		25	0.5	1	0.264	0.101	0.225	0.072	0.118	0.037
			1	1.5	0.201	0.085	0.117	0.039	0.033	0.027
			1	2	0.350	0.151	0.218	0.075	0.148	0.050
0.5	25	15	0.5	1	0.315	0.129	0.247	0.081	0.121	0.040
			1	1.5	0.197	0.083	0.091	0.044	0.025	0.033
			1	2	0.385	0.181	0.259	0.094	0.193	0.059
		20	0.5	1	0.297	0.120	0.227	0.077	0.132	0.038
			1	1.5	0.223	0.079	0.106	0.035	0.015	0.024
			1	2	0.369	0.164	0.242	0.081	0.167	0.051
	30	20	0.5	1	0.285	0.117	0.216	0.076	0.111	0.041
			1	1.5	0.206	0.084	0.098	0.043	0.017	0.034
			1	2	0.397	0.189	0.266	0.097	0.162	0.067
		25	0.5	1	0.291	0.112	0.225	0.073	0.127	0.039
			1	1.5	0.205	0.084	0.095	0.042	0.010	0.034
			1	2	0.348	0.171	0.231	0.094	0.194	0.063
0.75	25	15	0.5	1	0.298	0.118	0.228	0.075	0.121	0.038
			1	1.5	0.250	0.096	0.093	0.046	0.046	0.034
			1	2	0.384	0.178	0.253	0.089	0.179	0.057
		20	0.5	1	0.314	0.117	0.237	0.071	0.121	0.029
			1	1.5	0.201	0.086	0.129	0.043	0.039	0.028
			1	2	0.350	0.158	0.219	0.077	0.136	0.047
	30	20	0.5	1	0.319	0.131	0.247	0.085	0.140	0.043
			1	1.5	0.212	0.096	0.104	0.052	0.023	0.042
			1	2	0.357	0.172	0.238	0.093	0.169	0.065
		25	0.5	1	0.264	0.101	0.201	0.065	0.102	0.036
			1	1.5	0.201	0.085	0.090	0.044	0.003	0.036
			1	2	0.350	0.151	0.224	0.073	0.148	0.044

TABLE 3
Bias and MSE of the Bayes estimates under squared error loss function $\hat{\beta}_{SB}$, LINEX loss function $\hat{\beta}_{LB}$ and entropy loss function $\hat{\beta}_{EB}$ for different choices of n, m, α, β and p=0.25, 0.5 and 0.75.

p	n	m	α	β	$\hat{\beta}_{SB}$		$\hat{\beta}_{LB}$		$\hat{\beta}_{EB}$	
					Bias	MSE	Bias	MSE	Bias	MSE
0.25	25	15	0.5	1	-0.117	0.037	-0.140	0.042	-0.183	0.056
			1	1.5	-0.084	0.042	-0.167	0.055	-0.208	0.070
			1	2	-0.309	0.130	-0.422	0.203	-0.454	0.231
		20	0.5	1	-0.116	0.030	-0.151	0.036	-0.200	0.054
			1	1.5	-0.060	0.039	-0.136	0.046	-0.172	0.057
			1	2	-0.262	0.107	-0.370	0.165	-0.396	0.184
	30	20	0.5	1	-0.107	0.036	-0.136	0.040	-0.174	0.051
			1	1.5	-0.094	0.045	-0.153	0.057	-0.186	0.068
			1	2	-0.281	0.115	-0.386	0.176	-0.413	0.197
		25	0.5	1	-0.118	0.030	-0.150	0.036	-0.194	0.051
			1	1.5	-0.074	0.040	-0.148	0.049	-0.184	0.061
			1	2	-0.264	0.113	-0.365	0.165	-0.388	0.182
0.5	25	15	0.5	1	-0.135	0.038	-0.167	0.045	-0.213	0.062
			1	1.5	-0.095	0.047	-0.168	0.058	-0.204	0.071
			1	2	-0.337	0.140	-0.444	0.217	-0.477	0.247
		20	0.5	1	-0.127	0.035	-0.166	0.043	-0.213	0.062
			1	1.5	-0.083	0.047	-0.141	0.055	-0.180	0.068
			1	2	-0.293	0.125	-0.396	0.186	-0.423	0.207
	30	20	0.5	1	-0.115	0.038	-0.142	0.042	-0.179	0.053
			1	1.5	-0.055	0.040	-0.141	0.047	-0.182	0.061
			1	2	-0.314	0.127	-0.414	0.192	-0.441	0.215
		25	0.5	1	-0.117	0.030	-0.147	0.035	-0.188	0.049
			1	1.5	-0.072	0.034	-0.141	0.042	-0.173	0.053
			1	2	-0.259	0.120	-0.361	0.169	-0.383	0.185
0.75	25	15	0.5	1	-0.124	0.034	-0.158	0.041	-0.205	0.058
			1	1.5	-0.073	0.038	-0.148	0.047	-0.186	0.059
			1	2	-0.296	0.127	-0.410	0.194	-0.441	0.220
		20	0.5	1	-0.116	0.031	-0.146	0.036	-0.187	0.050
			1	1.5	-0.063	0.037	-0.146	0.046	-0.184	0.059
			1	2	-0.298	0.123	-0.399	0.188	-0.425	0.210
	30	20	0.5	1	-0.117	0.031	-0.151	0.037	-0.197	0.053
			1	1.5	-0.092	0.046	-0.160	0.055	-0.199	0.067
			1	2	-0.300	0.131	-0.402	0.192	-0.428	0.214
		25	0.5	1	-0.097	0.026	-0.129	0.030	-0.171	0.043
			1	1.5	-0.074	0.042	-0.157	0.053	-0.193	0.066
			1	2	-0.256	0.095	-0.357	0.150	-0.381	0.167

TABLE 4
Bias and MSE of the Bayes estimates under squared error loss function $\hat{p}_{SB}$, LINEX loss function $\hat{p}_{LB}$ and entropy loss function $\hat{p}_{EB}$ for different choices n, m, α, β and p=0.25, 0.5 and 0.75.

p	n	m	α	β	$\hat{p}_{SB}$		$\hat{p}_{LB}$		$\hat{p}_{EB}$	
					Bias	MSE	Bias	MSE	Bias	MSE
0.25	25	15	0.5	1	0.038	0.001	0.036	0.001	0.026	0.001
			1	1.5	0.101	0.010	0.098	0.010	0.017	0.000
			1	2	0.101	0.010	0.098	0.010	0.083	0.007
		20	0.5	1	0.008	0.000	0.005	0.000	-0.007	0.000
			1	1.5	0.029	0.001	0.027	0.001	0.023	0.001
			1	2	0.083	0.007	0.079	0.006	0.055	0.003
	30	20	0.5	1	0.090	0.008	0.088	0.008	0.076	0.006
			1	1.5	0.083	0.007	0.081	0.006	0.066	0.004
			1	2	0.083	0.007	0.081	0.007	0.071	0.005
		25	0.5	1	0.015	0.000	0.013	0.000	0.000	0.000
			1	1.5	-0.034	0.001	-0.036	0.001	-0.035	0.001
			1	2	0.083	0.007	0.188	0.035	0.066	0.004
0.5	25	15	0.5	1	0.167	0.028	0.163	0.026	0.154	0.024
			1	1.5	-0.061	0.004	0.037	0.001	-0.033	0.001
			1	2	0.081	0.006	0.077	0.006	0.067	0.004
		20	0.5	1	0.000	0.000	-0.005	0.000	-0.020	0.000
			1	1.5	0.042	0.002	-0.036	0.001	0.022	0.000
			1	2	-0.036	0.001	-0.040	0.002	-0.036	0.001
	30	20	0.5	1	0.063	0.004	0.063	0.004	0.049	0.002
			1	1.5	-0.050	0.002	-0.053	0.003	-0.044	0.002
			1	2	0.042	0.002	-0.085	0.007	-0.041	0.002
		25	0.5	1	0.071	0.003	0.059	0.003	0.039	0.001
			1	1.5	0.020	0.000	0.015	0.000	0.000	0.000
			1	2	-0.026	0.001	0.037	0.001	0.022	0.000
0.75	25	15	0.5	1	-0.179	0.011	-0.187	0.011	-0.211	0.011
			1	1.5	-0.083	0.007	-0.092	0.008	-0.113	0.013
			1	2	-0.100	0.010	-0.105	0.011	-0.118	0.011
		20	0.5	1	0.000	0.000	-0.004	0.000	-0.011	0.000
			1	1.5	-0.083	0.007	-0.087	0.008	-0.096	0.009
			1	2	-0.083	0.007	-0.092	0.008	-0.113	0.010
	30	20	0.5	1	-0.131	0.017	-0.136	0.019	-0.150	0.023
			1	1.5	-0.179	0.012	-0.187	0.015	-0.202	0.011
			1	2	-0.131	0.011	-0.136	0.019	-0.150	0.013
		25	0.5	1	-0.058	0.003	-0.062	0.004	-0.070	0.005
			1	1.5	-0.066	0.004	-0.071	0.005	-0.083	0.007
			1	2	-0.179	0.010	-0.111	0.010	-0.211	0.011

TABLE 5
Progressive type-II censoring samples with binomial removals

				(p=0.25, m=30)					
0.417	0.667	0.917	1.417	2.250	3.167	3.250	3.500	4.083	4.167
4.250	4.250	4.500	4.667	4.667	4.667	4.750	4.917	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500
				(p=0.25, m=35)					
0.417	1.167	1.417	1.583	2.250	2.333	2.667	2.667	3.333	3.417
3.500	4.083	4.167	4.250	4.250	4.500	4.667	4.667	4.667	4.750
4.917	5.000	5.167	5.167	5.250	5.333	5.583	5.917	6.000	6.000
6.083	6.083	6.417	6.417	6.500					
				(p=0.25, m=40)					
0.417	0.917	0.917	1.167	1.583	2.250	2.250	2.333	2.583	2.667
3.083	3.167	3.333	3.417	3.500	4.083	4.083	4.167	4.250	4.250
4.250	4.667	4.667	4.667	4.750	4.917	5.000	5.167	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500
				(p=0.5, m=30)					
0.417	2.250	3.167	3.417	4.083	4.083	4.167	4.250	4.250	4.250
4.500	4.667	4.667	4.667	4.750	4.917	5.000	5.167	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500
				(p=0.5, m=35)					
0.417	1.500	2.333	2.667	2.667	3.250	3.333	3.417	3.500	4.083
4.083	4.167	4.250	4.250	4.250	4.500	4.667	4.667	4.667	4.750
4.917	5.000	5.167	5.167	5.250	5.333	5.583	5.917	6.000	6.000
6.083	6.083	6.417	6.417	6.500					
				(p=0.5, m=40)					
0.417	1.167	1.417	1.917	2.250	2.333	2.583	2.667	3.083	3.167
3.250	3.333	3.417	3.500	4.083	4.083	4.167	4.250	4.250	4.250
4.500	4.667	4.667	4.667	4.750	4.917	5.000	5.167	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500
				(p=0.75, m=30)					
0.417	2.000	2.667	3.417	4.083	4.083	4.167	4.250	4.250	4.250
4.500	4.667	4.667	4.667	4.750	4.917	5.000	5.167	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500
				(p=0.75, m=35)					
0.417	1.250	2.583	2.667	3.167	3.250	3.333	3.417	3.500	4.083
4.083	4.167	4.250	4.250	4.250	4.500	4.667	4.667	4.667	4.750
4.917	5.000	5.167	5.167	5.250	5.333	5.583	5.917	6.000	6.000
6.083	6.083	6.417	6.417	6.500					
				(p=0.75, m=40)					
0.417	1.417	2.250	2.250	2.333	2.583	2.667	2.667	3.083	3.167
3.250	3.333	3.417	3.500	4.083	4.083	4.167	4.250	4.250	4.250
4.500	4.667	4.667	4.667	4.750	4.917	5.000	5.167	5.167	5.250
5.333	5.583	5.917	6.000	6.000	6.083	6.083	6.417	6.417	6.500

TABLE 6

Maximum likelihood estimates of α, β and p based on cancer patient data for different choices of $p = 0.25, 0.5$ and 0.75.

p	m	$\hat{\alpha}$	$\hat{\beta}$	$\hat{p}$
0.25	30	0.012	0.829	0.201
	35	0.018	0.794	0.270
	40	0.033	0.701	0.216
0.5	30	0.003	1.058	0.690
	35	0.010	0.887	0.615
	40	0.024	0.761	0.487
0.75	30	0.004	1.037	0.644
	35	0.010	0.896	0.632
	40	0.022	0.776	0.826

TABLE 7

Bayes estimates of α, β and p based on cancer patient data under squared error, LINEX and entropy loss functions for different choices of $p = 0.25, 0.5$ and 0.75.

p	m	$\hat{\alpha}_{SB}$	$\hat{\alpha}_{LB}$	$\hat{\alpha}_{EB}$	$\hat{\beta}_{SB}$	$\hat{\beta}_{LB}$	$\hat{\beta}_{EB}$	$\hat{p}_{SB}$	$\hat{p}_{LB}$	$\hat{p}_{EB}$
0.25	30	0.054	0.046	0.026	0.662	0.653	0.607	0.208	0.207	0.202
	35	0.050	0.048	0.032	0.657	0.651	0.631	0.276	0.275	0.269
	40	0.105	0.081	0.054	0.588	0.581	0.523	0.226	0.225	0.217
0.5	30	0.021	0.019	0.009	0.837	0.826	0.796	0.659	0.657	0.652
	35	0.051	0.039	0.021	0.723	0.714	0.656	0.591	0.589	0.582
	40	0.069	0.062	0.040	0.639	0.633	0.600	0.477	0.475	0.465
0.75	30	0.027	0.022	0.011	0.812	0.802	0.758	0.620	0.617	0.612
	35	0.043	0.036	0.021	0.729	0.721	0.682	0.605	0.602	0.595
	40	0.061	0.055	0.037	0.656	0.650	0.623	0.750	0.747	0.741

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SUMMARY

In this paper, the problem of estimation of parameters for a two-parameter Gompertz distribution is considered based on a progressively type-II censored sample with binomial removals. Together with the unknown parameters, the removal probability is also estimated. The maximum likelihood estimators of the parameters and the asymptotic variance-covariance matrix of the estimates are obtained. Bayes estimators are also obtained using different loss functions such as squared error, LINEX and general entropy. A simulation study is performed for comparison between various estimators developed in this paper. A real data set is also used for illustration.

Keywords: Gompertz distribution; Progressive type-II censoring; Binomial removals; Bayes estimates; MCMC method.