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THE MARSHALL-OLKIN WEIBULL TRUNCATED NEGATIVE BINOMIAL DISTRIBUTION AND ITS APPLICATIONS

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1. Introduction

The Weibull distribution is considered as a versatile family of life distributions for modelling data, especially in reliability and survival analysis. Many extensions and modified forms of the Weibull distribution have been introduced and studied by various researchers, see Ghitany *et al.* (2005) and Tahir and Nadarajah (2015).
The probability density function (pdf) and cumulative distribution function (cdf) of Weibull distribution are respectively, given by

$$g(x;\beta) = \beta x^{\beta-1} e^{-x^{\beta}};\;\; x>0,\; \beta>0, \tag{1}$$

$$G(x;\beta) = 1 - e^{-x^{\beta}};\;\; x>0,\; \beta>0. \tag{2}$$

Marshall and Olkin (1997) introduced a method of including an extra shape parameter to a given baseline model having cdf G(x), thus defined an extended distribution with survival function

$$\bar{F}(x;\alpha) = \frac{\alpha\bar{G}(x)}{G(x)+\alpha\bar{G}(x)};\;\; -\infty<x<\infty,\; \alpha>0, \tag{3}$$

where $\bar{G}(x) = 1 - G(x)$.
Various authors studied several univariate distributions belonging to the family of Marshall -Olkin distributions, see Thomas and Jose (2003), Alice and Jose (2005), Ghitany *et al.* (2005), Ghitany *et al.* (2007), Jose *et al.* (2010) and Ristić and Kundu (2015). Jayakumar and Mathew (2008) introduced a generalization of the family of Marshall-Olkin distributions using Lehman alternative 1 approach. Sankaran and Jayakumar (2008) presented a detailed discussion on the physical interpretation of the Marshall-Olkin family,

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by considering proportional odds model. Tahir and Nadarajah (2015) proposed another generalization of the family of Marshall-Olkin distributions using Lehmann alternative 2 approach. Krishnan and George (2017) introduced a generalization of Marshall-Olkin Weibull distribution. Nadarajah *et al.* (2013) introduced a generalization of Marshall-Olkin family of distributions as follows.

Let $X_1, X_2,...$ be a sequence of independent and identically distributed (i.i.d) random variables with survival function $\bar{G}(x)$ and N be a truncated negative binomial random variable, independent of X_i's, with parameters $\gamma \epsilon (0,1)$ and $\theta > 0$, such that

$$P(N=n) = \frac{\gamma^\theta}{1-\gamma^\theta} \begin{pmatrix} \theta+n-1 \\ \theta-1 \end{pmatrix} (1-\gamma)^n; \;\; n=1,2,3,....$$

If $U_N = \min(X_1, X_2, ..., X_N)$, then the survival function of U_N is

$$\begin{aligned} \bar{F}(x;\alpha) &= \frac{\gamma^\theta}{1-\gamma^\theta} \sum_{n=0}^{\infty} \begin{pmatrix} \theta+n-1 \\ \theta-1 \end{pmatrix} ((1-\gamma)\bar{G}(x))^n \\ &= \frac{\gamma^\theta}{1-\gamma^\theta} [(G(x)+\gamma\bar{G}(x))^{-\theta}-1]; \quad x\epsilon\mathbf{R}. \end{aligned} \tag{4}$$

If $\gamma > 1$ and N is a truncated negative binomial random variable with parameters $\frac{1}{\gamma}$ and $\theta > 0$, then $V_N = \max(X_1, X_2, ..., X_N)$ has the same survival function given by Equation (4). Jayakumar and Sankaran (2016b) defined a generalized uniform distribution using the approach of Nadarajah *et al.* (2013). Babu (2016) introduced the Weibull truncated negative binomial distribution. Further, Jayakumar and Sankaran (2017) introduced the generalized exponential truncated negative binomial distribution and studied its properties .

The paper is outlined as follows. In Section 2, we introduce a new generalization of the Weibull distribution and discuss the shapes of density function and hazard rate function. Structural properties of the new distribution such as moments, quantile function, median, random number generation and entropies are derived in Section 3. The distribution of order statistics is investigated in Section 4. The maximum likelihood estimation of the model parameters are discussed in Section 5. In Section 6, we analyse a real data set to illustrate the usefulness of the proposed distribution. A first order autoregressive minification process with new distribution as marginal is developed in Section 7. Finally, concluding remarks are presented in Section 8.

2. MOWTNB DISTRIBUTION

Here, we introduce a new family of distribution, namely, the Marshall-Olkin Weibull truncated negative binomial (MOWTNB) distribution with four parameters $(\alpha, \beta, \gamma, \delta)$, obtained by substituting the survival function of the Marshall-Olkin Weibull distribution

$$\bar{G}(x) = \frac{\alpha e^{-x^\beta}}{1-(1-\alpha)e^{-x^\beta}}; \;\; x > 0, \alpha, \beta > 0,$$

in the family of distribution given by Nadarajah *et al.* (2013).
The new survival function thus obtained is

$$\bar{F}(x;\alpha,\beta,\gamma,\theta) = \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right], \tag{5}$$

for $x > 0$ and $\alpha, \beta, \gamma, \theta > 0$.
The cdf and pdf of MOWTNB$(\alpha, \beta, \gamma, \theta)$ are respectively given by

$$F(x;\alpha,\beta,\gamma,\theta) = \frac{1}{1-\gamma^\theta} - \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta\right] \tag{6}$$

and

$$f(x;\alpha,\beta,\gamma,\theta) = \frac{\alpha\beta(1-\gamma)\theta\gamma^\theta x^{\beta-1}e^{-x^\beta}}{1-\gamma^\theta}\frac{\left(1-(1-\alpha)e^{-x^\beta}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^\beta}\right)^{\theta+1}}, \tag{7}$$

for $x > 0$ and $\alpha, \beta, \gamma, \theta > 0$.
We can see that the MOWTNB family contains many distributions as special cases. The distributions that arise as special cases of the MOWTNB$(\alpha, \beta, \gamma, \theta)$ distribution are given below.

(i) When $\beta = 1$, the MOWTNB distribution reduces to the Marshall-Olkin exponential truncated negative binomial distribution.

(ii) When $\theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the Marshall-Olkin Weibull distribution.

(iii) When $\beta = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the Marshall-Olkin exponential distribution.

(iv) When $\beta = 2, \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the Marshall-Olkin Rayleigh distribution.

(v) When $\beta = \theta = 1, \gamma = 2$, the MOWTNB distribution reduces to the Marshall-Olkin half logistic distribution.

(vi) When $\alpha = 1$, the MOWTNB distribution reduces to the Weibull truncated negative binomial distribution.

(vii) When $\alpha = \beta = 1$, the MOWTNB distribution reduces to the exponential truncated negative binomial distribution.

(viii) When $\alpha = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the Weibull distribution.

(ix) When $\alpha = \beta = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the exponential distribution.

(x) When $\alpha = \theta = 1, \beta = 2, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to the Rayleigh distribution.

(xi) When $\alpha = \beta = \theta = 1, \gamma = 2$, the MOWTNB distribution reduces to the half logistic distribution.

Graphs of pdf for various values of parameters are presented in Figures 1 and 2.

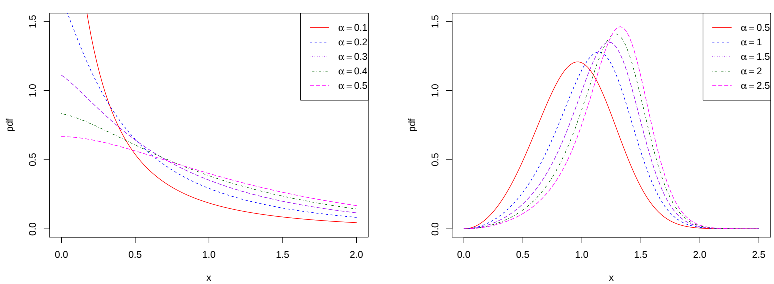

Figure 1 – Plots of pdf of MOWTNB($\alpha, \beta, \gamma, \theta$) when (i) $\beta = 1, \gamma = 2, \theta = 2$ (left) and (ii) $\beta = 3, \gamma = 2, \theta = 2$ (right), for various values of α.

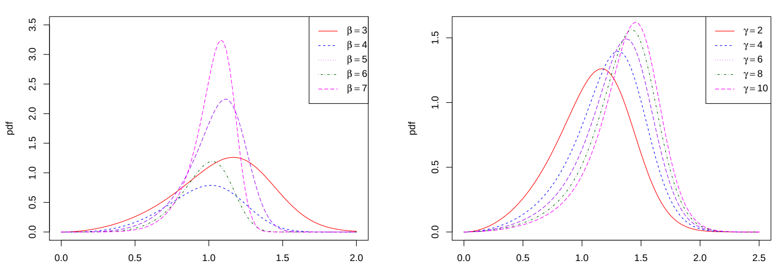

Figure 2 – Plots of pdf of MOWTNB($\alpha, \beta, \gamma, \theta$) when (i) $\alpha = 1.5, \gamma = 2, \theta = 1$ (left), for various values of β and (ii) $\alpha = 1.5, \beta = 3, \theta = 1$ (right), for various values of γ.

The hazard rate function and the reverse hazard rate function of the MOWTNB $(\alpha,\beta,\gamma,\theta)$ are given by

$$h(x)=\frac{\alpha\beta(1-\gamma)\theta x^{\beta-1}e^{-x^{\beta}}\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left[\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta}\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)-\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}\right]}$$

and

$$r(x)=\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{-x^{\beta}}\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left[\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}-\gamma^{\theta}\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta}\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)\right]},$$

respectively.

The graph of the hazard rate function is presented in Figure 3.

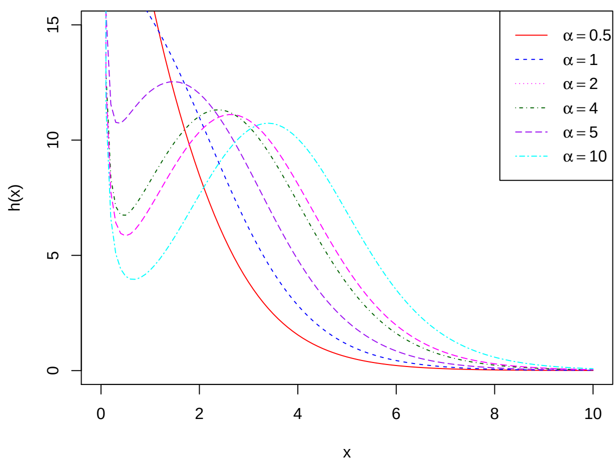

Figure 3 – Plots of hazard rate of MOWTNB$(\alpha,\beta,\gamma,\theta)$ when (i) $\beta=1,\gamma=10,\theta=20$ (left) and (ii) $\beta=1,\gamma=2,\theta=10$ (right), for various values of α.

3. Statistical properties

Moments are necessary and important in any statistical analysis, especially in applications. If X has MOWTNB $(\alpha,\beta,\gamma,\theta)$ distribution, then its r^{th} moment is given by

$$\begin{aligned}
\mu_r^{'} &= E[X^r] \\
&= \int_0^\infty \frac{\alpha\beta(1-\gamma)\theta\gamma^\theta x^{\beta-1}e^{-x^\beta}}{1-\gamma^\theta} \frac{\left(1-(1-\alpha)e^{-x^\beta}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^\beta}\right)^{\theta+1}} dx \\
&= \frac{\alpha(1-\gamma)\theta\gamma^\theta}{1-\gamma^\theta}\int_0^1 (-\log\ t)^{\frac{r}{\beta}} \frac{(1-(1-\alpha)t)^{\theta-1}}{(1-(1-\alpha\gamma)t)^{\theta+1}} dt,
\end{aligned}$$

by substituting $t = e^{-x^\beta}$.

Case 1.

If $|1-\alpha| < 1, |1-\alpha\gamma| < 1$, then

$$\begin{aligned}
\mu_r^{'} &= \frac{\alpha(1-\gamma)\gamma^\theta\Gamma(\theta+1)}{1-\gamma^\theta} \\
&\quad \sum_{i=0}^\infty\sum_{j=0}^\infty \binom{\theta+j}{\theta} \frac{(-1)^j(1-\alpha)^i(1-\alpha\gamma)^j}{\Gamma(\theta-i)i!} \int_0^1 (-\log\ t)^{\frac{r}{\beta}} t^{i+j} dt \\
&= \frac{\alpha(1-\gamma)\gamma^\theta\Gamma(\theta+1)\frac{r}{\beta}(\frac{r}{\beta}-1)...(\frac{r}{\beta}-(r-1))}{1-\gamma^\theta} \\
&\quad \sum_{i=0}^\infty\sum_{j=0}^\infty \binom{\theta+j}{\theta} \frac{(-1)^j(1-\alpha)^i(1-\alpha\gamma)^j}{\Gamma(\theta-i)i!(i+j+1)^{\frac{r}{\beta}+1}}.
\end{aligned}$$

Case 2. If $|1-\alpha| < \alpha, |1-\alpha\gamma| < \alpha$, by letting $t = 1-u$,

$$\begin{aligned}
\mu_r^{'} &= \frac{\alpha(1-\gamma)\theta\gamma^\theta}{1-\gamma^\theta}\int_0^1 (-\log\ (1-u))^{\frac{r}{\beta}} \frac{[1-(1-\alpha)(1-u)]^{\theta-1}}{[1-(1-\alpha\gamma)(1-u)]^{\theta+1}} du, \\
&= \frac{(1-\gamma)\theta}{\alpha\gamma(1-\gamma^\theta)}\int_0^1 (-\log\ (1-u))^{\frac{r}{\beta}} \frac{[1+(\frac{1-\alpha}{\alpha})u]^{\theta-1}}{[1+(\frac{1-\alpha\gamma}{\alpha\gamma})u]^{\theta+1}} du, \\
&= \frac{(1-\gamma)\Gamma(\theta+1)}{(1-\gamma^\theta)\alpha\gamma}\sum_{i=0}^\infty\sum_{j=0}^\infty \binom{\theta+j}{\theta} \frac{(-1)^j(1-\alpha)^i(1-\alpha\gamma)^j}{\alpha^{i+j}\gamma^j} \\
&\quad \int_0^1 (1-v)^{i+j}(-\log\ v)^{\frac{r}{\beta}} dv \\
&= \frac{(1-\gamma)\Gamma(\theta+1)\frac{r}{\beta}(\frac{r}{\beta}-1)...(\frac{r}{\beta}-(r-1))}{(1-\gamma^\theta)\alpha\gamma} \\
&\quad \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^{i+j} \binom{\theta+j}{\theta} \frac{(-1)^{j+k}(1-\alpha)^i(1-\alpha\gamma)^j(i+j-k+1)_k}{k!(k+1)^{\frac{r}{\beta}+1}}.
\end{aligned}$$

The mean, variance, skewness and kurtosis of the MOWTNB distribution for various values of parameters are presented in Table 1.

TABLE 1
Mean, Variance, Skewness and Kurtosis of MOWTNB($\alpha,\beta,\gamma,\theta$) distribution for different values of α and θ when $\beta=1,\gamma=2$.

α	θ	Mean	Variance	Skewness	Kurtosis
1.0	0.1	1.206	1.212	1.725	7.452
	0.2	1.226	1.230	1.703	7.340
	0.3	1.246	1.249	1.681	7.232
	0.4	1.265	1.266	1.660	7.128
	0.5	1.285	1.284	1.639	7.028
1.5	0.1	1.448	1.436	1.480	6.298
	0.2	1.470	1.455	1.461	6.215
	0.3	1.492	1.474	1.442	6.136
	0.4	1.514	1.492	1.423	6.061
	0.5	1.536	1.510	1.405	5.988
2.0	0.1	1.635	1.599	1.326	5.672
	0.2	1.658	1.618	1.309	5.606
	0.3	1.682	1.637	1.291	5.543
	0.4	1.705	1.655	1.274	5.483
	0.5	1.729	1.673	1.258	5.425
2.5	0.1	1.788	1.725	1.217	5.275
	0.2	1.813	1.744	1.200	5.220
	0.3	1.838	1.762	1.184	5.167
	0.4	1.862	1.780	1.168	5.117
	0.5	1.887	1.798	1.153	5.069
3.0	0.1	1.919	1.827	1.133	4.999
	0.2	1.944	1.846	1.118	4.952
	0.3	1.970	1.864	1.102	4.906
	0.4	1.996	1.881	1.088	4.863
	0.5	2.021	1.898	1.073	4.821

The quantile function of the random variable X, is obtained by inverting Equation (6) as

$$x_q = F^{-1}(q) = \left[\log\left(\frac{\gamma(1-\alpha)-(1-\alpha\gamma)(1-q(1-\gamma^{\theta}))^{\frac{1}{\theta}}}{\gamma-(1-q(1-\gamma^{\theta}))^{\frac{1}{\theta}}}\right)\right]^{\frac{1}{\beta}},$$

where $X \sim MOWTNB(\alpha,\beta,\gamma,\theta)$ and $F^{-1}(.)$ is the inverse function.

In particular, the median is given by

$$X=\left[\log\left(\frac{2^{\frac{1}{\theta}}\gamma(1-\alpha)-(1-\alpha\gamma)(1+\gamma^{\theta})^{\frac{1}{\theta}}}{2^{\frac{1}{\theta}}\gamma-(1+\gamma^{\theta})^{\frac{1}{\theta}}}\right)\right]^{\frac{1}{\beta}}.$$

A random sample X with MOWTNB$(\alpha,\beta,\gamma,\theta)$ distribution can be simulated as

$$X=\left[\log\left(\frac{\gamma(1-\alpha)-(1-\alpha\gamma)(1-Y(1-\gamma^{\theta}))^{\frac{1}{\theta}}}{\gamma-[1-Y(1-\gamma^{\theta})]^{\frac{1}{\theta}}}\right)\right]^{\frac{1}{\beta}}.$$

where Y∼U(0,1).

The entropy is the measure of variation or the uncertainty of a random variable X for the pdf from the lifetime distribution. The Rényi entropy of a random variable X with pdf f(.) is defined as

$$I_R(\eta)=\frac{1}{1-\eta}\log\int_0^{\infty}f^{\eta}(x)dx;\eta>0,\eta\neq 1.$$

The Rényi entropy of MOWTNB$(\alpha,\beta,\gamma,\theta)$ is

$$I_R(\eta)=\frac{1}{1-\eta}\log\int_0^{\infty}\left[\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{-x^{\beta}}}{1-\gamma^{\theta}}\frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}}\right]^{\eta}dx.$$

By letting $t=e^{-x^{\beta}}$, the above integral reduces to

$$\begin{aligned} I_R(\eta) &= \frac{1}{1-\eta}\log\left[\frac{1}{\beta}\left(\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}}{1-\gamma^{\theta}}\right)^{\eta}\right]+ \\ & \frac{1}{1-\eta}\log\left[\int_0^1 t^{\eta-1}(-\log t)^{(\eta-1)(\frac{\beta-1}{\beta})}\left[\frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}}\right]^{\eta}dx\right]. \end{aligned}$$

The Shannon entropy of X∼ MOWTNB$(\alpha,\beta,\gamma,\theta)$ is given by

$$\begin{aligned} E[-\log f(X)] &= E\left[-\log\left(\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}X^{\beta-1}e^{-X^{\beta}}}{1-\gamma^{\theta}}\frac{\left(1-(1-\alpha)e^{-X^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-X^{\beta}}\right)^{\theta+1}}\right)\right] \\ &= \log\left[\frac{1-\gamma^{\theta}}{\alpha\beta(1-\gamma)\theta\gamma^{\theta}}\right]-(\beta-1)E[\log X]+E[X^{\beta}] \\ & -(\theta-1)E[\log(1-(1-\alpha)e^{-X^{\beta}})] \\ & +(\theta+1)E[\log(1-(1-\alpha\gamma)e^{-X^{\beta}})]. \end{aligned}$$

4. Order statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from MOWTNB $(\alpha, \beta, \gamma, \theta)$ and let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the corresponding order statistics. Then, the pdf of i^{th} order statistic is

$$
\begin{aligned}
f_{X_{(i)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n!}{(i-1)!(n-i)!}\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}}{(1-\gamma^{\theta})e^{x^{\beta}}}\frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}} \\
&\quad \left[\frac{1}{1-\gamma^{\theta}}-\frac{\gamma^{\theta}}{1-\gamma^{\theta}}\left[\left(\frac{1-(1-\alpha)e^{-x^{\beta}}}{1-(1-\alpha\gamma)e^{-x^{\beta}}}\right)^{\theta}\right]\right]^{i-1} \\
&\quad \left[\frac{\gamma^{\theta}}{1-\gamma^{\theta}}\left[\left(\frac{1-(1-\alpha)e^{-x^{\beta}}}{1-(1-\alpha\gamma)e^{-x^{\beta}}}\right)^{\theta}-1\right]\right]^{n-i} \\
&= \frac{n!}{(i-1)!(n-i)!}\frac{\alpha\beta\theta\gamma^{(n+1-i)\theta}(1-\gamma)}{(1-\gamma^{\theta})^{n}}x^{\beta-1}e^{-x^{\beta}} \\
&\quad (1-(1-\alpha)e^{-x^{\beta}})^{\theta-1} \\
&\quad \left[(1-(1-\alpha\gamma)e^{-x^{\beta}})^{\theta}-\gamma^{\theta}(1-(1-\alpha)e^{-x^{\beta}})^{\theta}\right]^{i-1} \\
&\quad \frac{\left[(1-(1-\alpha)e^{-x^{\beta}})^{\theta}-(1-(1-\alpha\gamma)e^{-x^{\beta}})^{\theta}\right]^{n-i}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

The pdf of the largest order statistic, $X_{(n)}$, is

$$
\begin{aligned}
f_{X_{(n)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{x^{-\beta}}\left[1-(1-\alpha)e^{-x^{\beta}})\right]^{\theta-1}}{(1-\gamma^{\theta)^{n}}} \\
&\quad \frac{\left[\left((1-(1-\alpha\gamma)e^{-x^{\beta}})\right)^{\theta}-\gamma^{\theta}\left((1-(1-\alpha)e^{-x^{\beta}})\right)^{\theta}\right]^{n-1}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

The pdf of the smallest order statistic, $X_{(1)}$, is

$$
\begin{aligned}
f_{X_{(1)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{x^{-\beta}}\left[1-(1-\alpha)e^{-x^{\beta}})\right]^{\theta-1}}{(1-\gamma^{\theta)^{n}}} \\
&\quad \frac{\left[\left((1-(1-\alpha)e^{-x^{\beta}})\right)^{\theta}-\left((1-(1-\alpha\gamma)e^{-x^{\beta}})\right)^{\theta}\right]^{n-1}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

5. Estimation of parameters

The maximum likelihood estimators (MLEs) for the parameters of the MOWTNB distribution are discussed in this section.
Let $x_1, x_2, ..., x_n$ be an observed random sample from MOWTNB$(\alpha, \beta, \gamma, \theta)$ with unknown parameter vector $\nu = (\alpha, \beta, \gamma, \theta)^T$. Then, the likelihood function is

$$\begin{aligned}\ell(x;\nu) &= \prod_{i=1}^{n} f(x;\alpha,\beta,\gamma,\theta)\\ &= \frac{\alpha^n\beta^n\theta^n\gamma^{n\theta}(1-\gamma)^n}{(1-\gamma^\theta)^n}\frac{(\prod_{i=1}^n x_i)^{\beta-1}e^{-\sum x_i^\beta}\left[\prod_{i=1}^n(1-(1-\alpha)e^{-x_i^\beta})\right]^{\theta-1}}{\left[\prod_{i=1}^n(1-(1-\alpha\gamma)e^{-x_i^\beta})\right]^{\theta+1}},\end{aligned}$$

so that the log-likelihood function becomes

$$\begin{aligned}\log\,\ell &= n\log\alpha + n\log\beta + n\log\theta + n\log(1-\gamma) + n\theta\log\gamma - n\log(1-\gamma^\theta)\\ &+(\beta-1)\sum_{i=1}^n \log x_i - \sum_{i=1}^n x_i^\beta + (\theta-1)\sum_{i=1}^n \log(1-(1-\alpha)e^{-x_i^\beta})\\ &-(\theta+1)\sum_{i=1}^n \log(1-(1-\alpha\gamma)e^{-x_i^\beta}).\end{aligned}$$

Hence the score vector is

$$U(\nu) = \left(\frac{\partial \log\ell}{\partial\alpha}, \frac{\partial \log\ell}{\partial\beta}, \frac{\partial \log\ell}{\partial\gamma}, \frac{\partial \log\ell}{\partial\theta}\right)^T.$$

The partial derivatives of log-likelihood function with respect to the parameters are

$$\begin{aligned}\frac{\partial \log\,\ell}{\partial\alpha} &= \frac{n}{\alpha} + (\theta-1)\sum_{i=1}^n \frac{e^{-x_i^\beta}}{(1-(1-\alpha)e^{-x_i^\beta})} - (\theta+1)\sum_{i=1}^n \frac{\gamma e^{-x_i^\beta}}{(1-(1-\alpha\gamma)e^{-x_i^\beta})},\\ \frac{\partial \log\,\ell}{\partial\beta} &= \frac{n}{\beta} + \sum_{i=1}^n \log x_i - \sum_{i=1}^n x_i^\beta \log x_i + (\theta-1)\sum_{i=1}^n \frac{(1-\alpha)x_i^\beta e^{-x_i^\beta}\log x_i}{(1-(1-\alpha)e^{-x_i^\beta})}\\ &-(\theta+1)\sum_{i=1}^n \frac{(1-\alpha\gamma)x_i^\beta e^{-x_i^\beta}\log x_i}{(1-(1-\alpha\gamma)e^{-x_i^\beta})},\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \log \ell}{\partial \gamma} &= \frac{-n}{1-\gamma}+\frac{n\theta}{\gamma}+\frac{n\theta\gamma^{\theta-1}}{1-\gamma^{\theta}}-(\theta+1)\sum_{i=1}^{n}\frac{\alpha e^{-x_i^{\beta}}}{1-(1-\alpha\gamma)e^{-x_i^{\beta}}},\\
\frac{\partial \log \ell}{\partial \theta} &= n\log\gamma+\frac{n}{\theta}+\frac{n\gamma^{\theta}\log\gamma}{1-\gamma^{\theta}}+\sum_{i=1}^{n}\log(1-(1-\alpha)e^{-x_i^{\beta}})\\
&\quad -\sum_{i=1}^{n}\log(1-(1-\alpha\gamma)e^{-x_i^{\beta}}).
\end{aligned}
$$

We can find the estimates of the unknown parameters by setting the score vector equal to zero, $U(\nu)=0$ and solving them simultaneously to obtain the ML estimators $\hat{\alpha},\hat{\beta},\hat{\gamma}$ and $\hat{\theta}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm. For the four-parameter MOWTNB distribution, all the second order derivatives exist.

6. Data analysis

The data set represents failure time of 50 items reported in Aarset (1987), as shown in Table 2. Recently, Elbatal and Aryal (2013), Elbatal *et al.* (2016), Jayakumar and Sankaran (2016a) analysed this data using transmuted additive Weibull distribution, the additive Weibull geometric distribution and discrete Mittag-Leffler additive Weibull distribution, respectively. We compare the results of MOWTNB distribution with those of the following distributions:

(a) Weibull (W) with pdf

$$f(x;\beta,\lambda)=\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}};\ \ \beta,\lambda>0,$$

(b) exponential (E) with pdf

$$f(x;\alpha)=\alpha e^{-\alpha x};\ \ \alpha>0,$$

(c) gamma (G) with pdf

$$f(x;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma\alpha}x^{\alpha-1}e^{-\beta x};\ \ \alpha,\beta>0,$$

(d) generalized exponential (GE) with pdf

$$f(x;\alpha,\lambda)=\alpha\lambda(1-e^{-\lambda x})^{\alpha-1};\ \ \alpha,\lambda>0,$$

(e) Marshall-Olkin Weibull (MOW) with pdf

$$f(x;\alpha,\lambda,\beta)=\frac{\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{(1-(1-\alpha)e^{-(\lambda x)^{\beta}})^{2}};\ \alpha,\beta,\lambda>0,$$

(f) Marshall-Olkin exponential (MOE) with pdf

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda e^{-\lambda x}}{(1-(1-\alpha)e^{-\lambda x})^{2}};\ \alpha,\lambda>0,$$

(g) exponentiated Weibull (EW) with pdf

$$f(x;\gamma,\lambda,\beta)=\beta\gamma\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}(1-e^{-(\lambda x)^{\beta}})^{\gamma-1};\ \gamma,\lambda,\beta>0,$$

(h) exponentiated Weibull geometric (EWG) with pdf

$$f(x;\alpha,\beta,\gamma,p)=\frac{\alpha\gamma\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}(1-e^{-(\beta x)^{\alpha}})^{\gamma-1}}{(1-p(1-e^{-(\beta x)^{\alpha}})^{\gamma})^{2}};\ \alpha,\beta,\gamma>0,p\epsilon(0,1),$$

(i) Marshall-Olkin additive Weibull (MOAW) with pdf

$$f(x;\theta,\alpha,\beta,p)=\frac{p(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1})e^{-(\alpha x^{\theta}+\gamma x^{\beta})}}{[p+(1-p)(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})})]^{2}};\ \alpha,\theta,\gamma,\beta,p>0,$$

(j) Weibull truncated negative binomial (WTNB) with pdf

$$f(x;\alpha,\theta,\beta,\lambda)=\frac{(1-\alpha)\theta\alpha^{\theta}\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{(1-\alpha^{\theta})(1-(1-\alpha)e^{-(\lambda x)^{\beta}})^{\theta+1}};\ \alpha,\theta,\beta,\lambda>0.$$

Also, for comparison purpose, we consider the scale parameter $\lambda>0$ so that the corresponding pdf of MOWTNB $(\alpha,\beta,\gamma,\theta,\lambda)$ is given by

$$f(x;\alpha,\beta,\gamma,\theta)=\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{1-\gamma^{\theta}}\frac{\left(1-(1-\alpha)e^{-(\lambda x)^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-(\lambda x)^{\beta}}\right)^{\theta+1}}.$$

TABLE 2
Aarset data.

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 3 provides some descriptive statistics of the data.

TABLE 3
Descriptive Statistics of Aarset data.

n	Min	Max	Mean	Median	Var	Q_1	Q_3
50	0.10	86.00	45.69	48.50	1078.15	13.50	81.25

Plots of the estimated pdf of the MOWTNB model fitted to this data is given in Figure 4.

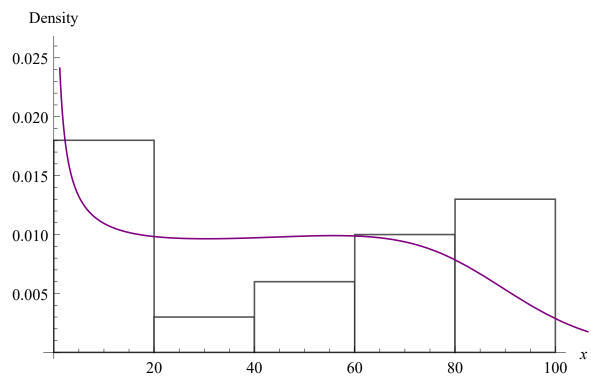

Figure 4 – Histogram and fitted pdf for the Aarset data.

The MLEs of the model parameters and numerical values of log $\hat{\ell}$, AIC, AICC, BIC, HQIC and K-S statistic are presented in Table 4. Since the values of $-\log\hat{\ell}$, AIC, AICC, BIC, HQIC and K-S are smaller for the MOWTNB distribution compared with those values of the other models such as W, E, G, GE, MOW, MOE, EW, EWG, MOAW and WTNB, the new distribution seems to be a very competitive model to this data.

TABLE 4
Parameter estimates and goodness of fit for various models fitted for Aarset data.

Model	Estimates	$-\log \hat{l}$	AIC	AICC	BIC	HQIC	K-S
MOWTNB	$\hat{\alpha}=0.902$, $\hat{\beta}=0.537$, $\hat{\gamma}=2138.54$, $\hat{\theta}=0.202$ $\hat{\lambda}=0.543$	231.5	472.9	474.3	482.5	476.6	0.07
EWG	$\hat{\gamma}=1.011$, $\hat{\beta}=0.026$, $\hat{\alpha}=1.069$, $\hat{p}=0.037$	234.8	477.7	478.6	485.3	480.5	0.14
MOAW	$\hat{\alpha}=0.001$, $\hat{\gamma}=0.016$, $\hat{\theta}=1.972$, $\hat{\beta}=0.658$, $\hat{p}=0.627$	235.1	480.1	481.5	489.7	483.8	0.16
MOW	$\hat{\lambda}=0.713$, $\hat{\beta}=0.068$, $\hat{\alpha}=6.373$	238.9	483.8	484.4	489.6	486.0	0.12
MOE	$\hat{\alpha}=2.622$, $\hat{\lambda}=0.033$,	239.6	483.1	483.4	486.9	484.6	0.12
GE	$\hat{\alpha}=0.780$, $\hat{\lambda}=0.019$	240.0	484.0	484.2	487.8	485.4	0.19
G	$\hat{\alpha}=0.799$, $\hat{\beta}=0.018$	240.2	484.4	484.6	488.2	485.8	0.58
W	$\hat{\beta}=0.948$, $\hat{\lambda}=0.022$	241.0	486.0	486.2	489.8	490.1	0.17
E	$\hat{\alpha}=0.022$	241.1	484.1	484.2	486.0	484.9	0.64
WTNB	$\hat{\alpha}=0.588$, $\hat{\theta}=0.452$, $\hat{\beta}=0.925$, $\hat{\lambda}=0.026$	245.2	498.4	499.3	506.1	501.3	0.37
EW	$\hat{\gamma}=8.070$, $\hat{\beta}=782.099$, $\hat{\lambda}=0.163$	261.1	528.1	528.6	533.8	530.4	0.92

7. Application in autoregressive time series modeling

Now, we develop a first order autoregressive (AR(1)) minification process with MOWTNB distribution as marginal distribution.
Consider an AR(1) minification process with structure

$$X_n = \begin{cases} \varepsilon_n & \text{w.p} \quad \rho \\ \min(X_{n-1}, \varepsilon_n) & \text{w.p} \quad 1-\rho \end{cases} \qquad 0 < \rho < 1; n \geq 1, \tag{8}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d random variables. In order to develop the time series model with MOWTNB marginal distribution, we need the following survival function. The Marshall-Olkin form of MOWTNB$(\alpha, \beta, \gamma, \theta)$ distribution with parameter ϑ has the survival function given by

$$\bar{F}_{\varepsilon_n}(x) = \frac{1}{1 + \frac{1}{\vartheta}\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta - \gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta - (1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}. \tag{9}$$

THEOREM 1. *The AR(1) process given by (8), defines a stationary AR(1) minification process with MOWTNB $(\alpha, \beta, \gamma, \theta)$ as marginal distribution if and only if ε_n's are i.i.d random variables having survival function (9) with $X_0 \stackrel{d}{=} MOWTNB(\alpha, \beta, \gamma, \theta)$.*

PROOF. We have, for MOWTNB$(\alpha, \beta, \gamma, \theta)$,

$$\begin{aligned} \bar{F}_X(x) &= \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right] \\ &= \frac{1}{1 + \left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta - \gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta - (1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}. \end{aligned}$$

The model (8) can be written in terms of survival function as

$$P(X_n > x) = P(\varepsilon_n > x)[\rho + (1-\rho)P(X_{n-1} > x)].$$

That is,

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)[\rho + (1-\rho)\bar{F}_{X_{n-1}}(x)]. \tag{10}$$

If $\{X_n\}$ is stationary with MOWTNB $(\alpha,\beta,\gamma,\theta)$ marginals, then

$$\begin{aligned}
\bar{F}_{\varepsilon_n}(x) &= \frac{\bar{F}_X(x)}{\rho+(1-\rho)\bar{F}_X(x)} \\
&= \frac{\frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta-1\right]}{\rho+(1-\rho)\left\{\frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta-1\right]\right\}} \\
&= \frac{1}{\rho\left[\frac{(1-\gamma^\theta)(1-(1-\alpha\gamma)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]+(1-\rho)} \\
&= \frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}.
\end{aligned}$$

That is, ε_n's are i.i.d random variables having survival function (9) with $\vartheta=\frac{1}{\rho}$.

Conversely, if ε_n's are i.i.d random variables having survival function (9) with $X_0 \stackrel{d}{=}$ MOWTNB $(\alpha,\beta,\gamma,\theta)$, then from (10), we have

$$\begin{aligned}
\bar{F}_{X_1}(x) &= \rho\bar{F}_{\varepsilon_1}(x)+(1-\rho)\bar{F}_{\varepsilon_1}(x)\bar{F}_{X_0}(x) \\
&= \rho\left[\frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right]+(1-\rho) \\
&\quad \left[\frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right] \\
&\quad \left[\frac{1}{1+\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right] \\
&= \frac{1}{1+\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}, \text{on simplification} \\
&= \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta-1\right].
\end{aligned}$$

That is, $X_1 \overset{d}{=}$ MOWTNB $(\alpha, \beta, \gamma, \theta)$. If we assume that $X_{n-1} \overset{d}{=}$ MOWTNB$(\alpha, \beta, \gamma, \theta)$, then by induction, we can establish that $X_n \overset{d}{=}$ MOWTNB $(\alpha, \beta, \gamma, \theta)$. Hence the process $\{X_n\}$ is stationary with MOWTNB marginals. □

8. CONCLUSION

In this article, we have introduced a four-parameter model, called the Marshall-Olkin Weibull truncated negative binomial (MOWTNB) distribution, which extends the truncated negative binomial distribution pioneered by Nadarajah *et al.* (2013) and some other well-known distributions including the Marshall-Olkin Weibull distribution. We have derived explicit expressions for the ordinary moments, median, quantile function, Rényi entropy and Shannon entropy. The distribution of order statistics were obtained. Further, the maximum likelihood estimation of the model parameters are discussed. The new distribution is applied to a real data set. The results, compared with other known distributions, revealed that the MOWTNB distribution provides a better fit than several other competitive Weibull and exponential models. We hope that the proposed model will attract wider application in areas such as engineering, lifetime data analysis, meteorology, hydrology, and economics.

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SUMMARY

The Weibull distribution is one of the widely known lifetime distribution that has been extensively used for modelling data in reliability and survival analysis. A generalization of both the Marshall-Olkin Weibull distribution and the Weibull truncated negative binomial distribution is introduced and studied in this article. Various distributional properties of the new distribution are derived. Estimation of model parameters using the method of maximum likelihood is discussed. Applications to a real data set is provided to show the flexibility and potentiality of the new distribution over other Weibull models. The first order autoregressive minification process with the new distribution as marginal is also developed. We hope that the new model will serve as a good alternative to other models available in the literature for modeling positive real data in several areas.

Keywords: Autoregressive model; Hazard rate; Marshall-Olkin distribution; Minification process; Renyi entropy; Shannon entropy; Weibull distribution.