STATISTICA, anno LXXVIII, n. 1, 2018

STATISTICAL INFERENCE FOR THE RELIABILITY FUNCTIONS OF A FAMILY OF LIFETIME DISTRIBUTIONS BASED ON PROGRESSIVE TYPE II RIGHT CENSORING

Ajit Chaturvedi
Department of Statistics, University of Delhi, Delhi 110007, India
Narendra Kumar
Department of Statistics, University of Delhi, Delhi 110007, India
Kapil Kumar [1]
Department of Statistics, Central University of Haryana, Mahendergarh 123031, India

1. Introduction

The lifetime experiments are usually time consuming and expensive in nature. To reduce the cost and time of experimentation, various types of censoring schemes are used in life testing experiments. The two most common censoring schemes in literature are type I and type II censoring schemes. But these censoring schemes do not allow intermediate removal of the experimental units from the experiment other than the final termination point, see Chaturvedi and Vyas (2017a), Chaturvedi and Vyas (2017b), etc. For this reason, a more general purpose censoring scheme known as progressive censoring scheme is considered. Recently, the progressive censoring scheme has received considerable attention in life testing and reliability studies. Progressive censoring scheme is a useful method for deriving inferential conclusions for data which arise from such experiments and it was first discussed by Cohen (1963). Recently it has become very popular in the reliability and life testing experiments. An excellent monograph on progressive censoring is given by Balakrishnan and Aggarwala (2000). Some recent studies on progressive censoring can be found in Krishna and Kumar (2011), Krishna and Kumar (2013), Rastogi and Tripathi (2014), Kumar *et al.* (2017), Valiollahi *et al.* (2018) and references cited therein.

The progressive type II right censoring scheme can briefly be described as follows: Let n units are placed on test at time zero. Immediately following the first failure, R_1 surviving units are removed from the test at random. Then, immediately following the

[1] Corresponding Author. E-mail: kapilstats@gmail.com

second observed failure, R_2 surviving units are removed from the test at random. This process continues until, at the time of the m^{th} observed failure, the remaining $R_m = n - R_1 - R_2 - \ldots - R_{m-1} - m$ units are all removed from the experiment. We denote the m completely observed failure times by $X_{i:m:n}$, $i = 1, 2, \ldots, m$. The joint pdf of $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ is given by, see, Balakrishnan and Aggarwala (2000)

$$f_{X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}}(x_1, x_2, \ldots, x_m) = c \prod_{j=1}^{m} f(x_j)\{1 - F(x_j)\}^{R_j}, \tag{1}$$

where, $c = n(n - R_1 - 1)(n - R_1 - R_2 - 2)\ldots(n - R_1 - \ldots - R_{m-1} - m + 1)$. Also, $f(x_j)$ and $F(x_j)$ are the probability density and distribution functions of $X_{j:m:n}$, respectively.

If the random variable (rv) X denotes the lifetime of an item or system, the reliability function is $R(t) = P(X > t)$ and the stress-strength reliability is the probability $P = P(X > X^*)$. The stress-strength reliability represents the reliability of an item or system of random strength X subject to random stress X^*. The point estimation and testing procedures of $R(t)$ and P have considerable attention in the statistics literature. Some early and recent work on reliability and stress-strength reliability can be found in Pugh (1963), Tong (1974), Kelly *et al.* (1976), Awad and Gharraf (1986), Tyagi and Bhattacharya (1989), Chaturvedi and Kumar (1999), Chaturvedi and Tomer (2002), Chaturvedi and Singh (2006) and others. Also, an excellent review on stress-strength reliability can be found in Kotz *et al.* (2003).

Let the rv X follows the distribution having the pdf

$$f(x; a, \lambda, \theta) = \lambda G'(x, a, \theta)\exp(-\lambda G(x; a, \theta)); \quad x > a \geq 0,\ \lambda > 0, \tag{2}$$

where, $G(x; a, \theta)$ is a function of x and may also depend on the parameters a and θ. θ may be vector-valued. Moreover, $G(x; a, \theta)$ is monotonically increasing in x with $G(a; a, \theta) = 0$, $G(\infty; a, \theta) = \infty$ and $G'(x; a, \theta)$ denotes the derivative of $G(x; a, \theta)$ with respect to x. The family of lifetime distributions given in (2) covers fourteen distributions as specific cases was proposed by Chaturvedi and Kumari (2015).

This article serves many fold purposes. The point estimation and testing procedures under progressive type II right censoring are developed. Uniformly minimum variance, maximum likelihood and invariantly optimal estimators are derived. A new technique of obtaining these estimators, in which estimators of powers of parameter are obtained, under progressive type II censoring is developed. These estimators are used to obtain estimators of $R(t)$. Using the derivatives of estimators of $R(t)$, the estimators of probability density function (pdf) (at a specified point) are obtained, which are subsequently used to obtain estimators of P. The estimators for P are derived for the cases when X and X^* belong to same and different families of distributions. The rest of the paper is organized as follows: in Section 2, the uniformly minimum variance unbiased estimators (UMVUE) of reliability function $R(t)$ and stress-stress reliability P are provided. The maximum likelihood estimators of $R(t)$ and P are derived in Section 3. In section 4, the invariantly optimal estimators of $R(t)$ and P are developed. Section 5 deals with the testing of hypotheses procedures. For comparing various estimators developed a Monte Carlo simulation study is carried out in Section 6. Section 7 deals with a real data analysis for illustration purposes. Finally, concluding remarks on this article are given in Section 8.

2. Uniformly minimum variance unbiased estimation

This section is devoted to UMVUE of powers of λ, $R(t)$ and P based on progressively type II censored sample $x_1, x_2, \ldots, x_m$ from model in (2). Now, from model (2), it is easy to see that

$$F(x_j;a,\lambda,\theta) = 1 - \exp(-\lambda G(x_j;a,\theta)), \quad j = 1,2,\ldots,m.$$

Thus, denoting by $Y_j = G(x_j;a,\theta)$,

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(x_1,x_2,\ldots,x_m) = c\lambda^m \exp\left(-\lambda\sum_{j=1}^{m}(R_j+1)Y_j\right). \tag{3}$$

Let us consider the transformations

$$Z_1 = nY_1,\ Z_2 = (n-R_1-1)(Y_2-Y_1),\ Z_3 = (n-R_1-R_2-2)(Y_3-Y_2),\ldots,$$
$$Z_m = (n-R_1-\ldots-R_{m-1}-m+1)(Y_m-Y_{m-1}).$$

The Jacobian of transformation is c^{-1}. Z_j's are exponential with mean life λ^{-1} and the distribution of $S_m = \sum_{j=1}^{m}(R_j+1)Y_j = \sum_{j=1}^{m} Z_j$ is gamma with pdf

$$g(s_m;a,\lambda,\theta) = \frac{\lambda^m s_m^{m-1}\exp(-\lambda s_m)}{\Gamma(m)} \tag{4}$$

It follows from Equation (3) and Equation (4) the S_m is a complete sufficient statistic for λ. The following theorem provides UMVUES of powers of λ.

Theorem 1. *For $p \in (-\infty,\infty)$, $p \neq 0$, the UMVUE of λ^p is given by*

$$\tilde{\lambda}^p = \begin{cases} \frac{\Gamma(m)}{\Gamma(m-p)} S_m^{-p}, & m-p > 0, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The result follows from (4) that

$$E(S_m^{-p}) = \frac{\Gamma(m-p)}{\Gamma(m)}\lambda^p, \quad m > p. \qquad \square$$

In the following theorem, we obtain UMVUE of the reliability function $R(t)$.

THEOREM 2. *The UMVUE of the reliability function is*

$$\tilde{R}(t)=\begin{cases}\left(1-\frac{G(t;a,\theta)}{S_m}\right)^{m-1}, & G(t;a,\theta)<S_m\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF. It is easy to see that

$$\begin{aligned} R(t) &= \exp(-\lambda G(t;a,\theta)) \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i\lambda^i G^i(t;a,\theta)}{i!}. \end{aligned} \tag{5}$$

Applying Theorem 1 and using (5) the UMVUE of $R(t)$ is obtained as

$$\begin{aligned} \tilde{R}(t) &= \sum_{i=0}^{\infty}\frac{(-1)^i\tilde{\lambda}^i G^i(t;a,\theta)}{i!} \\ &= \sum_{i=0}^{m-1}\frac{(-1)^i}{i!}\left\{\frac{\Gamma(m)}{\Gamma(m-i)}\right\}\left\{\frac{G(t;a,\theta)}{S_m}\right\}^i \end{aligned}$$

and the theorem follows. □

COROLLARY 3. *The UMVUE of the sampled pdf* (1) *at a specified point 'x' is*

$$\tilde{f}(x;a,\lambda,\theta)=\begin{cases}\frac{(m-1)G'(x;a,\theta)}{S_m}\left(1-\frac{G(x;a,\theta)}{S_m}\right)^{m-2}, & G(x;a,\theta)<S_m,\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF. We note that the expectation of $\int_t^\infty \tilde{f}(x;a,\lambda,\theta)dx$ with respect to S_m is $R(t)$. Hence

$$\tilde{R}(t)=\int_t^\infty \tilde{f}(x;a,\lambda,\theta)dx$$

or

$$-\frac{d\tilde{R}(t)}{dt}=\tilde{f}(t;a,\lambda,\theta).$$

The result now follows from Theorem 2. □

THEOREM 4. *The variance of* $\tilde{R}(t)$ *is given by*

$$Var\{\tilde{R}(t)\} = (\lambda G(t;a,\theta))^m \exp(-\lambda G(t;a,\theta)) \cdot \Bigg[\frac{a_{m-1}}{\lambda G(t;a,\theta)} + a_{m-2} \exp(\lambda G(t;a,\theta))\{-E_l(-\lambda G(t;a,\theta))\} + \sum_{l=0}^{m-3} a_l \Bigg\{ \sum_{k=1}^{m-l-2} \frac{(k-1)!\,(-\lambda G(t;a,\theta))^{m-l-k-2}}{(m-l-2)!} + \frac{(-\lambda G(t;a,\theta))^{m-l-2}}{(m-l-2)!} \times \exp(\lambda G(t;a,\theta))\{-E_l(-\lambda G(t;a,\theta))\} \Bigg\} + \sum_{l=m}^{2m-2} a_l \sum_{w=0}^{l-m+1} \binom{l-m+1}{w} w!(\lambda G(t;a,\theta))^{-(w+1)} \Bigg] - \exp(-2\lambda G(t;a,\theta)),$$

where $a_l = (-1)^l \binom{2m-2}{l}$ *and* $-E_l(-x) = \int_x^\infty \frac{e^{-u}}{u} du$.

PROOF. Using Equation (4) and Theorem 2

$$\begin{aligned} E\{\tilde{R}(t)\}^2 &= \frac{\lambda^m}{\Gamma(m)} \int_{G(t;a,\theta)}^\infty \left(1 - \frac{G(t;a,\theta)}{s_m}\right)^{2m-2} s_m^{m-1} \exp(-\lambda s_m) ds_m \\ &= \frac{1}{\Gamma(m)} (\lambda G(t;a,\theta))^m \exp(-\lambda G(t;a,\theta)) \int_0^\infty \frac{u^{2m-2}}{(1+u)^{m-1}} \exp(-\lambda G(t;a,\theta)u) du \\ &= \frac{1}{\Gamma(m)} (\lambda G(t;a,\theta))^m \exp(-\lambda G(t;a,\theta)) I, \text{ say,} \end{aligned} \tag{6}$$

where

$$\begin{aligned} I &= \sum_{l=0}^{2m-2} a_l \int_0^\infty \frac{(1+u)^l}{(1+u)^{m-1}} \exp(-\lambda G(t;a,\theta)u) du \\ &= \sum_{l=0}^{m-1} a_l \int_0^\infty \frac{1}{(1+u)^{m-l-1}} \exp(-\lambda G(t;a,\theta)u) du \\ &+ \sum_{l=m}^{2m-2} a_l \int_0^\infty (1+u)^{l-m+1} \exp(-\lambda G(t;a,\theta)u) du. \end{aligned} \tag{7}$$

Using the result $\int_0^\infty \frac{1}{(u+a)^n} \exp(-up) du = \sum_{k=1}^{n-1} \frac{(k-1)!(-p)^{n-k-1}}{(n-1)!a^k} - \frac{(-p)^{n-1}}{(n-1)!} \exp(ap) E_l(-ap)$

given by Erdélyi (1954), we have

$$\int_0^\infty \frac{1}{(1+u)^{m-l-1}} \exp(-\lambda G(t;a,\theta)u)du$$
$$= \sum_{k=1}^{m-l-2} \frac{(k-1)!}{(m-l-2)!}(-\lambda G(t;a,\theta))^{m-l-k-2} - \frac{1}{(m-l-2)!}(-\lambda G(t;a,\theta))^{m-l-2}$$
$$\times \exp(\lambda G(t;a,\theta))E_l(-\lambda G(t;a,\theta)), l = 0,1,\ldots,m-3. \quad (8)$$

and

$$\int_0^\infty \frac{1}{(1+u)} \exp(-\lambda G(t;a,\theta)u)du$$
$$= \exp(\lambda G(t;a,\theta)) \int_0^\infty \frac{1}{(u+1)} \exp(-\lambda G(t;a,\theta)(u+1))du$$
$$= \exp(\lambda G(t;a,\theta)) \int_{\lambda G(t;a,\theta)}^\infty \frac{e^{-z}}{z} dz$$
$$= -\exp(\lambda G(t;a,\theta))E_l(-\lambda G(t;a,\theta)), \quad (9)$$

we have

$$\int_0^\infty \exp(-\lambda G(t;a,\theta)u)du = \frac{1}{\lambda G(t;a,\theta)}. \quad (10)$$

Finally,

$$\int_0^\infty (1+u)^{l-m+1} \exp(-\lambda G(t;a,\theta)u)du$$
$$= \sum_{w=0}^{l-m+1} \binom{l-m+1}{w} \int_0^\infty u^w \exp(-\lambda G(t;a,\theta)u)du$$
$$= \sum_{w=0}^{l-m+1} \binom{l-m+1}{w} w!(-\lambda G(t;a,\theta))^{-(w+1)}. \quad (11)$$

The theorem now follows on making substitutions from Equations (8), (9), (10) and (11) in (7) and then using (6). □

Suppose two independent rv's X and X^* are following two different families of distributions $f_1(x;a_1,\lambda_1,\theta_1)$ and $f_2(x^*;a_2,\lambda_2,\theta_2)$, respectively, i.e.

$$f_1(x;a_1,\lambda_1,\theta_1) = \lambda_1 G'(x;a_1,\theta_1)\exp(-\lambda_1 G(x;a_1,\theta_1)); \; x > a_1 \geq 0, \; \lambda_1 > 0$$

and

$$f_2(x^*;a_2,\lambda_2,\theta_2) = \lambda_2 H'(x^*;a_2,\theta_2)\exp(-\lambda_2 H(x^*;a_2,\theta_2)); \; x^* > a_2 \geq 0, \lambda_2 > 0.$$

Let n and n^* experimental units are put on life testing experiment at time zero on X and X^*, respectively, and R_i, $i = 1,2,\ldots,m-1$, R_i^*, $i = 1,2,\ldots,m^*-1$ units are removed at the i^{th} stage on X and X^*, respectively. Obviously, denoting by $Y_j = G(X_j; a_1, \theta_1)$ and $Y_j^* = H(X_j^*; a_2, \theta_2)$, we have

$$f^{(1)}_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(x_1,x_2,\ldots,x_m) = c\lambda_1^m \exp\left(-\lambda_1 \sum_{j=1}^{m}(R_j+1)Y_j\right)$$

and

$$f^{(2)}_{X^*_{1:m^*:n^*},X^*_{2:m^*:n^*},\ldots,X^*_{m^*:m^*:n^*}}(x_1^*,x_2^*,\ldots,x_{m^*}^*) = c^*\lambda_2^{m^*} \exp\left(-\lambda_2 \sum_{j=1}^{m^*}(R_j^*+1)Y_j^*\right),$$

where $c = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-R_1-\ldots-R_{m-1}-m+1)$ and $c^* = n^*(n^*-R_1^*-1)(n^*-R_1^*-R_2^*-2)\ldots(n^*-R_1^*-\ldots-R_{m^*-1}^*-m^*+1)$, respectively.

Let us consider the following transformations

$$Z_1 = nY_1,\ Z_2 = (n-R_1-1)(Y_2-Y_1),\ Z_3 = (n-R_1-R_2-2)(Y_3-Y_2)\ldots,$$
$$Z_m = (n-R_1-\ldots-R_{m-1}-m+1)(Y_m-Y_{m-1})$$

and

$$Z_1^* = n^*Y_1^*,\ Z_2^* = (n^*-R_1^*-1)(Y_2^*-Y_1^*),\ Z_3^* = (n^*-R_1^*-R_2^*-2)(Y_3^*-Y_2^*),\ldots,$$
$$Z_{m^*}^* = (n^*-R_1^*-\ldots-R_{m^*-1}^*-m^*+1)(Y_{m^*}^*-Y_{m^*-1}^*).$$

The Jacobian of transformation for Z's and Z^*'s are c^{-1} and c^{*-1}, respectively. Z_j's are exponential with mean life λ_1^{-1} and Z_j^*'s are exponential with mean life λ_2^{-1}. Denoting by $S_m = \sum_{j=1}^{m}(R_j+1)Y_j = \sum_{j=1}^{m} Z_j$ and $T_{m^*} = \sum_{j=1}^{m^*}(R_j^*+1)Y_j^* = \sum_{j=1}^{m^*} Z_j^*$, we note that S_m and T_{m^*} follow gamma distribution with pdf's

$$g(s_m; a_1, \lambda_1, \theta_1) = \frac{\lambda_1^m s_m^{m-1}}{\Gamma(m)} \exp(-\lambda_1 s_m)$$

and

$$h(T_{m^*}; a_2, \lambda_2, \theta_2) = \frac{\lambda_2^{m^*} T_{m^*}^{m^*-1}}{\Gamma(m^*)} \exp(-\lambda_2 T_{m^*}). \qquad \square$$

Now, we derive the UMVUE of stress-strength reliability P in the following theorem

THEOREM 5. *The UMVUE of P is given by*

$$\tilde{P} = \begin{cases} (m^* - 1)\int_0^{T_{m^*}^{-1} H(G^{-1}(S_m))} (1-z)^{m^*-2} \left[1 - \frac{G(H^{-1}(zT_{m^*}))}{S_m}\right]^{m-1} dz, \\ \qquad G^{-1}(S_m) < H^{-1}(T_{m^*}), \\ (m^* - 1)\int_0^1 (1-z)^{m^*-2} \left[1 - \frac{G(H^{-1}(zT_{m^*}))}{S_m}\right]^{m-1} dz, \\ \qquad H^{-1}(T_{m^*}) < G^{-1}(S_m). \end{cases}$$

PROOF. From Corollary 3 the UMVUES of $f_1(x; a_1, \lambda_1, \theta_1)$ and $f_2(x^*; a_2, \lambda_2, \theta_2)$ at specified points 'x' and 'x^*' are, given by, respectively,

$$\tilde{f}_1(x; a_1, \lambda_1, \theta_1) = \begin{cases} \frac{(m-1)G'(x; a_1, \theta_1)}{S_m} \left[1 - \frac{G(x; a_1, \theta_1)}{S_m}\right]^{m-2}, & G(x; a_1, \theta_1) < S_m, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\tilde{f}_2(x^*; a_2, \lambda_2, \theta_2) = \begin{cases} \frac{(m^*-1)H'(x^*; a_2, \theta_2)}{T_{m^*}} \left[1 - \frac{H(x^*; a_2, \theta_2)}{T_{m^*}}\right]^{m^*-2}, & H(x^*; a_2, \theta_2) < T_{m^*}, \\ 0, & \text{otherwise.} \end{cases}$$

From the arguments similar to those used in the proof of Corollary 3,

$$\begin{aligned} \tilde{P} &= \int_{x^*=a_2}^{\infty} \int_{x=x^*}^{\infty} \tilde{f}_1(x; a_1, \lambda_1, \theta_1)\, \tilde{f}_2(x^*; a_2, \lambda_2, \theta_2) dx dx^* \\ &= \int_{x^*=a_2}^{\infty} \tilde{R}_1(x^*; a_1, \lambda_1, \theta_1) \left\{ -\frac{d\tilde{R}_2(x^*; a_2, \lambda_2, \theta_2)}{dx^*} \right\} dx^* \\ &= (m^* - 1) \int_{x^*=a_2}^{\min\{G^{-1}(S_m), H^{-1}(T_{m^*})\}} \left[1 - \frac{G(x^*; a_1, \theta_1)}{S_m}\right]^{m-1} \\ &\quad \cdot \frac{H'(x^*; a_2, \theta_2)}{T_{m^*}} \left[1 - \frac{H(x^*; a_2, \theta_2)}{T_{m^*}}\right]^{m^*-2} dx^*. \end{aligned}$$

The theorem now follows on considering the two cases and putting $T_{m^*}^{-1} H(x^*; a_2, \theta_2) = z$ □.

Now, the UMVUE of P when X and X^* belong to same families of distributions is obtained in the following theorem.

THEOREM 6. *When rv's X and X^* belong to the same families of distributions,*

$$\tilde{P} = \begin{cases} \sum_{i=0}^{m^*-2} (-1)^i \frac{(m^*-1)!(m-1)!}{(m^*-i-2)!(m+i)!} \left(\frac{S_m}{T_{m^*}}\right)^{i+1}, & S_m < T_{m^*}, \\ \sum_{i=0}^{m-1} (-1)^i \frac{(m^*-1)!(m-1)!}{(m^*+i-1)!(m-i-1)!} \left(\frac{T_{m^*}}{S_m}\right)^{i}, & T_{m^*} < S_m. \end{cases}$$

PROOF. In Theorem 5, taking $G(x;a_1,\theta_1)=H(x^*;a_2,\theta_2)$, we get, for $S_m < T_{m^*}$,

$$\begin{aligned}\tilde{P} &= (m^*-1)\int_0^{S_m/T_{m^*}}(1-z)^{m^*-2}\left(1-\frac{T_{m^*}}{S_m}z\right)^{m-1}dz\\ &=(m^*-1)\int_0^1\left(1-\frac{S_m}{T_{m^*}}u\right)^{m^*-2}(1-u)^{m-1}\left(\frac{S_m}{T_{m^*}}\right)du\\ &=(m^*-1)\sum_{i=0}^{m^*-2}(-1)^i\binom{m^*-2}{i}\left(\frac{S_m}{T_{m^*}}\right)^{i+1}\int_0^1 u^i(1-u)^{m-1}du\end{aligned}$$

and the first assertion follows. Similarly, we can prove the second assertion. □

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section, the MLEs of the reliability function of $R(t)$ and stress-strength reliability P, respectively, are derived. The following theorem provides the MLE of $R(t)$.

THEOREM 7. *The MLE of $R(t)$ is given by*

$$\hat{R}(t)=\exp\left(-\frac{m}{S_m}G(t;a,\theta)\right).$$

PROOF. Since the likelihood function is of the same form as (3), it can be easily seen that the MLE of λ is $\hat{\lambda}=\frac{m}{S_m}$. The theorem now follows from invariance property of the MLEs. □

COROLLARY 8. *The MLE of $f(x;a,\lambda,\theta)$ at a specified point 'x' is*

$$\hat{f}(x;a,\lambda,\theta)=\frac{mG'(x;a,\theta)}{S_m}\exp\left(-\frac{m}{S_m}G(x;a,\theta)\right).$$

PROOF. The result follows from the fact that

$$\hat{f}(t;a,\lambda,\theta)=-\frac{d\hat{R}(t)}{dt}.$$ □

In the following theorem , we obtain the expression for variance of $\hat{R}(t)$.

THEOREM 9. *The variance of $\hat{R}(t)$ is given by*

$$\begin{aligned}Var\{\hat{R}(t)\} &= \frac{2}{(m-1)!}(2m\lambda G(t;a,\theta))^{m/2}K_m(2\sqrt{2m\lambda G(t;a,\theta)})\\ &\quad-\left\{\frac{2}{(m-1)!}(m\lambda G(t;a,\theta))^{m/2}K_m(2\sqrt{m\lambda G(t;a,\theta)})\right\}^2,\end{aligned}$$

where, $K_m(\cdot)$ is the modified Bessel function of the second kind of order m.

PROOF. Using Equation (4), we have

$$\begin{aligned} E\{\hat{R}(t)\} &= \frac{\lambda^m}{\Gamma(m)} \int_0^\infty s_m^{m-1} \exp\left\{-\left(\frac{mG(t;a,\theta)}{s_m} + \lambda s_m\right)\right\} ds_m \\ &= \frac{1}{\Gamma(m)} \int_0^\infty u^{m-1} \exp\left\{-\left(u + \frac{m\lambda G(t;a,\theta)}{u}\right)\right\} du. \end{aligned} \quad (12)$$

Applying the result

$$\int_0^\infty u^{-m} \exp\left\{-\left(au + \frac{b}{u}\right)\right\} du = 2\left(\frac{a}{b}\right)^{(m-1)/2} K_{m-1}(2\sqrt{ab}), \quad [K_{-m}(\cdot) = K_m(\cdot),$$

for $m = 0, 1, 2, \ldots,$], given by Watson (1952), we obtain from (12) that

$$E\{\hat{R}(t)\} = \frac{2}{(m-1)!}(m\lambda G(t;a,\theta))^{m/2} K_m(2\sqrt{m\lambda G(t;a,\theta)}).$$

Similarly, we can obtain the expression for $E\{\hat{R}(t)\}^2$ and the result follows. □

The MLE of P when rv's X and X^* belong to different families of distributions is given by following theorem.

THEOREM 10. *The MLE of P is given by*

$$\hat{P} = \int_0^\infty \exp\left\{-\left(\frac{m}{S_m}\right) G\left(H^{-1}\left(\frac{z T_{m^*}}{m^*}\right)\right)\right\} e^{-z} dz.$$

PROOF. We have

$$\begin{aligned} \hat{P} &= \int_{x^*=a_2}^\infty \int_{x=x^*}^\infty \hat{f}_1(x;a_1,\lambda_1,\theta_1) \hat{f}_2(x^*;a_2,\lambda_2,\theta_2) dx\, dx^* \\ &= \int_{x^*=a_2}^\infty \hat{R}_1(x^*;a_1,\lambda_1,\theta_1) \left\{-\frac{d\hat{R}_2(x^*;a_2,\lambda_2,\theta_2)}{dx^*}\right\} dx^* \\ &= \int_{x^*=a_2}^\infty \exp\left(-\frac{m}{S_m} G(x^*;a_1,\theta_1)\right) \frac{m^* H'(x^*;a_2,\theta_2)}{T_{m^*}} \exp\left(-\frac{m^*}{T_{m^*}} H(x^*;a_2,\theta_2)\right) dx^*. \end{aligned}$$

The result now follows on putting $m^* T_{m^*}^{-1} H(x^*;a_2,\theta_2) = z$. □

The MLE of P when X and X^* belong to same family of distributions is given by following theorem.

THEOREM 11. *When X and X^* belong to same families of distributions, the MLE of P is given by*

$$\hat{P} = \frac{m^* S_m}{m^* S_m + m T_{m^*}}.$$

PROOF. The result follows from Theorem 10. □

4. Invariantly optimal estimation

This section deals with the IOE of $R(t)$ and P. The IOE of $R(t)$ is given by, see Hurt and Wertz (1983),

$$\check{R}(t)=\left[1+\frac{G(t;a,\theta)}{S_m}\right]^{-(m+1)}. \tag{13}$$

THEOREM 12. *The variance of $\check{R}(t)$ is given by*

$$\begin{aligned}
Var\{\check{R}(t)\} = &\Bigg\{(\lambda G(t;a,\theta))^m \frac{1}{\Gamma(m)}\Bigg[\frac{c_{2m+2}}{\lambda G(t;a,\theta)}+c_{2m+1}\\
&\cdot\exp(\lambda G(t;a,\theta))[-E_l(-\lambda G(t;a,\theta))]\\
&+\sum_{l=0}^{2m} c_l\Bigg\{\sum_{k=1}^{2m-l+1}\frac{(k-1)!}{(2m-l+1)!}(-\lambda G(t;a,\theta))^{2m-l+1-k}\\
&+\frac{(-\lambda G(t;a,\theta))^{2m-l+1}}{(2m-l+1)!}\exp(\lambda G(t;a,\theta))[-E_l(-\lambda G(t;a,\theta))]\Bigg\}\\
&+\sum_{l=2m+3}^{3m+1} c_l\sum_{w=0}^{l-2m-2}\binom{l-2m-2}{w}w!(\lambda G(t;a,\theta))^{-(w+1)}\Bigg]\Bigg\}\\
&-\Bigg\{(\lambda G(t;a,\theta))^m\frac{1}{\Gamma(m)}\Bigg\{\frac{b_{m+1}}{\lambda G(t;a,\theta)}+b_m\exp(\lambda G(t;a,\theta))\\
&\cdot[-E_l(-\lambda G(t;a,\theta))]+\sum_{l=0}^{m-1} b_l\Bigg\{\sum_{k=1}^{m-l}\frac{(k-1)!(-\lambda G(t;a,\theta))^{m-l-k}}{(m-l)!}\\
&+\frac{(-\lambda G(t;a,\theta))^{m-l}}{(m-l)!}\exp(\lambda G(t;a,\theta))[-E_l(-\lambda G(t;a,\theta))]\Bigg\}\\
&+\sum_{l=m+2}^{2m} b_l\sum_{w=0}^{l-m-1}\binom{l-m-1}{w}w!(\lambda G(t;a,\theta))^{-(w+1)}\Bigg\}\Bigg\}^2
\end{aligned}$$

Here, $b_l=(-1)^l\binom{2m}{l}$ *and* $c_l=\begin{cases}(-1)^l\binom{3m+1}{l}; & m \text{ odd}\\ (-1)^{l+1}\binom{3m+1}{l}; & m \text{ even.}\end{cases}$

PROOF. The proof is similar to that of Theorem 4. □

The IOE of pdf using Equation (13) is given by

$$\check{f}(x;a,\lambda,\theta)=\frac{(m+1)G'(x;a,\theta)}{S_m}\left[1+\frac{G(x;a,\theta)}{S_m}\right]^{-(m+2)}.$$

THEOREM 13. *When rv's X and X^* belong to different families of distributions, the IOE of P is given by*

$$\check{P} = (m^* + 1) \int_0^\infty (1 + z)^{-(m^*+2)} \left[1 + \frac{G(H^{-1}(zT_{m^*})}{S_m} \right]^{-(m+1)} dz.$$

PROOF. We have

$$\check{P} = \int_{x^*=a_2}^\infty \int_{x=x^*}^\infty \check{f}(x; a_1, \lambda_1, \theta_1) \check{f}(x^*; a_2, \lambda_2, \theta_2) dx dx^*$$

$$= (m^* + 1) \int_{x^*=a_2}^\infty \left[1 + \frac{G(x^*; a_1, \theta_1)}{S_m} \right]^{-(m+1)}$$

$$\cdot \frac{H'(x^*; a_2, \theta_2)}{T_{m^*}} \left[1 + \frac{H(x^*; a_2, \theta_2)}{T_{m^*}} \right]^{-(m^*+2)} dx^*.$$

The result now follows on putting $T_{m^*}^{-1} H(x^*; a_2, \theta_2) = z$. □

THEOREM 14. *When X and X^* belong to same families of distributions, the IOE of P is given by*

$$\check{P} = (m^* + 1) \int_0^\infty (1 + z)^{-(m^*+2)} \left[1 + \frac{T_{m^*}}{S_m} z \right]^{-(m+1)} dz.$$

PROOF. The result follows from Theorem 13. □

Note that in all the estimators, first we estimated powers of the parameter, which is used to estimate the pdf. This estimated pdf is subsequently used to estimate $R(t)$ and P. Thus, the basic role is played by the estimators of powers of parameter.

5. TESTING OF HYPOTHESES

Let we are to test the hypothesis $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$. It follows from (3) and MLE of λ that, under H_0,

$$\sup_{\Theta_0} L(\lambda \mid \underline{x}) = c \lambda_0^m \exp(-\lambda_0 S_m); \ \Theta_0 = \{\lambda : \lambda = \lambda_0\}$$

and

$$\sup_{\Theta} L(\lambda \mid \underline{x}) = c \left(\frac{m}{S_m} \right)^m \exp(-m); \ \Theta = \{\lambda : \lambda > 0\}.$$

Therefore, the likelihood ratio (LR) is given by

$$\phi(\underline{x}) = \frac{\sup_{\Theta_0} L(\lambda \mid \underline{x})}{\sup_{\Theta} L(\lambda \mid \underline{x})} = \left(\frac{S_m \lambda_0}{m} \right)^m \exp\{-S_m \lambda_0 + m\}. \tag{14}$$

Note that the first term on the right hand side of Equation (14) is monotonically increasing and the second term is monotonically decreasing in S_m. Denoting by $\chi^2_{2m}(\cdot)$, the chi-square statistic with $2m$ degrees of freedom and using the fact that $2\lambda_0 S_m \sim \chi^2_{(2m)}$, the critical region is given by $\{0 < S_m < k_0\} \cup \{k_0' < S_m < \infty\}$, where k_0 and k_0' are obtained such that $P[\chi^2_{(2m)} < 2\lambda_0 k_0 \text{ or } 2\lambda_0 k_0' < \chi^2_{(2m)}] = \alpha$.

Thus $k_0 = \frac{1}{2\lambda_0}\chi^2_{(2m)}\left(1-\frac{\alpha}{2}\right)$ and $k_0' = \frac{1}{2\lambda_0}\chi^2_{(2m)}\left(\frac{\alpha}{2}\right)$.

An important hypothesis in life testing experiments is $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$. It follows from (3) that, for $\lambda_1 > \lambda_2$,

$$\frac{f(x_1, x_2, \ldots, x_m; a, \lambda_1, \theta)}{f(x_1, x_2, \ldots, x_m; a, \lambda_2, \theta)} = \left(\frac{\lambda_1}{\lambda_2}\right)^m \exp\{-(\lambda_1 - \lambda_2)S_m\}. \tag{15}$$

It follows from (15) that $f(x; a, \lambda, \theta)$ has monotone likelihood ratio in S_m. Thus, the uniformly most powerful critical region for testing H_0 against H_1 is given by

$$\phi(\underline{x}) = \begin{cases} 1, & \text{if } S_m \leq k_0'', \\ 0, & \text{otherwise.} \end{cases}$$

where, k_0'' is obtained such that $P[\chi^2_{(2m)} < 2\lambda_0 k_0''] = \alpha$. Therefore, $k_0'' = \left(\frac{1}{2\lambda_0}\right)\chi^2_{(2m)}(1-\alpha)$. It can be seen that when X and X^* belong to same families of distributions

$$P = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

Suppose we want to test $H_0 : P = P_0$ against $H_1 : P \neq P_0$. It follows that H_0 is equivalent to $\lambda_2 = k\lambda_1$, where $k = \frac{P_0}{1-P_0}$. Thus, $H_0 : \lambda_2 = k\lambda_1$ and $H_1 : \lambda_2 \neq k\lambda_1$. It can be shown that, under H_0,

$$\hat{\lambda}_1 = \frac{m + m^*}{S_m + kT_{m^*}} \text{ and } \hat{\lambda}_2 = \frac{(m + m^*)k}{S_m + kT_{m^*}}.$$

For a generic constant K,

$$L(\lambda_1, \lambda_2 \mid \underline{x}, \underline{x}^*) = K\lambda_1^m \lambda_2^{m^*} \exp\{-(\lambda_1 S_m + \lambda_2 T_{m^*})\}.$$

Thus

$$\sup_{\Theta_0} L(\lambda_1, \lambda_2 \mid \underline{x}, \underline{x}^*) = \frac{K k^{m^*}}{(S_m + kT_{m^*})^{m+m^*}} \exp\{-(m + m^*)\};$$
$$\Theta_0 = \{\lambda_1, \lambda_2 : \lambda_2 = k\lambda_1\} \tag{16}$$

and

$$\sup_{\Theta} L(\lambda_1, \lambda_2 \mid \underline{x}, \underline{x}^*) = \frac{K}{(S_m)^m (T_{m^*})^{m^*}} \exp\{-(m + m^*)\};$$
$$\Theta = \{\lambda_1, \lambda_2 : \lambda_1 > 0, \lambda_2 > 0\}. \tag{17}$$

From (16) and (17), the LR is given by

$$\phi(\underline{x}, \underline{x}^*) = \frac{K\left(\frac{S_m}{kT_{m^*}}\right)^m}{\left(1+\frac{S_m}{kT_{m^*}}\right)^{m+m^*}}.$$

Denoting by $F_{a,b}(\cdot)$, the F-statistic with (a, b) degrees of freedom and using the fact that

$$\frac{S_m}{T_{m^*}} \sim \frac{m\lambda_2}{m^*\lambda_1} F_{2m,2m^*}(\cdot).$$

The critical region is given by

$$\left\{\frac{S_m}{T_{m^*}} < k_2 \text{ or } \frac{S_m}{T_{m^*}} > k_2'\right\},$$

where k_2 and k_2' are obtained such that

$$P\left[\frac{m^* S_m}{m\,k T_{m^*}} < F_{2m,2m^*} \text{ or } \frac{m^* S_m}{m\,k T_{m^*}} > F_{2m,2m^*}\right] = \alpha.$$

Thus $k_2 = \frac{mk}{m^*} F_{2m,2m^*}\left(1-\frac{\alpha}{2}\right)$ and $k_2' = \frac{mk}{m^*} F_{2m,2m^*}\left(\frac{\alpha}{2}\right)$.

6. Monte Carlo simulation study

This section deals with a Monte Carlo simulation study to judge the performance of different sample sizes, progressive censoring schemes and various estimates developed. Different progressively censored samples are generated using the algorithm proposed by Balakrishnan and Sandhu (1995). All the numerical computations are done on statistical software R 3.4.0. Eight different progressive censoring schemes with different sample sizes are considered and reported in Table 1. Different censoring schemes are denoted by short notation like $(0, 0, 0, 0, 1)$ by $(0 * 4, 1)$. Two different true values of the parameter $\lambda = 0.5$ and $\lambda = 1.5$ with mission time $t = 0.5$ so that $R(t) = 0.7788$ and 0.4724, respectively, are taken. In case of stress-strength reliability estimation, two sets of true values of the parameters $(\lambda_1, \lambda_2) = (0.5, 1.5)$ and $(1.5, 0.5)$ so that $P = 0.75$ and 0.25, respectively, are considered. Also, for stress-strength reliability different combinations of censoring schemes are used and reported in second column of Table 3. For a particular set of sample size, parameter value and progressive censoring schemes, we generate 1000 progressively censored samples. For each simulation, the ML, UMVU and IO estimates of reliability function $R(t)$ and stress-strength reliability P are computed. Then, the mean squared error (MSE) for all the estimators and average absolute bias (AB) for ML and IO estimators are computed based on the estimates from all 1000 simulations.

The results of simulation study are presented in Tables 2 and 3. From these Tables one can observe that as the sample size n and the effective sample size m increase, ABs

TABLE 1
Progressive censoring schemes.

	(n,m)	C.S.
R_1	$(20,5)$	$(3*5)$
R_2	$(20,5)$	$(0*4,15)$
R_3	$(20,15)$	$(1*2,0*5,1,0*5,1*2)$
R_4	$(20,15)$	$(0*14,5)$
R_5	$(40,10)$	$(3*10)$
R_6	$(40,10)$	$(0*9,30)$
R_7	$(40,25)$	$(1*5,0*5,1*5,0*5,1*5)$
R_8	$(40,25)$	$(0*24,15)$

TABLE 2
The ABs and MSEs for ML and IO estimates, and MSEs for UMVUE of the reliability function $R(t)$.

λ	C.S.	$\hat{R}(t)$		$\tilde{R}(t)$	$\check{R}(t)$	
		AB	MSE	MSE	AB	MSE
0.5	R_1	0.0816	0.0135	0.0108	0.0953	0.0181
	R_2	0.0801	0.0126	0.0099	0.0938	0.0171
	R_3	0.0422	0.0031	0.0027	0.0453	0.0036
	R_4	0.0426	0.0032	0.0028	0.0457	0.0037
	R_5	0.0527	0.0050	0.0043	0.0578	0.0062
	R_6	0.0537	0.0054	0.0046	0.0592	0.0067
	R_7	0.0321	0.0017	0.0016	0.0335	0.0019
	R_8	0.0322	0.0017	0.0016	0.0336	0.0019
1.5	R_1	0.1271	0.0255	0.0284	0.1280	0.0260
	R_2	0.1290	0.0264	0.0292	0.1297	0.0269
	R_3	0.0737	0.0087	0.0088	0.0743	0.0089
	R_4	0.0750	0.0089	0.0091	0.0755	0.0092
	R_5	0.0899	0.0129	0.0134	0.0907	0.0133
	R_6	0.0908	0.0132	0.0137	0.0915	0.0135
	R_7	0.0567	0.0051	0.0051	0.0571	0.0052
	R_8	0.0573	0.0051	0.0052	0.0575	0.0052

TABLE 3
The ABs and MSEs for ML and IO estimates, and MSEs for UMVUE of stress-strength reliability P.

(λ_1, λ_2)	C.S.	$\hat{P}$		$\tilde{P}$	$\check{P}$	
		AV	MSE	MSE	AV	MSE
(0.5, 1.5)	R_1, R_5	0.0862	0.0127	0.0126	0.0881	0.0135
	R_2, R_6	0.0859	0.0125	0.0123	0.0881	0.0134
	R_3, R_7	0.0493	0.0040	0.0039	0.0497	0.0041
	R_4, R_8	0.0504	0.0041	0.0041	0.0507	0.0042
	R_5, R_1	0.0814	0.0105	0.0114	0.0780	0.0096
	R_6, R_2	0.0806	0.0103	0.0112	0.0772	0.0095
	R_7, R_3	0.0491	0.0038	0.0039	0.0484	0.0037
	R_8, R_4	0.0486	0.0038	0.0039	0.0479	0.0037
(1.5, 0.5)	R_1, R_5	0.0809	0.0103	0.0113	0.0775	0.0095
	R_2, R_6	0.0816	0.0105	0.0115	0.0782	0.0097
	R_3, R_7	0.0485	0.0037	0.0038	0.0479	0.0036
	R_4, R_8	0.0482	0.0037	0.0038	0.0476	0.0036
	R_5, R_1	0.0862	0.0126	0.0124	0.0885	0.0135
	R_6, R_2	0.0871	0.0130	0.0129	0.0890	0.0138
	R_7, R_3	0.0495	0.0040	0.0040	0.0499	0.0041
	R_8, R_4	0.0497	0.0040	0.0039	0.0501	0.0041

and MSEs decrease. Progressive type II censoring schemes give less AB and MSEs in comparison to conventional type II censoring schemes in almost all cases. The UMVUE is better than MLE of the reliability function in all cases in terms of MSEs. The MLE is better than UMVUE and IOE for stress-strength parameter P in terms of MSEs in most of the cases. In case of estimation of reliability function and stress-strength reliability UMVUE and MLE, respectively, may be considered for all practical purposes.

7. Real data analysis

In this section, a pair of real data sets is analyzed for illustration purpose. The data on breaking strength of jute fibers were proposed by Xia *et al.* (2009). The Jute fibers were tested under tension at gauge lengths of 5, 10, 15, and 20 mm. In our study, we consider data on breaking strength of jute fibers under gauge lengths 15 mm and 20 mm. For ready reference of readers, these data are reported in Tables 4 and 5, respectively. First of all these two data sets are used to fit the exponential distribution, separately. The estimated parameters, Kolmogorov-Smirnov (K-S) and Anderson-Darling (A-D) statistics with corresponding p-values are presented in Table 6. This table represents that the K-S as well A-D tests accept the null hypothesis that each data set is drawn from exponential distribution. Now, three different progressively censored samples of sizes $m = 20$ and $m^* = 25$ are generated, respectively, from these two data sets. The progressive censoring schemes and the corresponding generated data sets are reported in Tables 7 and 8, respectively.

Now, based on these samples the ML, UMVU and IO estimates of reliability function $R(t)$ are computed for all data sets and results are presented in Table 9 assuming mission times median of complete samples. Again all the estimators are computed for stress-strength reliability P and results are reported in Table 10. Again, let us consider the testing of hypothesis $H_0 : P = 0.5$ against $H_1 : P \neq 0.5$ i.e. stress and strength are equally effective. The critical region is constructed according to the section 5. Also, the ratio of statistics S_m/T_{m^*} is calculated for three different censoring schemes and reported in Table 11. All three censoring schemes show that stress and strength both are equally effective.

TABLE 4
Breaking strength of jute fiber under gauge length 15mm.

594.40	202.75	168.37	574.86	225.65	76.38	156.67	127.81	813.87	562.39
468.47	135.09	72.24	497.94	355.56	569.07	640.48	200.76	550.42	748.75
489.66	678.06	457.71	106.73	716.30	42.66	80.40	339.22	70.09	193.42

TABLE 5
Breaking strength of jute fiber under gauge length 20mm.

71.46	419.02	284.64	585.57	456.60	113.85	187.85	688.16	662.66	45.58
578.62	756.70	594.29	166.49	99.72	707.36	765.14	187.13	145.96	350.70
547.44	116.99	375.81	581.60	119.86	48.01	200.16	36.75	244.53	83.55

TABLE 6
MLEs, K-S and A-D goodness-of-fit tests for real data sets.

Data Set	Parameter λ	K-S Test		A-D Test	
		Statistic	p-value	Statistic	p-value
Gauge length 15 mm (X)	0.0027	0.1823	0.2403	1.3449	0.2182
Gauge length 20 mm (X^*)	0.0029	0.1328	0.6181	0.9029	0.4116

TABLE 7
The generated progressively censored samples corresponding to real data set of gauge length 15mm.

	(n,m)	Censoring Scheme	Data Set
R_1	$(30,20)$	$(0*19,10)$	42.66, 70.09, 72.24, 76.38, 80.40, 106.73, 127.81, 135.09, 156.67, 168.37, 193.42, 200.76, 202.75, 225.65, 339.22, 355.56, 457.71, 468.47, 489.66, 497.94.
R_2	$(30,20)$	$(1*5,0*10,1*5)$	42.66, 72.24, 80.40, 127.81, 156.67, 193.42, 200.76, 202.75, 225.65, 339.22, 355.56, 457.71, 468.47, 489.66, 497.94, 550.42, 569.07, 594.40, 678.06, 748.75.
R_3	$(30,20)$	$(10,0*19)$	42.66, 200.76, 202.75, 225.65, 339.22, 355.56, 457.71, 468.47, 489.66, 497.94, 550.42, 562.39, 569.07, 574.86, 594.40, 640.48, 678.06, 716.30, 748.75, 813.87.

TABLE 8
The generated progressively censored samples corresponding to real data set of gauge length 20mm.

	(n^*,m^*)	Censoring Scheme	Data Set
R_1^*	$(30,25)$	$(0*24,5)$	36.75, 45.58, 48.01, 71.46, 83.55, 99.72, 113.85, 116.99, 119.86, 145.96, 166.49, 187.13, 187.85, 200.16, 244.53, 284.64, 350.70, 375.81, 419.02, 456.60, 547.44, 578.62, 581.60, 585.57, 594.29.
R_2^*	$(30,25)$	$(1*2,0*10,1,0*10,1*2)$	36.75, 48.01, 83.55, 99.72, 113.85, 116.99, 119.86, 145.96, 166.49, 187.13, 187.85, 200.16, 244.53, 350.70, 375.81, 419.02, 456.60, 547.44, 578.62, 581.60, 585.57, 594.29, 662.66, 688.16, 756.70.
R_3^*	$(30,25)$	$(5,0*24)$	36.75, 113.85, 116.99, 119.86, 145.96, 166.49, 187.13, 187.85, 200.16, 244.53, 284.64, 350.70, 375.81, 419.02, 456.60, 547.44, 578.62, 581.60, 585.57, 594.29, 662.66, 688.16, 707.36, 756.70, 765.14.

TABLE 9
The ML, UMVU and IO estimates of the reliability function corresponding to the real data sets. Here, mission times are 347.39 and 264.585, respectively.

CS	$\hat{R}(t)$	$\tilde{R}(t)$	$\check{R}(t)$	CS	$\hat{R}(t)$	$\tilde{R}(t)$	$\check{R}(t)$
R_1	0.4793	0.4907	0.4684	R_1^*	0.5026	0.5118	0.4937
R_2	0.5215	0.5333	0.5103	R_2^*	0.5202	0.5296	0.5113
R_3	0.5045	0.5162	0.4935	R_3^*	0.5181	0.5274	0.5091

TABLE 10

The ML, UMVU and IO estimates of the stress-strength reliability corresponding to the real data sets.

CS	$\hat{P}$	$\tilde{P}$	$\check{P}$
R_1, R_1^*	0.5512	0.5537	0.5490
R_2, R_2^*	0.5686	0.5715	0.5660
R_3, R_3^*	0.5579	0.5606	0.5555

TABLE 11

Testing of hypothesis $H_0 : P = 0.5$ against $H_1 : P \neq 0.5$ corresponding to the real data sets.

CS	Critical Region	S_m/T_{m^*}	Decision
R_1, R_1^*	$\{0 < S_m/T_{m^*} < 0.4366$ or $S_m/T_{m^*} > 1.4370\}$	0.9827	H_0 may be accepted
R_2, R_2^*	$\{0 < S_m/T_{m^*} < 0.4366$ or $S_m/T_{m^*} > 1.4370\}$	1.0543	H_0 may be accepted
R_3, R_3^*	$\{0 < S_m/T_{m^*} < 0.4366$ or $S_m/T_{m^*} > 1.4370\}$	1.0097	H_0 may be accepted

8. Concluding remarks

In this article, the problem of statistical inference for the reliability function and stress-strength reliability of a family of distributions based on progressive type II censoring was considered. Uniformly minimum variance, maximum likelihood and invariantly optimal estimators of the reliability function and stress-strength reliability were derived. All these estimators were in closed form. The comparisons of all estimators were done by extensive simulation study. Also, a real data analysis was performed for applicability of the proposed methods. This work was mainly associated with progressive type II right censoring case, the same methods can be extended for other censoring schemes also.

References

A. M. Awad, M. K. Gharraf (1986). *Estimation of $P(Y < X)$ in the Burr case: A comparative study*. Communications in Statistics - Simulation and Computation, 15, no. 2, pp. 389–403.

N. Balakrishnan, R. Aggarwala (2000). *Progressive Censoring-Theory, Methods and Applications*. Birkhäuser Publishers, Boston.

N. Balakrishnan, R. A. Sandhu (1995). *A simple simulation algorithm for generating progressive type-II censored samples*. American Statistician, 49, no. 2, pp. 229–230.

A. Chaturvedi, S. Kumar (1999). *Further remarks on estimating the reliability function of exponential distribution under type-I and type-II censorings*. Brazilian Journal of Probability and Statistics, 13, pp. 29–39.

A. Chaturvedi, T. Kumari (2015). *Estimation and testing procedures for the reliability* *functions of a family of lifetime distributions*. URL `interstat.statjournals.net/YEAR/ 2015/abstracts/ 1306001.php`.

A. CHATURVEDI, K. G. SINGH (2006). *Bayesian estimation procedures for a family of lifetime distributions under squared-error and entropy losses*. Metron, 64, no. 2, pp. 179–198.

A. CHATURVEDI, S. K. TOMER (2002). *Classical and Bayesian reliability estimation of the negative binomial distribution*. Journal of Applied Statistical Science, 11, pp. 33–43.

A. CHATURVEDI, S. VYAS (2017a). *Estimation and testing procedures for the reliability functions of exponentiated distributions under censorings*. Statistica, 77, no. 1, pp. 13–31.

A. CHATURVEDI, S. VYAS (2017b). *Estimation and testing procedures for the reliability functions of three parameter BURR distribution under censorings*. Statistica, 77, no. 3, pp. 207–235.

A. C. COHEN (1963). *Progressively censored sample in life testing*. Technometrics, 5, pp. 327–339.

J. HURT, W. WERTZ (1983). *Asymptotic properties of the invariantly optimal estimator of reliability in the exponential case*. Statistics and Decisions, 1, pp. 197–204.

G. D. KELLY, J. A. KELLY, W. R. SCHUCANY (1976). *Efficient estimation of $P(Y < X)$ in the exponential case*. Technometrics, 18, pp. 359–360.

S. KOTZ, Y. LUMELSKII, M. PENSKY (2003). *The Stress-Strength Model and its Generalization: Theory and Applications*. World Scientific Press, Singapore.

H. KRISHNA, K. KUMAR (2011). *Reliability estimation in Lindley distribution with progressively type II right censored sample*. Mathematics and Computers in Simulation, 82, no. 2, pp. 281–294.

H. KRISHNA, K. KUMAR (2013). *Reliability estimation in generalized inverted exponential distribution with progressively type-II censored sample*. Journal of Statistical Computation and Simulation, 83, no. 6, pp. 1007–1019.

K. KUMAR, R. GARG, H. KRISHNA (2017). *Nakagami distribution as a reliability model under progressive censoring*. International Journal of System Assurance Engineering and Management, 8, no. 1, pp. 109–122.

E. L. PUGH (1963). *The best estimate of reliability in the exponential case*. Operations Research, 11, pp. 57–61.

M. K. RASTOGI, Y. M. TRIPATHI (2014). *Parameter and reliability estimation for an exponentiated half-logistic distribution under progressive type-II censoring*. Journal of Statistical Computation and Simulation, 84, no. 8, pp. 1711–1727.

H. TONG (1974). *A note on the estimation of $P(Y < X)$ in the exponential case*. Technometrics, 16, p. 625.

R. K. TYAGI, S. K. BHATTACHARYA (1989). *A note on the MVU estimation of reliability for the Maxwell failure distribution*. Estadistica, 41, pp. 73–79.

R. VALIOLLAHI, M. Z. RAQAB, A. ASGHARZADEH, F. ALQALLAF (2018). *Estimation and prediction for power Lindley distribution under progressively type II right censored samples*. Mathematics and Computers in Simulation, 149, pp. 32 – 47.

Z. P. XIA, J. Y. YU, L. D. CHENG, L. F. LIU, W. M. WANG (2009). *Study on the breaking strength of jute fibres using modified Weibull distribution*. Composites: Part A, 40, pp. 54–59.

SUMMARY

In this article, a general family of lifetime distributions is considered under progressive type II right censoring. The classical point estimation and testing procedures are developed for reliability function and stress-strength reliability. The uniformly minimum variance unbiased, maximum likelihood and invariantly optimal estimators are considered. Testing procedures are developed for the hypotheses related to scale parameter, reliability and stress-strength reliability functions. A Monte Carlo simulation study is performed for comparison of various estimators developed. Finally, the use of proposed estimators is shown in an illustrative example.

Keywords: Progressive type II right censoring; Uniformly minimum variance unbiased estimation; Maximum likelihood estimation; Invariantly optimal estimator; Testing procedures.