*STATISTICA, anno LXXVIII, n. 3, 2018*

# THE MARSHALL-OLKIN GENERALIZED-G FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

Haitham M. Yousof
*Department of Statistics, Mathematics and Insurance, Benha University, Egypt*
Ahmed Z. Afify
*Department of Statistics, Mathematics and Insurance, Benha University, Egypt*
Saralees Nadarajah
*School of Mathematics, University of Manchester, Manchester, UK*
Gholamhossein Hamedani
*Department of Mathematics, Statistics and CS, Marquette University, USA*
Gokarna R. Aryal[1]
*Department of Mathematics, Statistics and CS, Purdue University Northwest, USA*

## 1. Introduction

The statistical literature contains many new classes of distributions which have been constructed by extending common families of continuous distributions. These new families have been used for modeling data in many areas such as engineering, economics, biological studies, environmental sciences, to name a few. The general objectives of generalizing a new family of distributions include the following:

- produce skewness for symmetrical models;
- define special models with different shapes of hazard rate function;
- construct heavy-tailed distributions for modeling various real data sets;
- make the kurtosis more flexible compared to that of the baseline distribution;
- generate distributions which are skewed, symmetric, J-shaped or reversed-J shaped;
- provide consistently better fits than other generalized distributions with the same underlying model.

[1] Corresponding Author. E-mail: aryalg@pnw.edu

The Marshall-Olkin distribution has been generalized by using the genesis of other distributions to develop wider family of distributions and achieve one or more of the above characteristics. Some notable examples include Marshall-Olkin-G (MO-G) family by Marshall and Olkin (1997), beta Marshall-Olkin-G (BMO-G) by Alizadeh *et al.* (2015a), Kumaraswamy Marshall-Olkin-G (KwMO-G) by Alizadeh *et al.* (2015b), among others.

Consider a random variable $X$ with cumulative distribution function (cdf) $G(x;\psi)$ depending on a parameter vector $\psi$. The reliability function (rf) and probability density function (pdf) of $X$ are given by $\overline{G}(x;\psi) = 1 - G(x;\psi)$ and $g(x;\psi) = \frac{d}{dx}G(x;\psi)$, respectively. The generalized-G (G-G) family has cdf given by

$$H(x;a,\psi) = 1 - [\overline{G}(x;\psi)]^{a}, \qquad x \in \mathbb{R}. \tag{1}$$

The corresponding pdf is

$$h(x;a,\psi) = a\,g(x;\psi)\left[\overline{G}(x;\psi)\right]^{a-1}, \qquad x \in \mathbb{R}. \tag{2}$$

Marshall and Olkin (1997) introduced a new method of adding a parameter to a family of distributions to develop the MO-G family as follows. If $\overline{H}(x)$ and $h(x)$ denote the rf and pdf of a continuous random variable $X$, then the MO-G family has cdf defined by

$$F(x;\delta,\psi) = 1 - \frac{\delta \overline{H}(x;\psi)}{1-(1-\delta)\overline{H}(x;\psi)},\ x \in \mathbb{R},\ \delta > 0. \tag{3}$$

Clearly, when $\delta = 1$, we get the baseline distribution with cdf $H(x;\psi)$. The pdf corresponding to (3) is given by

$$f(x;\delta,\psi) = \frac{\delta\, h(x;\psi)}{\left[1-(1-\delta)\overline{H}(x;\psi)\right]^{2}},\ x \in \mathbb{R},\ \delta > 0. \tag{4}$$

In this paper, we propose and study a new generalized family called the Marshall-Olkin generalized-G (MOG-G) family and provide a comprehensive description of its mathematical properties along with some applications.

The rest of the paper is outlined as follows. In Section 2, we provide the structural derivation of the MOG-G family and study the shape of its density and hazard rate function. In Section 3, we define three special submodels of MOG-G family. In Section 4, we derive some mathematical properties including ordinary and incomplete moments, order statistics. Some characterization results are provided in Section 5. Parameter estimation and related issues are addressed in Section 6. In Section 7, we provide applications to two real data sets to illustrate the importance of the new family. Finally, some concluding remarks are presented in Section 8.

## 2. THE MOG-G FAMILY

Inserting (1) in (3), we obtain the cdf of the MOG-G class

$$F(x;\delta,a,\psi)=\frac{1-\left[\overline{G}(x;\psi)\right]^{a}}{1-(1-\delta)\left[\overline{G}(x;\psi)\right]^{a}},\quad x\in\mathbb{R}, \tag{5}$$

where $\delta$ and $a$ are two positive shape parameters representing the different patterns of the MOG-G family. The effectiveness of the parameters $a$ and $\delta$ on skewness and kurtosis are illustrated in Section 4. The additional shape parameter $a$ is pursued as a tool to furnish a more flexible family of distributions. The corresponding pdf of MOG-G is given by

$$f(x;\delta,a,\psi)=\frac{\delta a\, g(x;\psi)\left[\overline{G}(x;\psi)\right]^{a-1}}{\left[1-(1-\delta)[\overline{G}(x;\psi)]^{a}\right]^{2}},\quad x\in\mathbb{R}. \tag{6}$$

Henceforth, $X\sim$ MOG-G$(\delta,a,\psi)$ denotes a random variable with density function (6).
The rf ,$R(x)$, and hazard rate function (hrf), $\tau(x)$, of MOG-G are given by

$$R(x)=\frac{\delta[\overline{G}(x;\psi)]^{a}}{1-(1-\delta)[\overline{G}(x;\psi)]^{a}}\ \text{and}\ \tau(x)=\frac{a\,\varsigma(x;\psi)}{\left[1-(1-\delta)\left[\overline{G}(x;\psi)\right]^{a}\right]}$$

respectively, where $\varsigma(x;\psi)=g(x;\psi)/\overline{G}(x;\psi)$ is the hrf of the baseline model. We may use $G(x)$ and $g(x)$ for $G(x;\psi)$ and $g(x;\psi)$ interchangeably.

The MOG-G family contains the following sub-classes

- the generalized-G (G-G) family pioneered by Gupta *et al.* (1998), for $\delta=1$;
- the Marshall-Olkin-G (MO-G) class given by Marshall and Olkin (1997), for $a=1$;
- the baseline distribution, for $\delta=1$ and $a=1$.

The shapes of the density and hazard rate functions can be described analytically. The critical points of the MOG-G pdf are the roots of the equation

$$\frac{g'(x)}{g(x)}+(1-a)\frac{g(x)}{\overline{G}(x)}-2(1-\delta)\frac{g(x)}{1-(1-\delta)\overline{G}(x)}=0. \tag{7}$$

Let $\lambda(x)=\frac{d^{2}\log f(x)}{dx^{2}}$, then

$$\begin{aligned}\lambda(x)\ &=\frac{g''(x)g(x)-g'(x)^{2}}{g(x)^{2}}+(1-a)\frac{g'(x)\overline{G}(x)+g(x)^{2}}{\overline{G}(x)^{2}}\\ &\quad-2(1-\delta)\frac{g'(x)\left[1-(1-\delta)\overline{G}(x)\right]-(1-\delta)g(x)^{2}}{\left[1-(1-\delta)\overline{G}(x)\right]^{2}}.\end{aligned}$$

If $x = x_0$ is a root of (7), then it corresponds to a local maximum (minimum) if $\lambda(x) > 0(< 0)$ for all $x < x_0$ and $\lambda(x) < 0(> 0)$ for all $x > x_0$. It yields points of inflection if either $\lambda(x) > 0$ for all $x \neq x_0$ or $\lambda(x) < 0$ for all $x \neq x_0$.

The critical points of the hrf $h(x)$ are obtained from the equation

$$\frac{g'(x)}{g(x)} + \frac{g(x)}{\overline{G}(x)} - (1-\delta)\frac{g(x)}{1-(1-\delta)\overline{G}(x)} = 0. \tag{8}$$

Let $\tau(x) = d^2 \log[h(x)]/dx^2$. We have

$$\begin{aligned}\tau(x) \quad &= \frac{g''(x)g(x)-g'(x)^2}{g(x)^2} + \frac{g'(x)\overline{G}(x)+g(x)^2}{\overline{G}(x)^2} \\ &-(1-\delta)\frac{g'(x)\left[1-(1-\delta)\overline{G}(x)\right]-(1-\delta)g(x)^2}{\left[1-(1-\delta)\overline{G}(x)\right]^2}.\end{aligned}$$

If $x = x_0$ is a root of (8), then it corresponds to a local maximum (minimum) if $\tau(x) > 0(< 0)$ for all $x < x_0$ and $\tau(x) < 0(> 0)$ for all $x > x_0$. It yields points of inflection if either $\tau(x) > 0$ for all $x \neq x_0$ or $\tau(x) < 0$ for all $x \neq x_0$.

## 3. Special submodels

In this section, we discuss three special submodels of the MOG-G family. These submodels generalize some well-known distributions appeared in the literature.

### 3.1. *The MOG-Weibull (MOG-W) distribution*

Consider the Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ whose cdf is given by $G(x) = 1 - \exp\left[-(\alpha x)^{\beta}\right]$, $x \geq 0$. Then, the pdf of the MOGW model is given by

$$f(x) = \frac{\delta a \beta \alpha^{\beta} x^{\beta-1} \exp\left[-a(\alpha x)^{\beta}\right]}{\left\{1-(1-\delta)\exp\left[-a(\alpha x)^{\beta}\right]\right\}^2}, \quad x > 0.$$

The MOG-W distribution includes the generalized Weibull (GW) distribution when $\delta = 1$. For $a = 1$, we obtain the MO-Weibull (MOW) model. For $\beta = 1$, we have the MOG-exponential (MOGE) distribution. For $\beta = 2$, we obtain the MOG-Rayleigh (MOGR) distribution.

### 3.2. *The MOG-Lomax (MOG-Lo) distribution*

Consider the Lomax distribution with parameters $\alpha > 0$ and $\beta > 0$ whose cdf is given by $G(x) = 1 - [1 + (x/\beta)]^{-\alpha}$, $x \geq 0$. Then, the pdf of the MOGLo distribution becomes

$$f(x) = \frac{\delta a \alpha [1 + (x/\beta)]^{-(\alpha a+1)}}{\beta\left\{1-(1-\delta)[1+(x/\beta)]^{-\alpha a}\right\}^2}, \quad x > 0.$$

The MOG-Lo distribution includes the generalized Lomax (GLo) distribution for $\delta = 1$. For $a = 1$, we obtain the MO-Lomax (MOLo) distribution.

### 3.3. *The MOG-log-logistic (MOG-LL) distribution*

Consider the log-logistic distribution with parameters $\alpha > 0$ and $\beta > 0$ whose cdf is given by $G(x) = 1 - \left[1 + (x/\alpha)^{\beta}\right]^{-1}$, $x \geq 0$. Then, the pdf of the MOG-LL distribution is given by

$$f(x) = \frac{\delta a \beta \alpha^{-\beta}\, x^{\beta-1}\left[1+(x/\alpha)^{\beta}\right]^{-a-1}}{\left\{1-(1-\delta)\left[1+(x/\alpha)^{\beta}\right]^{-a}\right\}^{2}}, \quad x > 0.$$

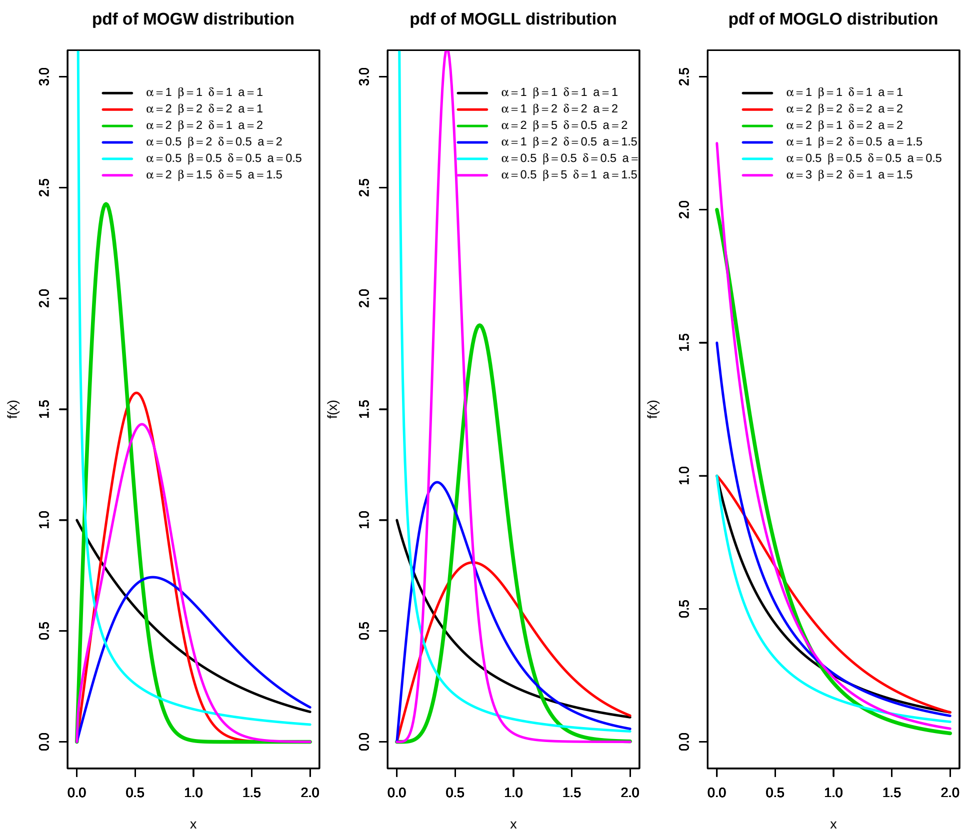


*Figure 1* – Pdfs of special MOG-G distributions.

The MOG-LL distribution includes the generalized log-logistic (GLL) distribution for $\delta = 1$. For $a = 1$, we obtain the MO-log-logistic (MOLL) distribution. Figure 1 displays possible shapes of the MOG-G distributions for selected models.

## 4. Mathematical properties

In this section we provide some mathematical properties of the MOG-G family of distributions including the moments and order statistics. First, we express the pdf and cdf of MOG-G family as a mixture of exp-G distributions which is useful to present the mathematical characteristics of MOG-G family analytically. Note that

$$1-\left[\overline{G}(x)\right]^{a} = 1+\sum_{k=0}^{\infty}(-1)^{1+k}\binom{a}{k}[G(x)]^{k} = \sum_{k=0}^{\infty}\alpha_k\,[G(x)]^{k}, \tag{9}$$

where $\alpha_0 = 2$ and $\alpha_k = (-1)^{1+k}\binom{a}{k}$ for $k \geq 1$, and

$$1-(1-\delta)-\overline{G}(x)^{a} = 1-(1-\delta)-\sum_{k=0}^{\infty}(-1)^{k}\binom{a}{k}G(x)^{k} = \sum_{k=0}^{\infty}\beta_k G(x)^{k}, \tag{10}$$

where $\beta_0 = \delta$ and $\beta_k = (1-\delta)(-1)^{1+k}\binom{a}{k}$. Using (9) and (10), the cdf of the MOG-G family can be expressed as

$$F(x) = \frac{\sum_{k=0}^{\infty}\alpha_k G(x)^{k}}{\sum_{k=0}^{\infty}\beta_k G(x)^{k}} = \sum_{k=0}^{\infty} t_k G(x)^{k},$$

where $t_0 = \frac{\alpha_0}{\beta_0}$ and for $k \geq 1$, we have

$$t_k = \frac{1}{\beta_0}\left(\alpha_k - \frac{1}{\beta_0}\sum_{r=1}^{k}\beta_r\; t_{k-r}\right).$$

The pdf of the MOG-G family can also be expressed as a mixture of exp-G densities. By differentiating $F(x)$, we obtain the same mixture representation

$$f(x) = \sum_{k=0}^{\infty} t_{k+1}\pi_{k+1}(x), \tag{11}$$

where $\pi_{k+1}(x) = (k+1)g(x;\psi)[G(x;\psi)]^{k}$ is the exp-G pdf with power parameter $k$.

### 4.1. *Moments*

The $r$th ordinary moment of $X$ is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx. \tag{12}$$

Using the series representation of $f(x)$ we obtain

$$\mu_r' = \sum_{j=0}^{\infty} t_{k+1} E\left(Y_{k+1}^r\right). \tag{13}$$

Henceforth, $Y_{k+1}$ denotes the exp-G distribution with power parameter $k+1$. The last integration can be computed numerically for most parent distributions. The $r$th order moment can be used to calculate the skewness and kurtosis. The effect of parameters $a$ and $\delta$ on skewness and kurtosis are displayed in Figures 2, 3 and 4, respectively. It can be observed that the skewness and kurtosis measures for all three distributions are highly influenced by these parameters.

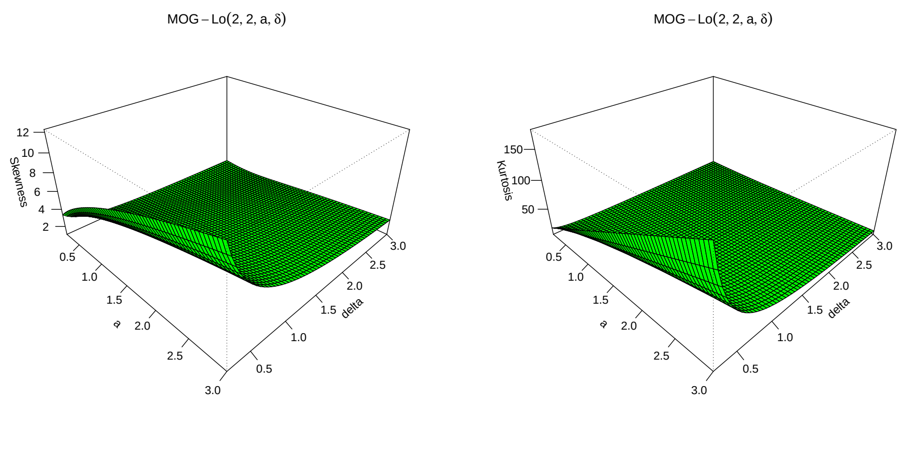


*Figure 2* – Skewness(left panel) and kurtosis (right panel) of MOG-Lo distribution.

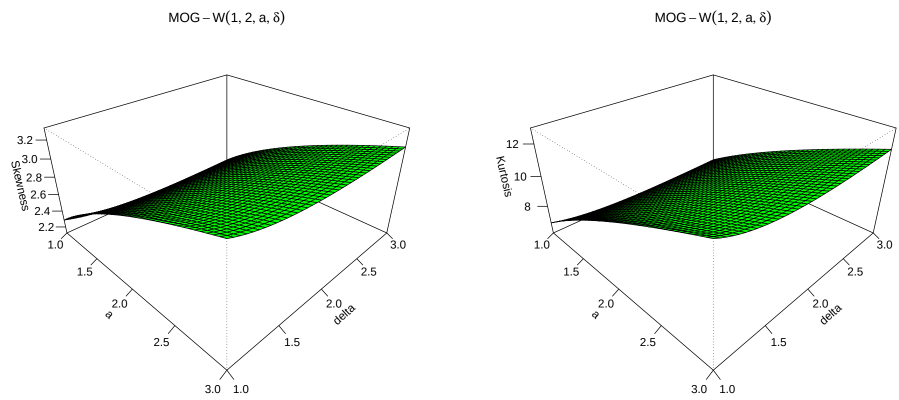


*Figure 3* – Skewness (left panel) and Kurtosis (right panel) of MOG-W distribution.

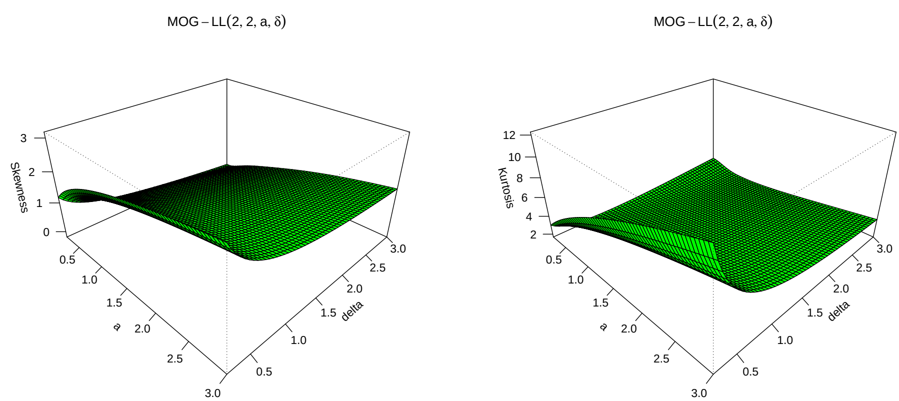


*Figure 4* – Skewness (left panel) and kurtosis (right panel) of MOG-LL distribution.

Similarly, the $s$th incomplete moment, say $\varphi_s(t)$, of $X$ can be expressed from (11) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\,dx = \sum_{k=0}^{\infty} t_{k+1} \int_{-\infty}^{t} x^s\, \pi_{k+1}(x)\,dx. \tag{14}$$

### 4.2. *Order statistics*

Suppose $X_1,\ldots,X_n$ is a random sample from a MOG-G distribution. Let $X_{i:n}$ denote the $i$th order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}F(x)^{j+i-1}. \tag{15}$$

Following similar algebraic developments of Nadarajah *et al.* (2015), we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x)=\sum_{r,k=0}^{\infty} b_{r,k}\ \pi_{r+k+1}(x), \tag{16}$$

where

$$b_{r,k}=\frac{n!\,(r+1)\,(i-1)!\,t_{r+1}}{(r+k+1)}\sum_{j=0}^{n-i}\frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!\,j!},$$

with $f_{j+i-1,k}$ defined recursively by $f_{j+i-1,0}=t_0^{j+i-1}$ and for $k\geq 1$

$$f_{j+i-1,k}=(k\,t_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]\,t_m f_{j+i-1,k-m}.$$

Observe that the pdf of the MOG-G order statistic is a combination of exp-G density functions. So, several mathematical quantities of the MOG-G order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the exp-G distribution.

## 5. CHARACTERIZATIONS

This section deals with various characterizations of the MOG-G distribution. These characterizations are based on: $(i)$ a simple relationship between two truncated moments; $(ii)$ the hazard function. It should be mentioned that for characterization $(i)$ the cdf need not have a closed form. We believe, due to the nature of the cdf of the MOG-G distribution, there may not be other possible characterizations than the ones presented in this section.

### 5.1. *Characterizations based on two truncated moments*

In this subsection, we present characterizations of the MOG-G distribution in terms of a simple relationship between two truncated moments. Our first characterization result borrows a theorem due to Glánzel (1987), see Theorem 1 below. Note that the result holds also when the interval $I$ is not closed. Moreover, as mentioned above, it could be also applied when the cdf $F$ does not have a closed form. As shown in Glánzel (1990), this characterization is stable in the sense of weak convergence. Again, by a continuous random variable, we mean the one whose cdf is a continuous function on $\mathbb{R}$.

THEOREM 1. *Let $(\Omega, \mathscr{F}, \boldsymbol{P})$ be a given probability space and let $I=[d,e]$ be an interval for some $d<e$ ($d=-\infty$, $e=\infty$ might as well be allowed). Let $X:\Omega\rightarrow I$ be a continuous random variable with the distribution function $F$ and let $q_1$ and $q_2$ be two real functions defined on $I$ such that*

$$\boldsymbol{E}[q_2(X)\mid X\geq x]=\boldsymbol{E}[q_1(X)\mid X\geq x]\eta(x),\quad x\in I$$

*is defined with some real function $\eta$. Assume that $q_1$, $q_2\in C^1(H)$, $\eta\in C^2(H)$ and $F$ is twice continuously differentiable and strictly monotone function on the set $I$. Finally, assume that the equation $q_1\eta=q_2$ has no real solution in the interior of $I$. Then $F$ is uniquely determined by the functions $q_1$, $q_2$ and $\eta$, particularly*

$$F(x)=\int_d^x C\left|\frac{\eta'(u)}{\eta(u)q_1(u)-q_2(u)}\right|\exp(-s(u))\,du,$$

*where the function $s$ is a solution of the differential equation $s'=\frac{\eta' q_1}{\eta q_1-q_2}$ and $C$ is the normalization constant, such that $\int_I dF=1$.*

PROPOSITION 2. *Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)=\left[1-\overline{\delta G}(x;\psi)^a\right]^2$ and $q_2(x)=q_1(x)\overline{G}(x;\psi)$ for $x\in\mathbb{R}$. The random variable $X$ belongs to MOG-G family* (6) *if and only if the function $\eta$ defined in Theorem 1 has the form*

$$\eta(x)=\frac{a}{a+1}\overline{G}(x;\psi),\quad x\in\mathbb{R}.$$

PROOF. Let $X$ be a random variable with density (6), then

$$(1-F(x))E[q_1(x)\mid X\geq x]=\delta\overline{G}(x;\psi)^a,\ \ x\in\mathbb{R},$$

and

$$(1-F(x))E[q_2(x)\mid X\geq x]=\frac{a\delta}{a+1}\overline{G}(x;\psi)^{a+1},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)q_1(x)-q_2(x)=-\frac{1}{a+1}q_1(x)\overline{G}(x;\psi)<0 \text{ for } x\in\mathbb{R}.$$

Conversely, if $\eta$ is given as above, then

$$s'(x)=\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x)-q_2(x)}=\frac{a\,g(x;\psi)}{\overline{G}(x;\psi)},\ x\in\mathbb{R},$$

and hence

$$s(x) = -\ln\left\{\overline{G}(x;\psi)^{a}\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem 1, $X$ has density (6). □

COROLLARY 3. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 2. The pdf of X is* (6) *if and only if there exist functions $q_2$ and $\eta$ defined in Theorem 1 satisfying the differential equation*

$$\frac{\eta'(x)\, q_1(x)}{\eta(x)\, q_1(x) - q_2(x)} = \frac{a\, g(x;\psi)}{\overline{G}(x;\psi)}, \quad x \in \mathbb{R}.$$

*The general solution of the differential equation in Corollary 3 is*

$$\eta(x) = \overline{G}(x;\psi)^{-a}\left[-\int a\, g(x;\psi)\,\overline{G}(x;\psi)^{a-1}\,(q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

*where $D$ is a constant. Note that a set of functions satisfying the differential equation in Corollary 3, is given in 2 with $D = 0$. However, it should be also noted that there are other triplets $(q_1, q_2, \eta)$ satisfying the conditions of Theorem 1.*

### 5.2. *Characterization based on hazard function*

It is known that the hazard function, $h_F$, of a twice differentiable distribution function, $F$, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization for the MOG-G distribution. This characterization produces a bridge between two fields of probability and differential equations.

PROPOSITION 4. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is* (6) *if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h_F'(x) - a\left(\frac{g'(x;\psi)}{g(x;\psi)}\right) h_F(x) = \frac{g(x;\psi)^2\left\{1 - \overline{\delta}(a+1)\,\overline{G}(x;\psi)^{a}\right\}}{\left\{\overline{G}(x;\psi)\left[1 - \overline{\delta}\,\overline{G}(x;\psi)^{a}\right]\right\}^2},$$

*with the boundary condition* $\lim_{x\to 0^+} h_F(x) = \frac{a}{\delta}\lim_{x\to 0^+} g(x;\psi)$.

PROOF. If $X$ has pdf (6), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\frac{1}{g(x;\psi)}h_F(x)\right\} = a\frac{d}{dx}\left\{\overline{G}(x;\psi)\left[1-\overline{\delta}\,\overline{G}(x;\psi)^a\right]\right\}^{-1},$$

or, equivalently,

$$h_F(x) = \frac{a\,g(x;\psi)}{\left\{\overline{G}(x;\psi)\left[1-\overline{\delta}\,\overline{G}(x;\psi)^a\right]\right\}} = \frac{a\,g(x;\psi)}{\overline{G}(x;\psi)} + \frac{\overline{\delta}a\,g(x;\psi)\overline{G}(x;\psi)^{a-1}}{1-\overline{\delta}\,\overline{G}(x;\psi)^a}.$$

Integrating both sides of the above equation, we arrive at

$$\ln[1-F(x)] = \ln\left[\frac{\delta\overline{G}(x;\psi)^a}{1-\overline{\delta}\,\overline{G}(x;\psi)^a}\right]$$

or

$$1-F(x) = \frac{\delta\overline{G}(x;\psi)^a}{1-\overline{\delta}\,\overline{G}(x;\psi)^a}.$$ □

## 6. Parameter estimation and simulation

Let $x_1,\ldots,x_n$ be a random sample from the MOG-G family with parameters $\delta, a$ and $\psi$. Let $\Theta = (\delta, a, \psi^\intercal)^\intercal$ be the $(p+2)\times 1$ parameter vector, where $p$ is the number of parameters of the G- distribution. To estimate the parameters $\Theta$ using maximum likelihood method we express the log-likelihood function as

$$\ell = \ell(\Theta) = n\log\delta + n\log a + \sum_{i=1}^{n}\log g(x_i;\psi) + (a-1)\sum_{i=1}^{n}\log\overline{G}(x_i;\psi) - 2\sum_{i=1}^{n}\log p_i,$$

where $p_i = 1-(1-\delta)\overline{G}(x_i;\psi)^a$. The components of the score vector, $U(\Theta) = \frac{\partial\ell}{\partial\Theta} = \left(\frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial a}, \frac{\partial\ell}{\partial\psi}\right)^\intercal$, and the elements of $J(\Theta) = \left\{\frac{\partial^2\ell}{\partial r\,\partial s}\right\}$ can be derived routinely but are quite complicated to solve analytically. Therefore, some numerical methods should be adopted to estimate the parameters.

Usually, it is more efficient to obtain the MLEs by maximizing $\ell$ directly. We used the routine *optim* in the *R* software for direct numerical maximization of $\ell$. *optim* is based on a quasi-Newton algorithm. The initial values for numerical maximization can be determined by the method of moments. The simultaneous roots of these $(p+2)$ equations are determined by the routine *multiroot* in the *R* software. The *optim* routine always converged when the method of moments estimates are used as initial values. The method of re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

Now we present some simulation results using different models to assess the reliability of the MLEs. For illustration purpose, we first choose the MOG-W distribution. An ideal technique for simulating from the MOG-W distribution is the inversion method. We can simulate $X$ by

$$X = \frac{1}{\alpha}\left\{-\frac{1}{a}\ln\left[\frac{1-U}{1-(1-\delta)U}\right]\right\}^{\frac{1}{\beta}},$$

where $U$ has a uniform distribution on $(0,1)$. For different values of $\alpha,\beta,\delta$ and $a$, samples of sizes $n = 100, 200, 300, 500$ and 1000 were generated from the MOG-W distribution. We repeated the simulation $k = 1000$ times and calculated the mean and the root mean square errors (RMSEs). The empirical results obtained using the $R$ software are given in Table 1.

*TABLE 1*
*Empirical means and the RMSEs of the MOG-W distribution.*

| $n$ | $\alpha=1, \beta=2, \delta=1, a=2$ | | | | $\alpha=1.5, \beta=2.5, \delta=2, a=2.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\delta}$ | $\widehat{a}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\delta}$ | $\widehat{a}$ |
| 100 | 1.458 | 2.049 | 5.133 | 1.945 | 1.899 | 2.663 | 8.921 | 2.488 |
| | (3.981) | (0.489) | (34.698) | (0.257) | (2.802) | (0.761) | (37.458) | (2.130) |
| 200 | 1.239 | 1.995 | 3.032 | 1.960 | 1.666 | 2.520 | 4.392 | 2.425 |
| | (1.722) | (0.350) | (14.878) | (0.157) | (0.875) | (0.536) | (11.519) | (0.386) |
| 300 | 1.114 | 2.017 | 2.044 | 1.976 | 1.590 | 2.543 | 3.193 | 2.451 |
| | (1.222) | (0.286) | (11.434) | (0.130) | (0.650) | (0.437) | (8.138) | (0.237) |
| 500 | 1.028 | 2.010 | 1.366 | 1.998 | 1.576 | 2.495 | 3.059 | 2.465 |
| | (0.372) | (0.214) | (5.335) | (0.347) | (0.431) | (0.367) | (6.283) | (0.539) |
| 1000 | 1.007 | 1.200 | 1.056 | 1.993 | 1.519 | 2.507 | 2.242 | 2.479 |
| | (0.076) | (0.140) | (0.365) | (0.041) | (0.126) | (0.235) | (1.784) | (0.096) |

Similarly, we simulate MOG-LL random variable by

$$X = \alpha\left[\left(\frac{1-(1-\delta)U}{1-U}\right)^{1/a} - 1\right]^{1/\beta},$$

where $U$ has a uniform distribution on $(0,1)$. For different values of $\alpha,\beta,\delta$ and $a$, samples of sizes $n = 100, 200, 300, 400$ and 500 were generated from the MOGLL distribution. The corresponding results from 1,000 simulations are given in Table 2. It can be observed from the estimated parameters and the RMSEs that the maximum likelihood method works well to estimate the model parameters of both the MOG-W and MOG-LL distributions.

*TABLE 2*
*Empirical means and the RMSEs of the MOG-LL distribution.*

| $n$ | $\alpha=2.5, \beta=1.5, \delta=2, a=1.5$ | | | | $\alpha=2, \beta=2.5, \delta=2.5, a=1.5$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\delta}$ | $\widehat{a}$ | $\widehat{\alpha}$ | $\widehat{\beta}$ | $\widehat{\delta}$ | $\widehat{a}$ |
| 100 | 2.973 | 1.479 | 2.043 | 2.043 | 2.435 | 2.634 | 2.330 | 1.838 |
| | (1.365) | (0.504) | (0.849) | (0.978) | (1.114) | (0.522) | (1.068) | (1.073) |
| 200 | 2.841 | 1.462 | 1.983 | 1.592 | 2.227 | 2.537 | 2.368 | 1.672 |
| | (0.959) | (0.480) | (0.718) | (0.653) | (0.666) | (0.354) | (0.786) | (0.603) |
| 300 | 2.742 | 1.462 | 1.955 | 1.541 | 2.141 | 2.535 | 2.425 | 1.598 |
| | ( 0.781) | (0.464) | (0.675) | (0.582) | (0.515) | (0.283) | (0.718) | (0.453) |
| 400 | 2.742 | 1.435 | 1.956 | 1.535 | 2.118 | 2.506 | 2.395 | 1.578 |
| | (0.754) | (0.546) | (0.648) | (0.591) | (0.449) | (0.326) | (0.623) | (0.386) |
| 500 | 2.567 | 1.313 | 1.970 | 1.377 | 2.090 | 2.494 | 2.439 | 1.553 |
| | (0.491) | (0.856) | (0.431) | ( 0.944) | (0.367) | (0.344) | (0.465) | (0.459) |

## 7. Applications

In this section, we provide applications to two real data sets to illustrate the flexibility of the MOG-LL, MOG-Lo and MOG-W models presented in Section 3.

### *7.1. Cancer patient data*

The first data set describes the remission times (in months) of a random sample of 128 bladder cancer patients studied by Lee and Wang (2003). For these data, we compare the fit of the MOG-LL and MOG-Lo distributions with the following distributions: Kumaraswamy Lomax (KwLo) of Lemonte and Cordeiro (2013), generalized transmuted log-logistic (GT-LL) of Nofal *et al.* (2017), Kumaraswamy-log-logistic (KwLL)of Flor de Santana *et al.* (2012), transmuted complementary Weibull geometric (TCWG) of Afify *et al.* (2014), transmuted Weibull Lomax (TWLo) of Afify *et al.* (2015c), Kumaraswamy exponentiated Burr XII (KwEBXII) of Mead and Afify (2017), generalized inverse gamma of Mead (2015), beta exponentiated Burr XII (BEBXII) of Mead (2014), beta Fréchet (BFr) of Nadarajah and Gupta (2004) and exponentiated transmuted generalized Rayleigh (ETGR) of Afify *et al.* (2015b). The probability density functions of these distributions are provided in the Appendix. The estimated parameters of these distributions for the cancer data are provided in Table 3.

*TABLE 3*
*MLEs and their standard errors (in parentheses) for cancer data.*

| Model | Estimates | | | | |
|---|---|---|---|---|---|
| MOG-LL | $\widehat{\alpha}=2.325$ (0.005) | $\widehat{\beta}=1.058$ (0.067) | $\widehat{a}=2.159$ (0.002) | $\widehat{\delta}=17.646$ (3.091) | |
| MOG-Lo | $\widehat{\alpha}=1.184$ (0.022) | $\widehat{\beta}=2.053$ (0.246) | $\widehat{a}=1.927$ (0.036) | $\widehat{\delta}=23.647$ (15.619) | |
| KwLo | $\widehat{\alpha}=0.390$ (1.228) | $\widehat{\beta}=12.295$ (11.754) | $\widehat{a}=1.516$ (0.266) | $\widehat{b}=12.055$ (45.016) | |
| GT-LL | $\widehat{\alpha}=9.229$ (1.908) | $\widehat{\beta}=2.172$ (0.321) | $\widehat{\lambda}=-0.0002$ (0.048) | $\widehat{a}=0.585$ (0.160) | $\widehat{b}=0.001$ (0.087) |
| KwLL | $\widehat{\alpha}=4.658$ (13.163) | $\widehat{\beta}=0.298$ (0.167) | $\widehat{a}=7.866$ (4.493) | $\widehat{b}=112.881$ (243.364) | |
| TCWG | $\widehat{\alpha}=106.069$ (124.800) | $\widehat{\beta}=1.712$ (0.099) | $\widehat{\lambda}=0.217$ (0.610) | $\widehat{\gamma}=0.009$ (0.007) | |
| TWLo | $\widehat{\alpha}=0.201$ (0.180) | $\widehat{\beta}=5.495$ (5.401) | $\widehat{\lambda}=-0.001$ (0.505) | $\widehat{a}=10.571$ (21.344) | $\widehat{b}=1.519$ (0.297) |
| KwEBXII | $\widehat{a}=2.780$ (44.510) | $\widehat{b}=67.636$ (104.728) | $\widehat{c}=0.338$ (0.385) | $\widehat{\beta}=3.083$ (49.353) | $\widehat{k}=0.839$ (1.723) |
| BEBXII | $\widehat{a}=22.186$ (21.956) | $\widehat{b}=20.277$ (17.296) | $\widehat{c}=0.224$ (0.144) | $\widehat{\beta}=1.780$ (1.076) | $\widehat{k}=1.306$ (1.079) |
| GIG | $\widehat{a}=2.327$ (0.369) | $\widehat{b}=0.0002$ (0.0002) | $\widehat{c}=17.931$ (7.385) | $\widehat{\beta}=0.543$ (0.042) | $\widehat{k}=0.001$ (0.0003) |
| BFr | $\widehat{\alpha}=27.753$ (71.507) | $\widehat{\beta}=0.169$ (0.104) | $\widehat{a}=12.526$ (24.469) | $\widehat{b}=33.342$ (36.348) | |
| ETGR | $\widehat{\alpha}=7.376$ (5.389) | $\widehat{\beta}=0.047$ (0.004) | $\widehat{\lambda}=0.118$ (0.260) | $\widehat{\delta}=0.049$ (0.036) | |

### 7.2. *Failure times of aircraft windshield*

The second data set was studied by Murthy *et al.* (2004), which represents the failure times for a particular windshield device. For these data, we shall compare the fits of the MOG-Lo and MOG-W distributions with the following distributions: TCWG of Afify *et al.* (2014), McDonald Lomax (McLo), KwLL, ETGR, TWLo, Kumaraswamy Lomax (KwLo) of Lemonte and Cordeiro (2013), Kumaraswamy Weibull (KwW) of Cordeiro *et al.* (2010), McDonald Weibull (McW) of Cordeiro *et al.* (2014), beta Weibull (BW) of Lee *et al.* (2007) and transmuted Marshall-Olkin Fréchet (TMOFr) of Afify *et al.* (2015a). The probability density functions of these distributions are provided in the Appendix. The estimated parameters of these distributions for windshield data are provided in Table 4.

*TABLE 4*
*MLEs and standard errors (in parentheses) for failure times of windshield data.*

| Model | | | Estimates | | |
|---|---|---|---|---|---|
| MOG-Lo | $\widehat{\alpha}$= 21.120 (0.044) | $\widehat{\beta}$= 258.503 (47.232) | $\widehat{a}$= 17.682 (0.037) | $\widehat{\delta}$= 35.902 (15.627) | |
| MOG-W | $\widehat{\alpha}$= 0.734 (0.003) | $\widehat{\beta}$= 1.058 (0.034) | $\widehat{a}$= 1.754 (0.008) | $\widehat{\delta}$= 27.611 (3.346) | |
| TCWG | $\widehat{\alpha}$= 0.019 (0.061) | $\widehat{\beta}$= 0.960 (0.716) | $\widehat{\gamma}$= 1.408 (2.749) | $\widehat{\lambda}$= 0.665 (0.294) | |
| McLo | $\widehat{\alpha}$= 19.924 (71.621) | $\widehat{\beta}$= 75.661 (212.774) | $\widehat{\gamma}$= 12.417 (21.374) | $\widehat{a}$= 2.188 (0.483) | $\widehat{b}$= 119.175 (170.536) |
| KwLL | $\widehat{\alpha}$= 9.294 (4.481) | $\widehat{\beta}$= 4.636 (2.023) | $\widehat{a}$= 0.497 (0.201) | $\widehat{b}$= 14.501 (19.016) | |
| ETGR | $\widehat{\alpha}$= 0.034 (0.048) | $\widehat{\beta}$= 0.379 (0.025) | $\widehat{\lambda}$= −0.354 (0.815) | $\widehat{\delta}$= 26.430 (40.252) | |
| TWLo | $\widehat{\alpha}$= 8.052 (28.393) | $\widehat{\beta}$= 387.740 (1439.700) | $\widehat{\lambda}$= 0.673 (0.259) | $\widehat{a}$= 571.516 (1446) | $\widehat{b}$= 2.435 (0.189) |
| KwLo | $\widehat{\alpha}$= 5.277 (37.988) | $\widehat{\beta}$= 78.677 (799.338) | $\widehat{a}$= 2.615 (1.343) | $\widehat{b}$= 100.276 (404.095) | |
| KwW | $\widehat{\alpha}$= 14.433 (27.095) | $\widehat{\beta}$= 0.204 (0.042) | $\widehat{a}$= 34.660 (17.527) | $\widehat{b}$= 81.846 (52.014) | |
| McW | $\widehat{\alpha}$= 1.940 (1.011) | $\widehat{\beta}$= 0.306 (0.045) | $\widehat{a}$= 17.686 (6.222) | $\widehat{b}$= 33.639 (19.994) | $\widehat{c}$= 16.721 (9.622) |
| BW | $\widehat{\alpha}$= 1.360 (1.002) | $\widehat{\beta}$= 0.298 (0.060) | $\widehat{a}$= 34.180 (14.838) | $\widehat{b}$= 11.496 (6.730) | |
| TMOFr | $\widehat{\alpha}$= 200.747 (87.275) | $\widehat{\beta}$= 1.952 (0.125) | $\widehat{\sigma}$= 0.102 (0.017) | $\widehat{\lambda}$= −0.869 (0.101) | |

### 7.3. *Model comparisons*

In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion (AIC) consistent Akaike information criterion (CAIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC) and $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood. Further, we adopt the Anderson-Darling ($A^*$) and Cramér-von Mises ($W^*$) statistics in order to compare the fits of the two new models with other nested and non-nested models. Tables 5 and 6 list the values of these statistic for cancer patient data and windshield data, respectively.

In Table 5, we compare the fits of the MOG-LL and MOG-Lo distributions with the KwLo, GT-LL, KwLL, TCWG, TWLo, KwEBXII, BEBXII, GIG, BFr and ETGR models. We note that the MOG-LL and MOG-Lo models have the lowest values for goodness-of-fit statistics (for the cancer data) among the fitted models. So, the MOG-LL and MOG-Lo models could be chosen as the best models.

*TABLE 5*
*The statistics* $-2\widehat{\ell}$, $AIC$, $CAIC$, $HQIC$ ,$BIC$, $W^*$ *and* $A^*$ *for cancer data.*

| Model | Goodness of fit criteria | | | | | | |
|---|---|---|---|---|---|---|---|
| | $-2\widehat{\ell}$ | AIC | CAIC | HQIC | BIC | $W^*$ | $A^*$ |
| MOG-LL | 819.081 | 827.081 | 827.406 | 831.716 | 838.489 | 0.015 | 0.095 |
| MOG-Lo | 819.088 | 827.088 | 827.413 | 831.723 | 838.496 | 0.016 | 0.095 |
| KwLo | 819.873 | 827.873 | 828.198 | 832.508 | 839.281 | 0.028 | 0.186 |
| GT-LL | 819.398 | 829.398 | 829.89 | 835.192 | 843.658 | 0.016 | 0.106 |
| KwLL | 821.531 | 829.531 | 829.857 | 834.167 | 840.939 | 0.049 | 0.317 |
| TCWG | 821.995 | 829.995 | 830.320 | 834.63 | 841.403 | 0.043 | 0.306 |
| TWLo | 820.402 | 830.402 | 830.894 | 836.196 | 844.662 | 0.034 | 0.222 |
| KwEBXII | 821.651 | 831.651 | 832.143 | 837.445 | 845.911 | 0.048 | 0.320 |
| BEBXII | 831.268 | 841.268 | 841.760 | 847.062 | 855.528 | 0.134 | 0.900 |
| GIG | 829.824 | 839.824 | 840.316 | 845.618 | 854.085 | 0.410 | 2.618 |
| BFr | 834.965 | 842.965 | 843.290 | 847.600 | 854.373 | 0.168 | 1.121 |
| ETGR | 858.350 | 866.350 | 866.675 | 870.985 | 877.758 | 0.398 | 2.361 |

*TABLE 6*
*The statistics* $-2\widehat{\ell}$, $AIC$, $CAIC$, $HQIC$, $BIC$, $W^*$ *and* $A^*$ *for windshield data.*

| Model | Goodness of fit criteria | | | | | | |
|---|---|---|---|---|---|---|---|
| | $-2\widehat{\ell}$ | AIC | CAIC | HQIC | BIC | $W^*$ | $A^*$ |
| MOG-Lo | 256.54 | 264.54 | 265.047 | 268.449 | 274.264 | 0.065 | 0.509 |
| MOG-W | 256.491 | 264.491 | 264.997 | 268.399 | 274.214 | 0.067 | 0.519 |
| TCWG | 257.28 | 265.28 | 265.786 | 269.188 | 275.003 | 0.078 | 0.578 |
| McLo | 257.137 | 267.137 | 267.906 | 272.023 | 279.291 | 0.099 | 0.729 |
| KwLL | 259.287 | 267.287 | 267.793 | 271.195 | 277.010 | 0.079 | 0.706 |
| ETGR | 261.975 | 269.975 | 270.481 | 273.883 | 279.700 | 0.085 | 0.786 |
| TWLo | 261.743 | 271.743 | 272.513 | 276.629 | 283.897 | 0.085 | 0.786 |
| KwLo | 262.296 | 270.296 | 270.802 | 274.204 | 280.019 | 0.097 | 0.868 |
| KwW | 273.434 | 281.434 | 281.941 | 285.343 | 291.158 | 0.185 | 1.506 |
| McW | 273.899 | 283.899 | 284.669 | 288.785 | 296.053 | 0.199 | 1.591 |
| BW | 297.028 | 305.028 | 305.534 | 308.937 | 314.751 | 0.465 | 3.220 |
| TMOFr | 301.472 | 309.472 | 309.978 | 313.380 | 319.195 | 0.320 | 2.404 |

Similarly, in Table 6, we compare the fits of the MOG-Lo and MOG-W models with the TCWG, McLo, KwLL, ETGR, TWLo, KwLo, KwW, McW, BW and TMOFr models. The figures in this table reveal that the MOG-Lo and MOG-W models have the lowest values of goodness-of-fit statistics (for the windshield data) among all fitted models. So, the MOG-Lo and MOG-W distributions can be chosen as the best models.

The histogram and the estimated densities and cdfs of the cancer data and windshield data are displayed in Figures 5 and 6.

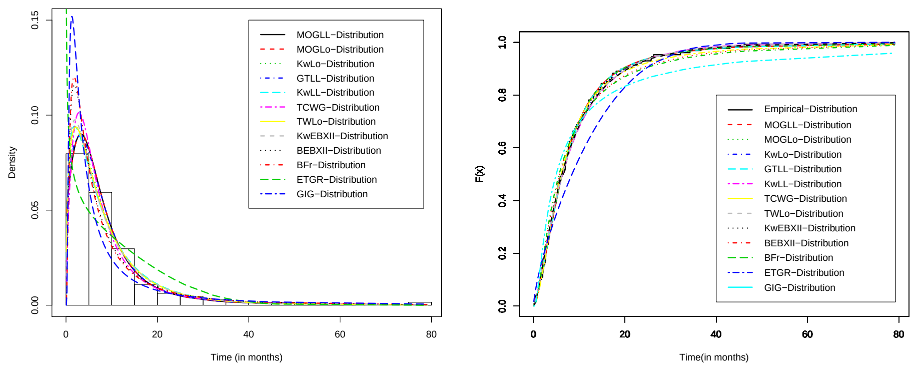


*Figure 5* – Fitted pdfs (left panel) and cdfs (right panel) for cancer data.

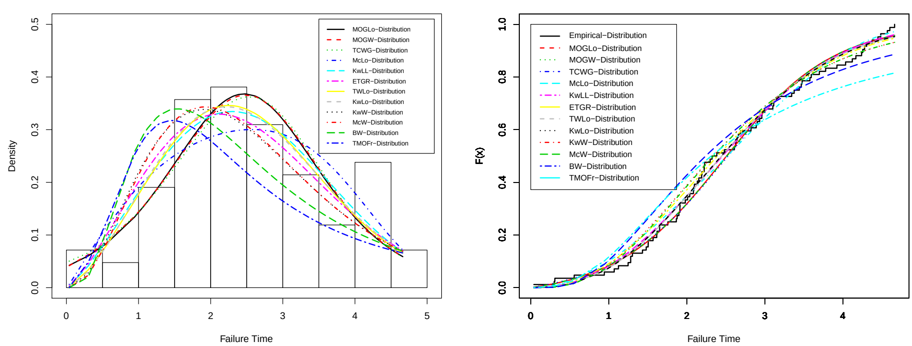


*Figure 6* – Fitted pdfs (left panel) and cdfs (right panel) for windshield data.

## 8. Conclusions

In this article, we have presented *Marshall-Olkin generalized-G* (MOG-G) family of distributions. Many well-known distributions emerge as special cases of the proposed family. We have provided some mathematical properties of the new family. The method of maximum likelihood estimation for model parameters has been investigated. By means of two real data sets, we have verified that special cases of the MOG-G family can provide better fits than other distributions generated from well-known families.

**Addendum**. *This work was completed in October 2015 and was first submitted for publication in November 2015 (not to STATISTICA). The article by Dias* et al. *(2016) was brought to our attention by one of the current reviewers, which generalizes the Marshall-Olkin family of distributions including ours. The properties and applications discussed in that paper, however, are different from ours. It seems to us that Dias* et al. *(2016) were not aware of our (2015) work, as we were not aware of their work before either.*

## Acknowledgements

The authors are grateful to the editor and anonymous reviewers for their constructive comments and valuable suggestions which certainly improved the presentation and quality of the paper.

## Appendix

Probability density functions of the distributions referenced in Section 7:

$$KwLo: \quad f(x) = \frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a-1} \times\left\{1-\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a}\right\}^{b-1}.$$

$$GTLL: \quad f(x) = \beta\alpha^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a-1} \times\left\{a(1+\lambda)-\lambda(a+b)\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{b}\right\}.$$

$$KwLL: \quad f(x) = \frac{ab\beta}{\alpha^{a\beta}}x^{a\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-a-1}\left\{1-\left[1-\frac{1}{1+\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{a}\right\}^{b-1}.$$

$$TCWG: \quad f(x) = \alpha\beta\gamma(\gamma x)^{\beta-1}\,e^{-(\gamma x)^{\beta}}\left[\alpha+(1-\alpha)\,e^{-(\gamma x)^{\beta}}\right]^{-3} \times\left[\alpha(1-\lambda)-(\alpha-\alpha\lambda-\lambda-1)\,e^{-(\gamma x)^{\beta}}\right].$$

$$TWLo: \quad f(x) = \frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{b\alpha-1}e^{-a\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}} \left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{b-1}\times\left\{1-\lambda+2\lambda e^{-a\left\{\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}}\right\}.$$

$$KwEBXII: \quad f(x) = \frac{a\,b\,ck\beta x^{c-1}}{(1+x^{c})^{k+1}}\left[1-(1+x^{c})^{-k}\right]^{a\beta-1} \times\left\{1-\left[1-(1+x^{c})^{-k}\right]^{a\beta}\right\}^{b-1}.$$

$$BEBXII: \quad f(x) = \frac{ck\beta}{B(a,b)}x^{c-1}(1+x^{c})^{-(k+1)}\left[1-(1+x^{c})^{-k}\right]^{a\beta-1} \times\left\{1-\left[1-(1+x^{c})^{-k}\right]^{\beta}\right\}^{b-1}.$$

$$GIG: \quad f(x) = \frac{\beta c^{a\beta}}{\Gamma_{b}(a,k)}x^{-(a\beta+1)}\left[\left(\frac{c}{x}\right)^{\beta}+k\right]^{-b}e^{-\left(\frac{c}{x}\right)^{\beta}}.$$

$$BFr: \quad f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}\,x^{-(\beta+1)}\,e^{-a\left(\frac{\alpha}{x}\right)^{\beta}}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{b-1}.$$

$$ETGR: \quad f(x) = 2\alpha\delta\beta^{2}\,x\,e^{-(\beta x)^{2}}\left\{1+\lambda-2\lambda\left[1-e^{-(\beta x)^{2}}\right]^{\alpha}\right\} \times\left[1-e^{-(\beta x)^{2}}\right]^{\alpha\delta-1}\left\{1+\lambda-\lambda\left[1-e^{-(\beta x)^{2}}\right]^{\alpha}\right\}^{\delta-1}.$$

$$McLo: \quad f(x) = \frac{\alpha\lambda}{\beta B(a\lambda^{-1},b)}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{a-1} \times\left\{1-\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}\right\}^{b-1}.$$

$$KwW: \quad f(x) = ab\beta\alpha^{\beta}x^{\beta-1}\, e^{-(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]^{a}\right\}^{b-1}.$$

$$McW: \quad f(x) = \frac{\beta c\alpha^{\beta}}{B(a/c,b)}\, x^{\beta-1}\, e^{-(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}\left\{1-\left[1-e^{-(\alpha x)^{\beta}}\right]^{c}\right\}^{b-1}.$$

$$BW: \quad f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}e^{-b(\alpha x)^{\beta}}\left[1-e^{-(\alpha x)^{\beta}}\right]^{a-1}.$$

$$TMOFr: \quad f(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\left[\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}\right]^{2}}\left[1+\lambda-\frac{2\lambda e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}{\alpha+(1-\alpha)e^{-\left(\frac{\sigma}{x}\right)^{\beta}}}\right].$$

The parameters of all distributions are positive real numbers except $|\lambda| \leq 1$.

## REFERENCES


A. Z. AFIFY, G. G. HAMEDANI, I. GHOSH, E. M. MEAD (2015a). *The transmuted Marshall-Olkin Fréchet distribution: Properties and applications*. International Journal of Probability and Statistics, 4, pp. 132–148.

A. Z. AFIFY, Z. M. NOFAL, N. S. BUTT (2014). *Transmuted complementary Weibull geometric distribution*. Pakistan Journal of Statistics and Operation Research, 10, pp. 435–454.

A. Z. AFIFY, Z. M. NOFAL, A. N. EBRAHEIM (2015b). *Exponentiated transmuted generalized Rayleigh: A new four parameter Rayleigh distribution*. Pakistan Journal of Statistics and Operation Research, 11, pp. 115–134.

A. Z. AFIFY, Z. M. NOFAL, H. M. YOUSOF, Y. M. EL GEBALY, N. S. BUTT (2015c). *The transmuted Weibull Lomax distribution: Properties and application*. Pakistan Journal of Statistics and Operation Research, 11, pp. 135–152.

M. ALIZADEH, G. M. CORDEIRO, E. BRITO, C. G. DEMÉTRIO (2015a). *The beta Marshall-Olkin family of distributions*. Journal of Statistical Distributions and Applications, 4, no. 2, pp. 1–18.

M. ALIZADEH, M. EMADI, M. DOOSTPARAST, G. M. CORDEIRO, E. ORTEGA, R. PESCIM (2015b). *A new family of distributions: The Kumaraswamy odd log-logistic, properties and applications*. Hacettepe Journal of Mathematics and Statistics, 44, pp. 1491–1512.

G. M. CORDEIRO, E. M. HASHIMOTO, E. M. ORTEGA (2014). *The McDonald Weibull model*. Statistics: A Journal of Theoretical and Applied Statistics, 48, pp. 256–278.

G. M. CORDEIRO, E. M. M. ORTEGA, S. NADARAJAH (2010). *The Kumaraswamy Weibull distribution with application to failure data*. Journal of the Franklin Institute, 349, pp. 1174–1197.

B. DIAS, G. M. CORDEIRO, M. ALIZADEH, P. R. D. MARINHO, H. F. C. COÊLHO (2016). *Exponentiated Marshall-Olkin family of distributions*. Journal of Statistical Distributions and Applications, 3, no. 15, pp. 1–21.

T. N. FLOR DE SANTANA, E. M. M. ORTEGA, G. M. CORDEIRO (2012). *Kumaraswamy log-logistic distribution*. Journal of Statistical Theory and Applications, 11, pp. 265–291.

W. GLÁNZEL (1987). *A characterization theorem based on truncated moments and its application to some distribution families*. In P. BAUER, F. KONECNY, W. WERTZ (eds.), *Mathematical Statistics and Probability Theory*, Springer, Dordrecht, pp. 75–84.

W. GLÁNZEL (1990). *Some consequences of a characterization theorem based on truncated moments*. Statistics: A Journal of Theoretical and Applied Statistics, 21, pp. 613–618.

R. C. GUPTA, P. L. GUPTA, R. D. GUPTA (1998). *Modeling failure time data by Lehmann alternatives*. Communications in Statistics - Theory and Methods, 27, pp. 887–904.

C. LEE, F. FAMOYE, O. OLUMOLADE (2007). *Beta-Weibull distribution: Some properties and applications to censored data*. Journal of Mordern Applied Statistical Methods, 6, no. 1, pp. 173–186.

E. T. LEE, J. W. WANG (2003). *Statistical Methods for Survival Data Analysis*. Wiley, New York.

A. LEMONTE, G. CORDEIRO (2013). *An extended Lomax distribution*. Statistics, 47, pp. 800–816.

A. W. MARSHALL, I. OLKIN (1997). *A new methods for adding a parameter to a family of distributions with application to the exponential and Weibull families*. Biometrika, 84, pp. 641–652.

M. MEAD (2014). *A new generalization of Burr XII distribution*. Journal of Statistics: Advances in Theory and Applications, 12, no. 2, pp. 53–73.

M. MEAD (2015). *Generalized inverse gamma distribution and its application in reliability*. Communications in Statistics - Theory and Methods, 44, pp. 1426–1435.

M. E. MEAD, A. Z. AFIFY (2017). *On five-parameter Burr XII distribution: Properties and applications*. South African Statistical Journal, 51, no. 1, pp. 67–80.

D. MURTHY, M. XIE, R. JIANG (2004). *Weibull Models*. John Wiley Sons, Hoboken, NJ.

S. NADARAJAH, G. CORDEIRO, E. ORTEGA (2015). *The Zografos-Balakrishnan-G family of distributions: Mathematical properties and applications*. Communications in Statistics - Theory and Methods, 44, pp. 186–215.

S. NADARAJAH, A. K. GUPTA (2004). *The beta Fréchet distribution*. Far East Journal of Theoretical Statistics, 14, pp. 15–24.

Z. M. NOFAL, A. Z. AFIFY, H. M. YOUSOF, G. M. CORDEIRO (2017). *The generalized transmuted-G family of distributions*. Communications in Statistics–Theory and Methods, 46, pp. 4119–4136.

SUMMARY

We introduce a new class of distributions called the Marshall-Olkin generalized-G family. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, order statistics are discussed. The maximum likelihood method is used for estimating the model parameters. The importance and flexibility of the new family are illustrated by means of two applications to real data sets.

*Keywords*: Marshall-Olkin distribution; Order statistics; Parameter estimation; Simulation.