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ESTIMATION OF $P(X > Y)$ FOR THE POSITIVE EXPONENTIAL FAMILY OF DISTRIBUTIONS

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1. Introduction

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes the lifetime of an item or a system, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength setup is the probability $P = P(X > Y)$, which represents the reliability of an item or a system of random strength X subject to random stress Y. A lot of work has been done in the literature for the point estimation and testing of $R(t)$ and P. For a brief review, one may refer to Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Kelly *et al.* (1976), Sathe and Shah (1981), Chao (1982), Chaturvedi and Surinder (1999), Awad and Gharraf (1986), Tyagi and Bhattacharya (1989) and Chaturvedi and Rani (1997, 1998), Chaturvedi and Tomer (2002, 2003), Chaturvedi and Singh (2006, 2008), Chaturvedi and Kumari (2015, 2016), Chaturvedi and Vyas (2017), Chaturvedi and Malhotra (2017) and Chaturvedi and Pathak (2012, 2013, 2014), Chaturvedi *et al.* (2018), Kotz *et al.* (2003), Baklizi (2008a,b), Eryilmaz (2008a,b, 2010, 2011), Krishnamoorthy *et al.* (2007), Krishnamoorthy *et al.* (2009), Krishnamoorthy and Lin (2010), Kundu and Raqab (2009), Rezaei *et al.* (2010) and Saracoglu and Kaya (2007).

Constantine *et al.* (1986) derived UMVUE and MLE of P when X and Y follow gamma distributions with shape parameters to be integer-valued. Huang *et al.* (2012) generalized these results for the case when shape parameters are positive-valued. Liang (2008) proposed a family of lifetime distributions, the positive exponential family, which covers gamma distribution as specific case.

In the present paper, we consider estimation of $R(t)$ and P for the positive exponential family of distributions. We derive UMVUES and MLES. In order to obtain these

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estimators, the basic role is played by the estimators of powers of parameters. The estimator of probability density function (pdf) is derived, which is subsequently used to estimate $R(t)$ and P. Thus, we have established an interrelationship between the three estimation problems. In the present approach, the expressions for $R(t)$ and P are not required. The case when all the parameters are unknown is also handled. The results of Constantine *et al.* (1986) and Huang *et al.* (2012) are shown to be particular cases of our results.

In Section 2, we derive UMVUES and MLES. In Section 3, we obtain MLES and MOMES assuming all the parameters to be unknown. In Section 4, we present numerical findings along with an example on real data and finally in Section 6, we conclude our study.

2. UMVUES AND MLES

The rv X follows a positive exponential family if its pdf is given by

$$f(x;\theta)=\frac{\alpha x^{\alpha\nu-1}e^{-\frac{x^\alpha}{\theta}}}{\Gamma(\nu)\theta^\nu};\quad x>0,\ \theta,\alpha,\nu>0. \tag{1}$$

When $\nu=1$ and $\alpha=1$, $f(x;\theta)$ reduces to an exponential density, and to the Weibull or gamma densities when $\nu=1$ or $\alpha=1$, respectively.

Let $X_1,X_2,\dots,X_n$ be a random sample of size n from the distribution given in (1). Then, assuming ν and α are known, the likelihood function of the parameter θ given the sample observations $\underline{x}=(x_1,x_2,\dots,x_n)$ is

$$L(\theta\mid\underline{x})=\left(\frac{\alpha}{\Gamma(\nu)}\right)^n\frac{1}{\theta^{n\nu}}e^{-\frac{1}{\theta}\sum_{i=1}^{n}x_i^\alpha}\prod_{i=1}^{n}x_i^{\alpha\nu-1}. \tag{2}$$

The following theorem provides UMVUES of powers of θ.

THEOREM 1. *For $q\in(-\infty,\infty)$, the UMVUE of θ^q is given by:*

$$\widetilde{\theta}^q=\begin{cases}\left\{\dfrac{\Gamma(n\nu)}{\Gamma(n\nu+q)}\right\}S^q; & n\nu+q>0\\ 0; & \textit{otherwise}\end{cases}$$

where $\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

PROOF. It follows from (1) and factorization theorem (see, p.361 Rohtagi and Saleh, 2012) that $S=\sum_{i=1}^{n}X_i^\alpha$ is a sufficient statistic for θ and the pdf of S is

$$f_s(s\mid\theta)=\frac{s^{n\nu-1}}{\Gamma(n\nu)\theta^{n\nu}}\exp\left(-\frac{s}{\theta}\right);\quad \nu>0,\theta>0,s\geq0. \tag{3}$$

From (2), since the distribution of S belongs to exponential family, it is also complete (see, p.367 Rohtagi and Saleh, 2012)]. The result now follows from (3) that

$$E[S^q] = \left\{ \frac{\Gamma(n\nu + q)}{\Gamma(n\nu)} \right\} \theta^q$$

$\square$

In the following theorem, we obtain UMVUE of the sampled pdf at a specified point.

THEOREM 2. *The UMVUE of the sampled pdf* (1) *at a specified point x is*

$$\widetilde{f}(x;\theta) = \begin{cases} \dfrac{\alpha x^{\alpha\nu-1}}{\beta(\nu,(n-1)\nu)S^\nu} \left[1 - \dfrac{x^\alpha}{S}\right]^{(n-1)\nu-1}; & x^\alpha < S \\ 0; & \textit{otherwise}, \end{cases}$$

where $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

PROOF. We can write,

$$f(x;\theta) = \frac{\alpha x^{\alpha\nu-1}}{\Gamma(\nu)\theta^\nu} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{x^\alpha}{\theta}\right)^i.$$

Applying Theorem 1,

$$\begin{aligned} \widetilde{f}(x;\theta) &= \frac{\alpha x^{\alpha\nu-1}}{\Gamma(\nu)} \sum_{i=0}^{\infty} \frac{(-1)^i x^\alpha i}{i!} (\widetilde{\theta}^{-(\nu+i)} \\ &= \frac{\alpha x^{\alpha\nu-1}}{S^\nu \beta(\nu,(n-1)\nu)} \sum_{i=0}^{(n-1)\nu-1} (-1)^i \binom{(n-1)\nu-1}{i} \left(\frac{x^\alpha}{S}\right)^i \end{aligned}$$

and the result follows. $\square$

THEOREM 3. *The UMVUE of $R(t)$ is*

$$\widetilde{R}(t) = \begin{cases} 1 - I_{\frac{t^\alpha}{S}}(\nu,(n-1)\nu); & t^\alpha < S \\ 0; & \textit{otherwise}, \end{cases}$$

where $I_x(p,q) = \frac{1}{\beta(p,q)} \int_0^x y^{p-1}(1-y)^{q-1} dy; 0 \le y \le 1,\ x < 1,\ p,q > 0$ is the incomplete beta function.

PROOF. We note that the expectation of $\int_t^\infty \widetilde{f}(x;\theta)dx$ with respect to S is $R(t)$. Thus, applying Theorem 2,

$$\begin{aligned}\widetilde{R}(t) &= \int_t^\infty \widetilde{f}(x;\theta)dx \\ &= \frac{\alpha}{\beta(\nu,(n-1)\nu)S^\nu}\int_t^\infty x^{\alpha\nu-1}\left[1-\frac{x^\alpha}{S}\right]^{(n-1)\nu-1}dx\end{aligned}$$

and the result follows by substituting $\frac{x^\alpha}{S}=z$.

Let X and Y be two independent random variables with respective pdf

$$f(x;\theta_1)=\frac{\alpha_1 x^{\alpha_1\nu_1-1}e^{-\frac{x^{\alpha 1}}{\theta_1}}}{\Gamma(\nu_1)(\theta_1)^{\nu_1}};\quad x>0,\theta_1,\alpha_1,\nu_1>0$$

and

$$f(y;\theta_2)=\frac{\alpha_2 y^{\alpha_2\nu_2-1}e^{-\frac{x^{\alpha 2}}{\theta_2}}}{\Gamma(\nu_2)(\theta_2)^{\nu_2}};\quad y>0,\theta_2,\alpha_2,\nu_2>0.$$

Let $X_1,X_2,\ldots,X_n$ be a random sample of size n from $f(x;\theta_1)$ and $Y_1,Y_2,\ldots,Y_m$ be a random sample of size m from $f(y;\theta_2)$. Define, $S=\sum_{i=1}^n X_i^{\alpha_1}$ and $T=\sum_{i=1}^m Y_i^{\alpha_2}$. □

THEOREM 4. *The UMVUE of P is*

$$\widetilde{P}=\begin{cases}\displaystyle\int_{z=0}^{1}\frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}I_{\frac{(SZ)^{\frac{\alpha_2}{\alpha_1}}}{T}}(\nu_2,(m-1)\nu_2)dz; & S^{\frac{1}{\alpha_1}}\le T^{\frac{1}{\alpha_2}}\\ \displaystyle\int_{z=0}^{\frac{T^{\frac{\alpha_1}{\alpha_2}}}{S}}\frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}I_{\frac{(SZ)^{\frac{\alpha_2}{\alpha_1}}}{T}}(\nu_2,(m-1)\nu_2)dz & \\ \quad +1-I_{\frac{T^{\frac{\alpha_1}{\alpha_2}}}{S}}(\nu_1,(n-1)\nu_1); & S^{\frac{1}{\alpha_1}}>T^{\frac{1}{\alpha_2}}\end{cases}$$

PROOF. It follows from Theorem 2 that

$$\widetilde{f}(x;\theta_1)=\begin{cases}\dfrac{\alpha_1 x^{\alpha_1\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)S^{\nu_1}}\left[1-\dfrac{x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1}; & x^{\alpha_1}<S\\ 0; & \text{otherwise}\end{cases}$$

and

$$\widetilde{f}(y;\theta_2)=\begin{cases}\dfrac{\alpha_2 y^{\alpha_2\nu_2-1}}{\beta(\nu_2,(m-1)\nu_2)T^{\nu_2}}\left[1-\dfrac{y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1}; & y^{\alpha_2}<T\\ 0; & \text{otherwise}\end{cases}$$

From the arguments similar to those used in Theorem 2,

$$\widetilde{P}=\int_{x=0}^{\infty}\int_{y=0}^{x}\widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\,dy$$

$$=\int_{x=0}^{\min(S^{\frac{1}{\alpha_1}},T^{\frac{1}{\alpha_2}})}\frac{\alpha_1 x^{\alpha_1\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)S^{\nu_1}}\left[1-\frac{x^{\alpha_1}}{S}\right]^{(n-1)\nu_1-1}I_{\frac{x^{\alpha_2}}{T}}(\nu_2,(m-1)\nu_2)dx. \quad (4)$$

When $S^{\frac{1}{\alpha_1}}\leq T^{\frac{1}{\alpha_2}}$, we substitute $\frac{x^{\alpha_1}}{S}=z$ and the first assertion follows. For $S^{\frac{1}{\alpha_1}}>T^{\frac{1}{\alpha_2}}$, the integral in (4) can be expressed as the sum of integrals on $(0,T^{\frac{1}{\alpha_2}})$ and $(T^{\frac{1}{\alpha_2}})$, $(S^{\frac{1}{\alpha_1}})$, then on substituting $\frac{x^{\alpha_1}}{S}=z$, we get

$$\widetilde{P}=\int_{z=0}^{\frac{T^{\frac{\alpha_1}{\alpha_2}}}{S}}\frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}I_{\frac{(SZ)^{\frac{\alpha_2}{\alpha_1}}}{T}}(\nu_2,(m-1)\nu_2)dz$$
$$+\int_{z=\frac{T^{\frac{\alpha_1}{\alpha_2}}}{S}}^{1}\frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)}dz$$

and the second assertion follows. It is interesting to note that on putting $\alpha_1=\alpha_2=1$, we get the UMVUE of $P(X>Y)$ obtained by Huang *et al.* (2012). Hence we were able to obtain a generalized expression of UMVUE of $P(X>Y)$ by a different yet simpler approach. Assuming both shape parameters ν_1 and ν_2 to be integers, Constantine *et al.* (1986) showed that UMVUE of $P(X>Y)$ can be expressed in terms of an incomplete beta function and hypergeometric series. Following which, in Corollary 5, we derive a generalized expression of UMVUE of $P(X>Y)$ when both the shape parameters ν_1 and ν_2 are integers. □

COROLLARY 5. *The UMVUE of P when the shape parameters ν_1 and ν_2 are integers is*

$$\widetilde{P}=\begin{cases}\dfrac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)}\displaystyle\sum_{i=0}^{(m-1)\nu_2-1}\frac{(-1)^i}{\nu_2+i}\binom{(m-1)\nu_2-1}{i} \\ \cdot\displaystyle\int_0^1 z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}\left[\frac{(zS)^{\frac{\alpha_2}{\alpha_1}}}{T}\right]^{\nu_2+i}dz; \qquad S^{\frac{1}{\alpha_1}}\leq T^{\frac{1}{\alpha_2}} \\ 1-\dfrac{1}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)}\displaystyle\sum_{i=0}^{(n-1)\nu_1-1}\frac{(-1)^i}{\nu_1+i}\binom{(n-1)\nu_1-1}{i} \\ \cdot\displaystyle\int_0^1 z^{\nu_2-1}(1-z)^{(m-1)\nu_2-1}\left[\frac{(zT)^{\frac{\alpha_1}{\alpha_2}}}{S}\right]^{\nu_1+i}dz; \qquad S^{\frac{1}{\alpha_1}}>T^{\frac{1}{\alpha_2}}\end{cases}$$

PROOF. From Theorem 4, for $S^{\frac{1}{\alpha_1}} \leq T^{\frac{1}{\alpha_2}}$,

$$\widetilde{P} = \int_{z=0}^{1} \frac{z^{\nu_1-1}(1-z)^{(n-1)\nu_1-1}}{\beta(\nu_1,(n-1)\nu_1)\beta(\nu_2,(m-1)\nu_2)} \cdot \int_{w=0}^{\frac{(SZ)^{\frac{\alpha_2}{\alpha_1}}}{T}} w^{\nu_2-1}(1-w)^{(m-1)\nu_2-1} dw dz$$

and the first assertion follows by binomial expansion of $(1-w)^{(m-1)\nu_2-1}$. For $S^{\frac{1}{\alpha_1}} > T^{\frac{1}{\alpha_2}}$, consider

$$\begin{aligned}\widetilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{f}(x;\theta_1)\widetilde{f}(y;\theta_2)dx\, dy \\ &= \int_{y=0}^{T^{\frac{1}{\alpha_2}}} \frac{\alpha_2 y^{\alpha_2\nu_2-1}}{\beta(\nu_2,(m-1)\nu_2)T^{\nu_2}} \left[1-\frac{y^{\alpha_2}}{T}\right]^{(m-1)\nu_2-1} [1-I_{\frac{y^{\alpha_1}}{S}}(\nu_1,(n-1)\nu_1)]dy\end{aligned}$$

and the second assertion follows on substituting $y^{\frac{\alpha_2}{T}} = z$. It is interesting to note that on putting $\alpha_1 = \alpha_2 = 1$, we get the UMVUE of $P(X > Y)$ obtained by Constantine *et al.* (1986). Hence we were able to obtain another generalized expression of UMVUE of $P(X > Y)$ by a different yet simpler approach when the shape parameters ν_1 and ν_2 are assumed to be integers. □

THEOREM 6. *The MLE of $R(t)$ is given by:*

$$\widehat{R}(t) = 1 - \frac{\gamma\left(\nu, \frac{n\nu t^{\alpha}}{S}\right)}{\Gamma(\nu)}$$

where $\gamma(a,r) = \int_0^r y^{a-1}e^{-y}dy$ is the lower incomplete gamma function.

PROOF. It can be easily seen from (2) that the MLE of θ is $\widehat{\theta} = \frac{S}{n\nu}$. From invariance property of MLE, the MLE of sampled pdf is

$$\widehat{f}(x;\theta) = \frac{\alpha x^{\alpha\nu-1}}{\Gamma(\nu)}\left(\frac{n\nu}{S}\right)^{\nu} \exp\left\{\frac{-n\nu x^{\alpha}}{S}\right\}.$$

Thus, $\widehat{R}(t) = \int_t^{\infty} \widehat{f}(x;\theta)dx.$ □

THEOREM 7. *The MLE of P is*

$$\widehat{P}=1-\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)}\int_{z=0}^{\infty} z^{\nu_2-1}e^{-z}\gamma\left(\nu_1,\frac{n\nu_1\left(\frac{zT}{m\nu_2}\right)^{\frac{\alpha_1}{\alpha_2}}}{S}\right)dz.$$

PROOF. We have

$$\begin{aligned}\widehat{P}&=\int_{y=0}^{\infty}\int_{x=y}^{\infty}\widehat{f}(x;\theta_1)\widehat{f}(y;\theta_2)dx\,dy\\&=\int_{y=0}^{\infty}\widehat{R}_X(y)\widehat{f}(y;\theta_2)dy\\&=\int_{y=0}^{\infty}\left[1-\frac{\gamma\left(\nu_1,\frac{n\nu_1 y^{\alpha_1}}{S}\right)}{\Gamma(\nu_1)}\right]\frac{\alpha_2 y^{\alpha_2\nu_2-1}}{\Gamma(\nu_2)}\left(\frac{m\nu_2}{T}\right)^{\nu_2}\exp\left\{\frac{-m\nu_2 y^{\alpha_2}}{T}\right\}dy\end{aligned}$$

and the theorem follows on substituting $\frac{m\nu_2 y^{\alpha_2}}{T}=z$. □

3. MLES AND MOMES WHEN ALL THE PARAMETERS ARE UNKNOWN

Now we discuss the case when all the three parameters α, ν and θ are unknown. For MLES, the log-likelihood function of the parameters α, ν and θ given the sample observations $\underline{x}$ is

$$l(\alpha,\nu,\theta\mid\underline{x})=n\log(\alpha)-n\log(\Gamma(\nu))-n\nu\log(\theta)-\frac{1}{\theta}\sum_{i=1}^{n}x_i^{\alpha}+(\alpha\nu-1)\sum_{i=1}^{n}\log(x_i).$$

The MLES of α, ν and θ are given by the simultaneous solution of the following three equations

$$\frac{\partial l}{\partial\alpha}=\frac{n}{\alpha}-\frac{1}{\theta}\sum_{i=1}^{n}x_i^{\alpha}\log(x_i)+\nu\sum_{i=1}^{n}\log(x_i)=0, \tag{5}$$

$$\frac{\partial l}{\partial\nu}=\frac{-n}{\Gamma(\nu)}\frac{d\Gamma(\nu)}{d\nu}-n\log(\theta)+\alpha\sum_{i=1}^{n}\log(x_i)=0, \tag{6}$$

$$\frac{\partial l}{\partial\theta}=\frac{-n\nu}{\theta}+\frac{\sum_{i=1}^{n}x_i^{\alpha}}{\theta^2}=0. \tag{7}$$

Since these non-linear equations don't have a closed form solution, therefore we apply Newton-Raphson algorithm to compute MLES of α and ν. These values of MLES of α and ν so obtained can be substituted in (7) to obtain MLE of θ. From (7), the MLE of θ is $\widehat{\theta}=\frac{\sum_{i=1}^{n} x_i^{\widehat{\alpha}}}{n\widehat{\nu}}$, where $\widehat{\alpha}$ and $\widehat{\nu}$ are the MLES of α and ν respectively. It is to be noted that from Theorem 6, Theorem 7 and invariance property of MLE, the MLE of $R(t)$ is given as

$$\widehat{R}(t)=1-\frac{\gamma\left(\widehat{\nu},\frac{n\widehat{\nu}t^{\widehat{\alpha}}}{S}\right)}{\Gamma(\widehat{\nu})},$$

where $S=\sum_{i=1}^{n} X_i^{\widehat{\alpha}}$ and the MLE of P is given as

$$\widehat{P}=1-\frac{1}{\Gamma(\widehat{\nu}_1)\Gamma(\widehat{\nu}_2)}\int_{z=0}^{\infty} z^{(\widehat{\nu}_2-1)}e^{-z}\gamma\left(\widehat{\nu}_1,\frac{n\widehat{\nu}_1\left(\frac{zT}{m\widehat{\nu}_2}\right)^{\frac{\widehat{\alpha}_1}{\widehat{\alpha}_2}}}{S}\right)dz,$$

where $S=\sum_{i=1}^{n} X_i^{\widehat{\alpha}_1}$, $T=\sum_{i=1}^{m} Y_i^{\widehat{\alpha}_2}$.

Next we derive the moment estimators of the parameters α, ν and θ of PEF. From Equation (1), we obtain the rth moment as

$$\begin{aligned}E(X^r)&=\int_0^{\infty}\frac{\alpha x^{r+\alpha\nu-1}e^{-\frac{x^\alpha}{\theta}}}{\Gamma(\nu)\theta^\nu}dx\\&=\frac{\theta^{\frac{r}{\alpha}}a_r}{\Gamma(\nu)},\end{aligned}$$

on substituting $\frac{x^\alpha}{\theta}=u$, where $a_r=\Gamma\left(\frac{r}{\alpha}+\nu\right)$. For $r=1,2,3$ and denoting $E(X^r)$ by $\overline{X^r}$, we obtain the following equations

$$\overline{X^2}a_1^2-\overline{X}^2a_2\Gamma(\widehat{\nu}_M)=0, \tag{8}$$

$$\widehat{\theta}_M-\left(\frac{\overline{X}\Gamma(\widehat{\nu}_M)}{a_1}\right)^{\widehat{\alpha}_M}=0, \tag{9}$$

$$\frac{\overline{X}^3}{\overline{X^3}}-\frac{a_1^3}{a_3\Gamma^2(\widehat{\nu}_M}=0. \tag{10}$$

These equations can be simultaneously solved using uniroot function in R software to obtain MOMES of the parameters α, ν and θ of PEF.

4. SIMULATION STUDIES

Firstly, we conduct Monte Carlo simulation studies to compare the performance of $\widetilde{\theta}^q$ and $\widetilde{\theta}^q$ for different sample sizes and powers of parameter θ. For $\nu=3$, we generate

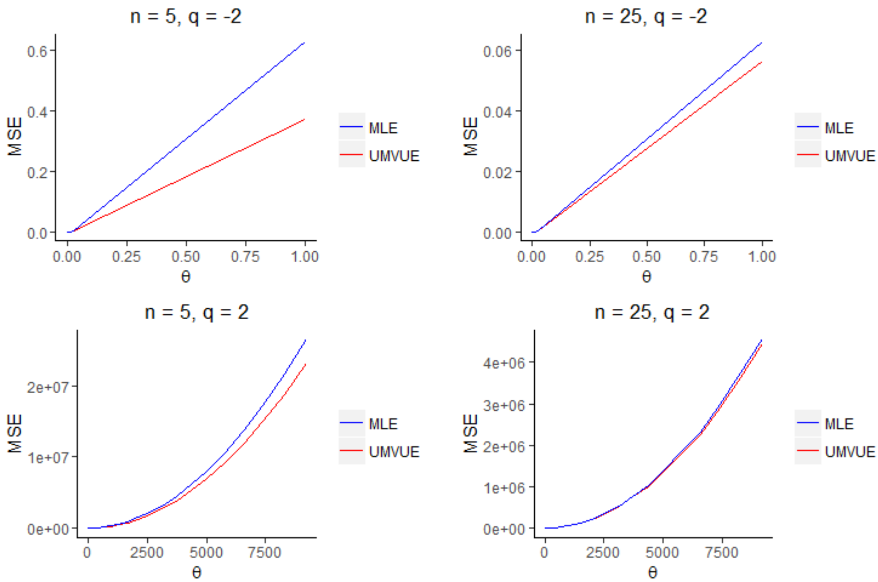

Figure 1 – MSE of the UMVUE and MLE of θ^q for different sample sizes and values of q.

10,000 samples each of size n from positive exponential family of distributions and repeat this procedure for several values of θ. Figure 1 shows the mean square error (MSE) of the UMVUE and MLE of θ^q. From these figures we note that the MSE of the MLE of θ^q is always greater than that of the UMVUE, however for large sample sizes these estimators of θ^q are better and almost equally efficient.

As an estimation of bias, Figure 2 shows the differences $\overline{\widetilde{\theta}^q} - \theta^q$ and $\overline{\widehat{\theta}^q} - \theta^q$ which are nothing but the difference between the average estimate of UMVUE and MLE and the true value of the parameter θ^q respectively. From these figures, it is clear that an average estimate of $\widetilde{\theta}^q$ based on 10,000 samples lies very close to the true value of θ^q. Certainly, it is consistent with the unbiasedness property of UMVUES. We may also note that $\widetilde{\theta}^q$ overestimates for large values of θ^q but its bias decreases for large sample sizes.

On similar lines, we perform simulation studies to compare the performance of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different sample sizes. For $t = 7$ and $\alpha = \nu = 2$, we generate 10,000 samples each of size n from positive exponential family of distributions and repeat this procedure for several values of $R(t)$. Figure 3 shows the MSE of the UMVUE and MLE of $R(t)$. From these figures we note that the MSE of the UMVUE of $R(t)$ is always greater than that of the MLE, however for large sample sizes these estimators of $R(t)$ are better and almost equally efficient. Figure 4 shows the estimated bias of $\widetilde{R}(t)$ and $\widehat{R}(t)$ for different

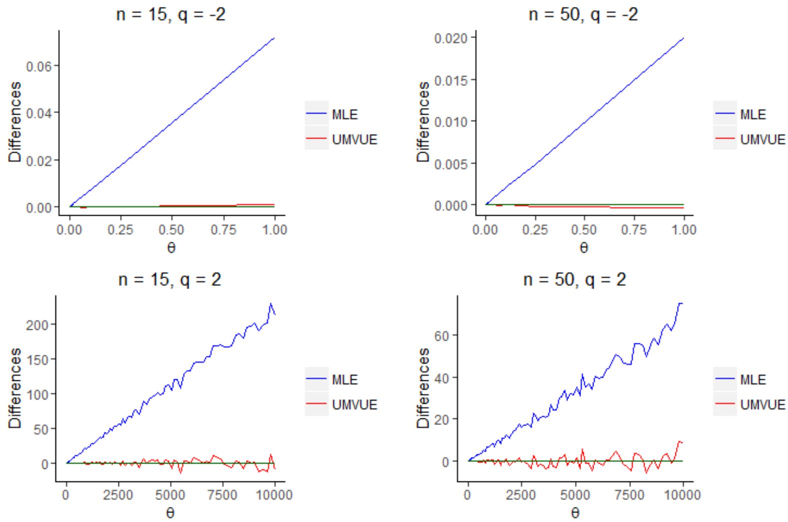

Figure 2 – Estimated bias of the UMVUE and MLE of θ^q for different sample sizes and values of q.

sample sizes. Similar to the results based on Figure 2, we can observe that $\overline{\widetilde{R}(t)}$ lies very close to the true parameter $R(t)$ and hence is in conformity with the unbiasedness property of UMVUES. As far as MLES are concerned, $\widehat{R}(t)$ overestimates for small values of $R(t)$ and underestimates for large values of $R(t)$, however this bias in the estimates of $\widehat{R}(t)$ decreases as sample size increases.

Now, we compare the performance of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. By Monte Carlo simulation, for $\alpha_1 = \nu_1 = 2$ and $\alpha_2 = \nu_2 = 3$, we generate 10,000 samples each of size n and m from positive exponential family of distributions and repeat this procedure for several values of P. Figure 5 shows the MSE of the UMVUE and MLE of P. From these figures we note that the MSE of the UMVUE of P is always greater than that of the MLE, however for large sample sizes these estimators of P are better and almost equally efficient. Note that this result has also been observed for estimators of $R(t)$ but not θ^q. Figure 6 shows the estimated bias of $\widetilde{P}$ and $\widehat{P}$ for different sample sizes. Similar to the results based on Figure 2 and Figure 4, we can observe that $\overline{\widetilde{P}}$ lies very close to the true parameter P and hence is in conformity with the unbiasedness property of UMVUES. As far as MLES are concerned, $\widehat{P}$ overestimates for small values of P and underestimates for large values of P, however this bias in the estimates of $\widehat{P}$ decreases as sample size increases. It is interesting to note that this result has also been observed for estimators of $R(t)$ but not θ^q.

Figure 7 shows the estimates of the pdf in (1) based on MLE and UMVUE which are good approximations of the pdf.

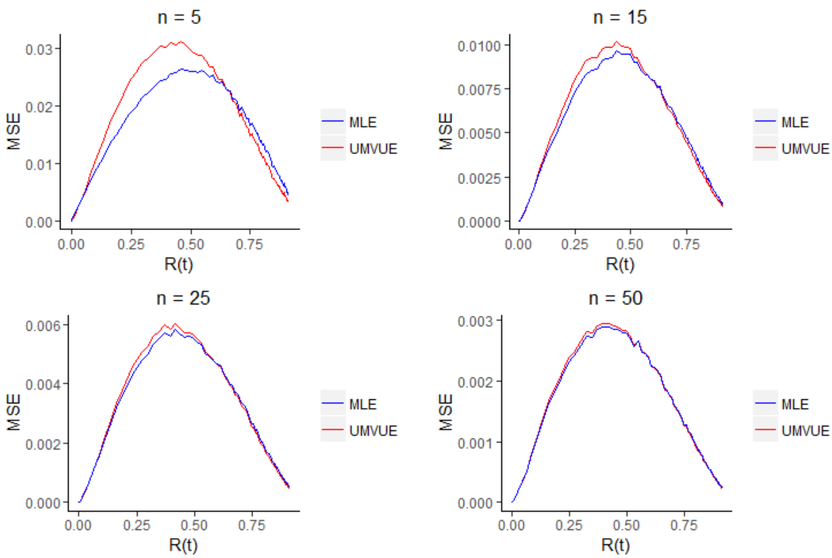

Figure 3 – MSE of the UMVUE and MLE of $R(t)$ for different sample sizes.

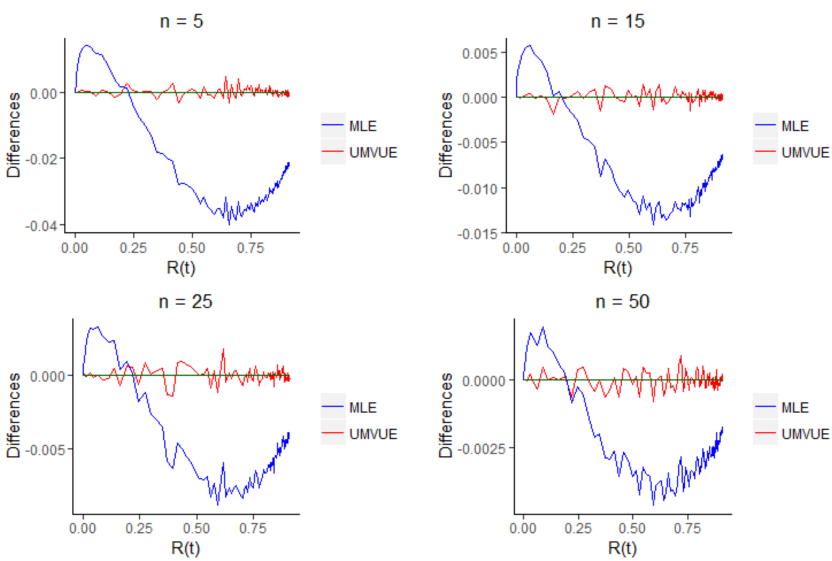

Figure 4 – Estimated bias of the UMVUE and MLE of $R(t)$ for different sample sizes.

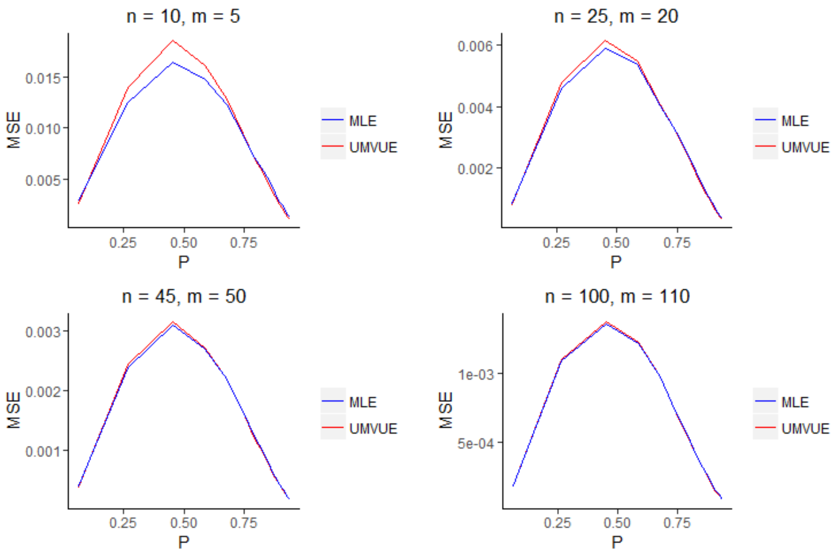

Figure 5 – MSE of the UMVUE and MLE of P for different sample sizes.

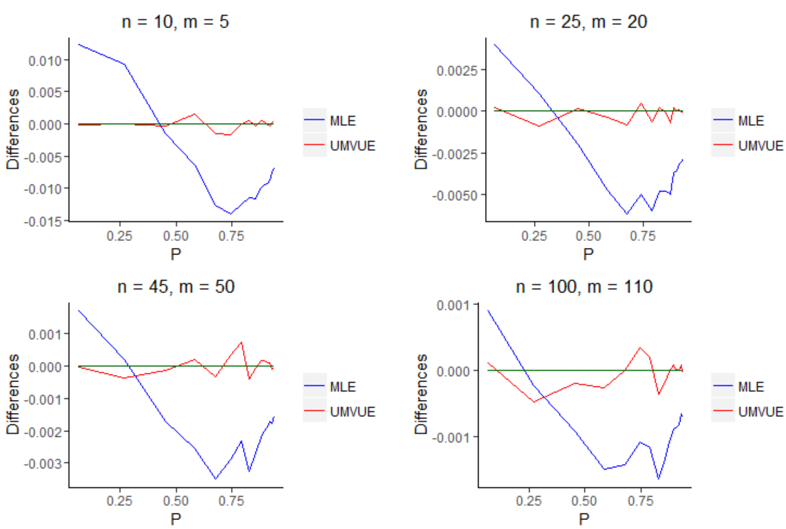

Figure 6 – Estimated bias of the UMVUE and MLE of P for different sample sizes.

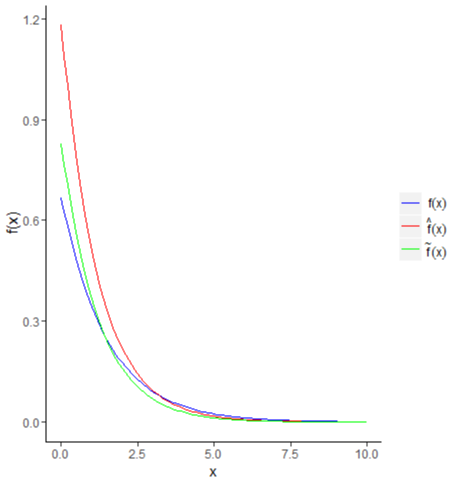

Figure 7 – MLE and UMVUE of sampled pdf.

5. AN EXAMPLE ON REAL DATA

This section deals with an example of real data to illustrate the proposed estimation methods. This data represents the strength of single carbon fibres (measured in GPA) and impregnated 1000-carbon fibres tows. Single fibres were tested under tension at gauge lengths of 20mm (Data set I) and 10mm (Data set II) with sample sizes 69 and 63 respectively. Kundu and Gupta (2006) analyzed these data sets by fitting two-parameter Weibull distribution. After subtracting 0.75 from each point of these data sets, Kundu and Gupta (2006) observed that the Weibull distributions with equal shape parameters fit to both these data sets. This has been confirmed in Figure 8 and Figure 9.

Let us assign the random variable $X \sim f(x;\alpha_1,\theta_1)$ to Data set I that has been reproduced in the following table.

Data Set I

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997	2.006
2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274
2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535
2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726	2.770	2.773
2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096	3.128
3.233	3.433	3.585	3.585									

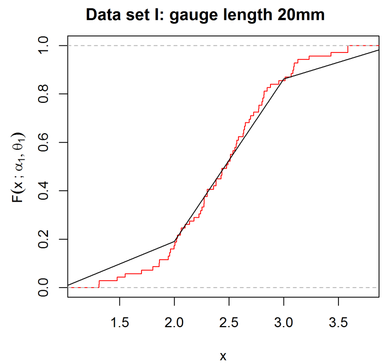

Figure 8 – The empirical and theoretical cdf of Weibull(α_1, θ_1) model.

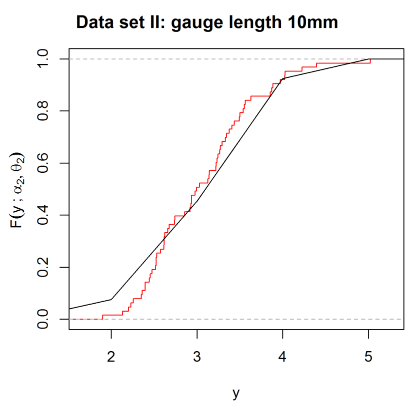

Figure 9 – The empirical and theoretical cdf of Weibull(α_2, θ_2) model.

Now let us assign the random variable $Y \sim f(y;\alpha_2,\theta_2)$ to Data set II that has been reproduced in the following table.

Data Set II

1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397	2.445	2.454	2.474	2.518
2.522	2.525	2.532	2.575	2.614	2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856
2.917	2.928	2.937	2.937	2.977	2.996	3.030	3.125	3.139	3.145	3.220	3.223	3.235
3.243	3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501	3.537	3.554
3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020		

The following Table 1 shows the different estimators of parameters α_1 and θ_1 of Weibull model and its corresponding reliability function $R_X(t)$ based on Data set I. Similarly Table 2 shows the different estimators of parameters α_2 and θ_2 of Weibull distribution and its corresponding reliability function $R_Y(t)$ based on Data set II.

Now, for the above two data sets, we obtain estimators of $P = P(X > Y)$ and the results are presented in Table 3.

TABLE 1
The MLE, UMVUE and MOME of parameters α_1 and θ_1 of Weibull distribution and its corresponding reliability function $R_X(t)$ for time $t = 2$ based on Data set I.

$\widehat{\alpha}_1$	$\widehat{\theta}_1$	$\widetilde{\theta}_1$	$\widehat{\alpha}_{1_M}$	$\widehat{\theta}_{1_M}$	$\widehat{R}_X(t)$	$\widetilde{R}_X(t)$
5.5049	214.1524	214.1515	5.7810	278.4287	0.8089	0.8111

TABLE 2
The MLE, UMVUE and Bayes estimator of parameters α_2 and θ_2 of Weibull distribution and its corresponding reliability function $R(t)$ for time $t = 2$ based on Data set II.

$\widehat{\alpha}_2$	$\widehat{\theta}_2$	$\widetilde{\theta}_2$	$\widehat{\alpha}_{2_M}$	$\widehat{\theta}_{2_M}$	$\widehat{R}_Y(t)$	$\widetilde{R}_Y(t)$
5.0462	422.7363	422.8160	5.7553	473.2841	0.9248	0.9259

TABLE 3
The MLE and UMVUE of $P(X > Y)$.

$\widehat{P}$	$\widetilde{P}$
0.2426	0.2409

6. Conclusion

A lot of work has been done in literature on the estimation of parametric functions of various lifetime distributions. In the present paper, we have discussed a positive exponential family of distributions which is quite useful in reliability theory. New techniques have been adopted to obtain the UMVUES of parametric functions. We have also derived generalized expressions for the UMVUE of $P(X > Y)$ using a simple technique.

From the above numerical findings, it is interesting to observe that even though the estimators of $R(t)$ and P based on MLE are biased estimators, their MSE is nearly same and in fact smaller than the MSE of the estimators of $R(t)$ and P based on UMVUE, most of the time. This indicates that the estimators of $R(t)$ and P based on MLE are more efficient than the estimators of $R(t)$ and P based on UMVUE. Considering this fact and the fact that computations of $\widehat{P}$ based on Theorem 7 are relatively simpler than the computations of $\widetilde{P}$ based on Theorem 4, the estimator of $P(X > Y)$ based on MLE is recommended over UMVUE for estimating $P(X > Y)$. Following the same arguments, the estimator of θ^q based on UMVUE is recommended over that of MLE for estimating θ^q.

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SUMMARY

A positive exponential family of distributions is taken into consideration. Two measures of reliability are discussed. Uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES) are developed for the reliability functions. In addition to the UMVUES and MLES, we derive the method of moment estimators (MOME). The performances of two types of estimators are compared through Monte Carlo simulation.

Keywords: Positive exponential family of distribution; Point estimation; Uniformly minimum variance unbiased estimator; Maximum likelihood estimator; Method of moment estimators; Monte Carlo simulation.