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A COMPENDIUM OF COPULAS

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1. Introduction

A p-dimensional copula is a function $C:[0,1]^p \to [0,1]$ that satisfies

i) $C\left(u_1,\ldots,u_{i-1},0,u_{i+1},\ldots,u_p\right)=0$ for all $1 \leq i \leq p$ and $0 \leq u_k \leq 1$, $k=1,\ldots,p$, $k \neq i$. That is, the copula is zero if any one of its arguments is zero;

ii) $C(1,\ldots,1,u,1,\ldots,1)=u$ for $0<u<1$ in each of the p arguments. That is, the copula is equal to u if one argument is u and all others are equal to 1;

iii) for any a_i, b_i ordered like $a_i \leq b_i$, $i=1,\ldots,p$,

$$\sum_{i_1=1}^{2}\cdots\sum_{i_p=1}^{2}(-1)^{i_1+\cdots+i_p} C\left(u_{1,i_1},\ldots,u_{p,i_p}\right) \geq 0,$$

where $u_{j,1}=a_j$ and $u_{j,2}=b_j$ for $j=1,\ldots,p$. That is, the copula is non-decreasing in p dimensions.

The concept of copulas was introduced by Sklar (1959). His fundamental theorem states the following:

> Let F be a p-dimensional cumulative distribution function with marginal distribution functions F_i, $i=1,\ldots,p$. Then there exists a copula C such that
>
> $$F\left(x_1,\ldots,x_p\right)=C\left(F_1(x_1),\ldots,F_p\left(x_p\right)\right).$$

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Conversely, for any univariate cumulative distribution functions $F_1, \ldots, F_p$ and any copula C, the function F is a p-dimensional cumulative distribution function with marginals $F_1, \ldots, F_p$. Furthermore, if $F_1, \ldots, F_p$ are continuous, then C is unique.

There are several proofs of Sklar's theorem (Moore and Spruill, 1975): see Carley and Taylor (2002) for a proof using the notion of checkerboard copula; Rüschendorf (2009) for a proof using distributional transforms; Durante *et al.* (2013) for a topological proof; Faugeras (2013) for a proof using probabilistic continuation and two consistency results; Oertel (2015) for a proof based on the use of right quantile functions.

Sklar's theorem has also been extended to other contexts: see Mayor *et al.* (2007) for a discrete extension involving copula-like operators defined on a finite chain; Durante *et al.* (2012) for an extension when at least one component of the copula is discrete; Montes *et al.* (2015) for an extension when there is imprecision about the marginals; Schmelzer (2015) for an extension for minitive belief functions.

There are two extreme types of dependence exhibited by a copula. A copula C is said to exhibit independence if

$$C\left(u_1, \ldots, u_p\right) = u_1 \cdots u_p$$

for all $0 \leq u_1, \ldots, u_p \leq 1$. A copula C is said to exhibit complete dependence if

$$C\left(u_1, \ldots, u_p\right) = \min\left(u_1, \ldots, u_p\right)$$

for all $0 \leq u_1, \ldots, u_p \leq 1$.

Sklar (1959)'s theorem allows one to model dependence between two or more variables by means of a copula. Many parametric, non-parametric and semi-parametric models have been proposed for copulas, including methods for constructing models for copulas. Most of the proposed models have been parametric models. There are fewer non-parametric models and even fewer semi-parametric models.

Applications of copulas are too numerous to list. Most applications have been based on parametric models for copulas. There are not many applications based on non-parametric or semi-parametric models. Furthermore, the parametric models used have been very limited (for example, Archimedean copulas). This is possibly due to the practitioners not being aware of the range of parametric copulas available.

The aim of this paper is to provide an up-to-date and a comprehensive collection of known parametric copulas. We feel that such a review is timely because most models for copulas have been proposed in the last few years. We feel also that such a review could serve as an important reference, encourage more of the copulas being applied and encourage further developments of copulas.

There are several books and review papers written on copulas. For books, we refer the readers to Dall'Aglio *et al.* (1991), Rüschendorf *et al.* (1996), Beneš and Stěpán (1997), Joe (1997), Drouet-Mari and Kotz (2001), Cuadras *et al.* (2002), Cherubini *et al.* (2004), Genest (2005a), Genest (2005b), McNeil *et al.* (2005), Schweizer and Sklar (2005), Alsina *et al.* (2006), Malevergne and Sornette (2006), Salvadori *et al.* (2007), Nelsen (2006), Balakrishnan and Lai (2009), Jaworski *et al.* (2010), Mai and Scherer (2012), Jaworski *et al.* (2013), Rüschendorf (2013), Joe (2014), Mai and Scherer (2014) and Durante and Sempi (2015). An appendix to Salvadori *et al.* (2007) written by Durante has a list of families of copulas, with graphs and level curves.

For review papers, we refer the readers to Schweizer (1991), Nelsen (2002), Embrechts *et al.* (2003), Kolev *et al.* (2006), Genest and Nešlehová (2007), Genest *et al.* (2009), Kolev and Paiva (2009), Manner and Reznikova (2012) and Patton (2012). But to the best of our knowledge, none of these have provided an up-to-date and a comprehensive review of the kind given in this paper.

Because of the length of this paper, we have not given details of copulas like probabilistic interpretations, analytical properties, estimation methods and simulation algorithms. These details can be read from the cited references. For many of the given copulas, details like analytical properties, estimation methods and simulation algorithms have not been worked out. Also many of the given copulas are not implemented in R or any of its contributed packages. These could be some open problems for the reader. Some other open problems are: selection criteria between two or more copulas; characterizations of copulas; efficient estimation methods; efficient simulation algorithms; estimation of copulas under misspecification; change point estimation of copulas; Bayesian copulas; copula density estimation; time series models based on copulas; compatibility of copulas; copula calibration; bounds for copulas; transformations to improve fits of copulas like those in Michiels and de Schepper (2012); extreme value behaviors of bivariate and multivariate copulas; further measures of asymmetry for bivariate and multivariate copulas like those in Rosco and Joe (2013); further tests for symmetry for bivariate and multivariate copulas like those in Genest and Nešlehová (2014); development of comprehensive R contributed packages for copulas; applications to novel areas of current interest; and so on. There are also many problems associated with copula processes, time varying copulas, space varying copulas, and copulas varying with respect to both time and space, concepts not discussed in this paper. Further open problems are stated throughout.

The copulas are grouped into five sections. The Archimedean copula, its particular cases and related copulas are given in Section 2. The elliptical copula and its particular cases are given in Section 3. The EFGM copula, its particular cases and related copulas are given in Section 4. The extreme value copula, its particular cases and related copulas are given in Section 5. Other copulas are given in Section 6. The copulas within each section are presented in chronological order. The list is by no means complete, but we believe we have covered most of the important parametric copulas.

2. Archimedean copula

After the work on associativity by Ling (1965), who continued a long line of investigations started by Abel (1826), Archimedean copulas were defined by

$$C\left(u_1,\ldots,u_p\right)=\psi\left(\sum_{i=1}^{p}\psi^{-1}(u_i)\right),$$

where $\psi:[0,1]\to[0,\infty)$ is a real valued function satisfying $(-1)^k d^k\psi(x)/d^k x\geq 0$ for all $x\geq 0$ and $k=1,\ldots,p-2$ and $(-1)^{p-2}\psi^{p-2}(x)$ is non-increasing and convex. For more on this definition, see McNeil and Nešlehová (2009). The use of Archimedean copulas was popularised by Genest and MacKay (1986). In the bivariate case, the Kendall tau rank correlation coefficient and the tail dependence coefficient are

$$1+4\int_0^1\frac{\psi(t)}{\psi'(t)}dt$$

and

$$2-2\lim_{t\to 0}\frac{\psi'(t)}{\psi'(2t)},$$

respectively. Particular cases of the Archimedean copula include Clayton's copula in Section 2.3, Plackett's copula in Section 2.1, Nelsen's copula in Section 2.6, the AMH copula in Section 2.2, Gumbel's copula in Section 5.1 and Frank copula in Section 2.4.

Extensions of the Archimedean copula include hierarchical/nested Archimedean copulas studied by Joe (1997), Whelan (2004), McNeil *et al.* (2005), Hofert (2008), McNeil (2008), Hering *et al.* (2010) and Savu and Trede (2010). An asymmetric Archimedean copula due to Wei and Hu (2002) is

$$C\left(u_1,\ldots,u_p\right)=\psi\left(\psi^{-1}\circ\phi\left(\sum_{i=1}^{k}\psi^{-1}(u_i)\right)+\sum_{i=k+1}^{p}\psi^{-1}(u_i)\right)$$

for $2\leq k\leq p$, where $\phi:[0,1]\to[0,\infty)$ satisfies the same properties as ψ. An extension due to Durante *et al.* (2007) is

$$C(u_1,u_2)=\phi^{-1}\left(\phi\left(\min(u_1,u_2)\right)+\psi\left(\max(u_1,u_2)\right)\right),$$

where $\phi:[0,1]\to[0,\infty)$ is continuous, strictly decreasing, convex and $\psi:[0,1]\to[0,\infty)$ is continuous, decreasing such that $\psi(1)=0$ and $\psi-\phi$ is increasing. Another extension due to Durante *et al.* (2007) is

$$C(u_1,u_2)=\phi^{-1}\left(\phi\left(\min(u_1,u_2)\right)\psi\left(\max(u_1,u_2)\right)\right),$$

where $\phi:[0,1]\to[0,1]$ is continuous, increasing, log-concave and $\psi:[0,1]\to[0,1]$ is continuous, increasing such that $\psi(1)=1$ and ϕ/ψ is increasing.

Archimedean copulas have received widespread applications. Some recent applications have included: bivariate rainfall frequency distributions (Zhang and Singh, 2007a); correlation smile matching for collateralized debt obligation tranches (Prange and Scherer, 2009); Collateralized Debt Obligations pricing (Hofert and Scherer, 2011); life expectancy estimation (Lee *et al.*, 2011); risk assessment of hydroclimatic variability on groundwater levels in the Manjara basin aquifer in India (Reddy and Ganguli, 2012); modeling of wind speed dependence in system reliability assessment (Xie *et al.*, 2012); models of tourists' time use and expenditure behavior with self-selection (Zhang *et al.*, 2012); simulation of multivariate sea storms (Corbella and Stretch, 2013).

2.1. Plackett's copula

Plackett (1965) has defined the copula

$$C(u_1, u_2) = \frac{1 + (\theta - 1)(u_1 + u_2) - \sqrt{[1 + (\theta - 1)(u_1 + u_2)]^2 - 4\theta(\theta - 1)u_1 u_2}}{2(\theta - 1)}$$

for $\theta > 0$. Independence corresponds to $\theta = 1$. The Spearman's rank correlation coefficient is $\frac{\theta+1}{\theta-1} - \frac{2\theta}{(\theta-1)^2}$.

Recent applications of Plackett's copula have included: analysis of extreme rainfall at several stations in Indiana (Kao and Govindaraju, 2008); frequency analysis of droughts in China (Song and Singh, 2010).

2.2. AMH copula

The Ali-Mikhail-Haq copula due to Ali *et al.* (1978) is defined by

$$C\left(u_1, \ldots, u_p\right) = (1 - \alpha)\left[\prod_{i=1}^{p}\left(\frac{1 - \alpha}{u_i} + \alpha\right) - \alpha\right]^{-1}$$

for $-1 \leq \alpha \leq 1$. Independence corresponds to $\alpha = 0$. In the bivariate case, the Kendall tau rank and Spearman's rank correlation coefficients are

$$\frac{3\alpha - 2}{3\alpha} - \frac{2(1 - \alpha)^2 \log(1 - \alpha)}{3\alpha^2}$$

and

$$\frac{12(1 + \alpha)\mathrm{dilog}(1 - \alpha) - 24(1 - \alpha)\log(1 - \alpha)}{\alpha^2} - \frac{3(\alpha + 12)}{\alpha},$$

respectively, where $\mathrm{dilog}(\cdot)$ denotes the dilogarithm function defined by

$$\mathrm{dilog}(x) = \int_1^x \frac{\log t}{1 - t} dt.$$

The former takes values in $(-0.1817, 0.3333)$. The latter takes values in $(-0.271, 0.478)$. The copula exhibits positively quadrant dependence, likelihood ratio dependence and positively regression dependence. Recent applications of the copula have included estimation in coherent reliability systems (Eryilmaz, 2011).

2.3. *Clayton's copula*

Clayton (1978), Cook and Johnson (1981) and Oakes (1982) have defined the copula

$$C\left(u_1, \ldots, u_p\right) = \left[\sum_{i=1}^{p} u_i^{-\alpha} - p + 1\right]^{-1/\alpha}$$

for $\alpha > 0$. Independence corresponds to $\alpha \to 0$. Complete dependence corresponds to $\alpha \to \infty$. The copula exhibits monotone regression dependence. It is one of the most popular copulas. Its recent applications have included: analysis of bivariate truncated data (Wang, 2007); tail dependence estimation in financial market risk management (Shamiri *et al.*, 2011); probable modeling of hydrology data (Bekrizadeh *et al.*, 2013); estimation of failure probabilities in hazard scenarios (Salvadori *et al.*, 2016).

2.4. *Frank's copula*

Frank (1979) has defined the copula

$$C\left(u_1, u_2\right) = \log_\alpha \left[1 + \frac{\left(\alpha^{u_1} - 1\right)\left(\alpha^{u_2} - 1\right)}{\alpha - 1}\right]$$

for $\alpha > 0$. Positive dependence corresponds to $0 < \alpha < 1$, independence corresponds to $\alpha \to 1$ and negative dependence corresponds to $\alpha > 1$. The copula exhibits positively likelihood ratio dependence if $0 < \alpha < 1$. The p-variate version is

$$C\left(u_1, \ldots, u_p\right) = \log_\alpha \left[1 + \frac{\prod_{i=1}^{p} \left(\alpha^{u_i} - 1\right)}{(\alpha - 1)^{p-1}}\right]$$

for $\alpha \geq 0$.

Frank's copula has received many applications. Some recent applications have included: intensity-duration model of storm rainfall (de Michele and Salvadori, 2003); analytical calculation of storm volume statistics (Salvadori and de Michele, 2004a); characterization of temporal structure of storms (Salvadori and de Michele, 2006); modeling of higher-order correlations of neural spike counts (Onken and Obermayer, 2009); drought frequency analysis (Wong, 2013); modeling of acoustic signal energies (García and González-Lópeza, 2014).

2.5. Order statistics copula

There are several order statistics copulas. One due to Schmitz (2004) is

$$C(u_1,u_2)=\begin{cases} u_2-\left[(1-u_1)^{1/n}+u_2^{1/n}-1\right]^n, & \text{if } 1-(1-u_1)^{1/n}<u_2^{1/n}, \\ u_2, & \text{if } 1-(1-u_1)^{1/n}\geq u_2^{1/n} \end{cases}$$

for n an integer greater than or equal to one. The Kendall tau rank and Spearman's rank correlation coefficients are

$$\frac{1}{2n-1}$$

and

$$3-\frac{12n}{\binom{2n}{n}}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k}\binom{2n}{n+k}+\frac{12(-1)^n(n!)^3}{(3n)!},$$

respectively. This copula is related to the Clayton copula, see Section 2.3.

2.6. Nelsen (2006)'s copulas

Nelsen (2006) has assembled a range of different copulas. Some of them are

$$C\left(u_1,\ldots,u_p\right)=\exp\left[1-\left\{1+\sum_{i=1}^{p}\left[(1-\log u_i)^{\theta}-1\right]\right\}^{\frac{1}{\theta}}\right]$$

for $\theta\geq 1$ with independence corresponding to $\theta=1$;

$$C(u_1,u_2)=\left[1+\frac{\left[(1+u_1)^{-\alpha}-1\right]\left[(1+u_2)^{-\alpha}-1\right]}{2^{-\alpha}-1}\right]^{-1/\alpha}-1$$

for $\alpha>0$;

$$C(u_1,u_2)=-\frac{1}{\theta}\log\left\{1+\frac{[\exp(-\theta u_1)-1][\exp(-\theta u_2)-1]}{\exp(-\theta)-1}\right\}$$

for $-\infty<\theta<\infty$ with independence corresponding to $\theta\to 0$;

$$C(u_1,u_2)=\frac{\theta^2u_1u_2-(1-u_1)(1-u_2)}{\theta^2-(\theta-1)^2(1-u_1)(1-u_2)}$$

for $1\leq\theta<\infty$;

$$C(u_1,u_2)=u_1u_2\exp\left[-\theta\log u_1\log u_2\right]$$

for $0 < \theta \leq 1$ with independence corresponding to $\theta \to 0$;

$$C(u_1, u_2) = \frac{u_1 u_2}{\left[1 + \left(1 - u_1^\theta\right)\left(1 - u_2^\theta\right)\right]^{1/\theta}}$$

for $0 < \theta \leq 1$;

$$C(u_1, u_2) = \frac{1}{2}\left[S + \sqrt{S^2 + 4\theta}\right]$$

for $0 \leq \theta < \infty$, where

$$S = u_1 + u_2 - 1 - \theta\left(\frac{1}{u_1} + \frac{1}{u_2} - 1\right);$$

$$C(u_1, u_2) = \left\{1 + \frac{\left[(1 + u_1)^{-\theta} - 1\right]\left[(1 + u_1)^{-\theta} - 1\right]}{2^{-\theta} - 1}\right\}^{-1/\theta}$$

for $-\infty < \theta < \infty$;

$$C(u_1, u_2) = 1 + \theta\left\{\log\left[\exp\left(\frac{\theta}{u_1 - 1}\right) + \exp\left(\frac{\theta}{u_2 - 1}\right)\right]\right\}^{-1}$$

for $2 \leq \theta < \infty$;

$$C(u_1, u_2) = \theta\left\{\log\left[\exp\left(\frac{\theta}{u_1}\right) + \exp\left(\frac{\theta}{u_2}\right) - \exp(\theta)\right]\right\}^{-1}$$

for $0 < \theta < \infty$;

$$C(u_1, u_2) = \left\{\log\left[\exp\left(u_1^{-\theta}\right) + \exp\left(u_2^{-\theta}\right) - \exp(1)\right]\right\}^{-1/\theta}$$

for $0 < \theta < \infty$;

$$C(u_1, u_2) = \left[1 - \left(1 - u_1^\theta\right) u_2^{\theta/2} - \left(1 - u_2^\theta\right) u_1^{\theta/2}\right]^{1/\theta}$$

for $0 \leq \theta < 1$;

$$C(u_1, u_2) = 1 - \left[(1 - u_1)^\theta + (1 - u_2)^\theta - (1 - u_1)^\theta (1 - u_2)^\theta\right]^{1/\theta}$$

for $1 \leq \theta < \infty$;

$$C(u_1, u_2) = \left[u_1^\theta u_2^\theta - 2\left(1 - u_1^\theta\right)\left(1 - u_2^\theta\right)\right]^{1/\theta}$$

for $0 < \theta \leq 1/2$;

$$C(u_1,u_2) = \left\{1 + \left[\left(u_1^{-1}-1\right)^{\theta} + \left(u_2^{-1}-1\right)^{\theta}\right]^{1/\theta}\right\}^{-1}$$

for $1 \leq \theta < \infty$;

$$C(u_1,u_2) = \left\{1 + \left[\left(u_1^{-1/\theta}-1\right)^{\theta} + \left(u_2^{-1/\theta}-1\right)^{\theta}\right]^{1/\theta}\right\}^{\theta}$$

for $0 < \theta \leq 1$;

$$C(u_1,u_2) = \left\{1 - \left[\left(1-u_1^{1/\theta}\right)^{\theta} + \left(1-u_2^{1/\theta}\right)^{\theta}\right]^{1/\theta}\right\}^{\theta}$$

for $1 \leq \theta < \infty$; and

$$C(u_1,u_2) = 1 - \left[1 - \left\{\left[1-(1-u_1)^{\theta}\right]^{1/\theta} + \left[1-(1-u_2)^{\theta}\right]^{1/\theta} - 1\right\}^{\theta}\right]^{1/\theta}$$

for $1 \leq \theta < \infty$.

This list of copulas also appears in Section 2.6 of Alsina *et al.* (2006). All of the copulas in the list are Archimedean copulas. Some of these copulas are due to Joe and Hu (1996). The original references for these copulas can be found in Alsina *et al.* (2006).

2.7. *Ahmadi-Clayton copula*

Let $\theta_i \geq 0$, $i = 1,\ldots,p$ be such that $\theta_i \geq \theta_{i-1}$, $i = 2,\ldots,p$. Let n_i, $i = 1,\ldots,p$ be integers summing to n. Let $\mathbf{\Pi}$ denote a permutation matrix with each row and each column containing only one element equal to one and the remaining elements equal to zero. Javid (2009) has defined the following extension of the Clayton copula

$$C(u_1,\ldots,u_n) = \prod_{i=1}^{p} \left[\sum_{j=n_{i-1}+1}^{n_i} u_j^{-\theta_i} - n_i + n_{i-1} + 1\right]^{-1/\theta_i},$$

where $(z_1,\ldots,z_n)^T = \mathbf{\Pi}(u_1,\ldots,u_n)^T$. This copula contains Clayton's copula in Section 2.3 as a particular case.

3. Elliptical copulas

A random vector $\mathbf{X} = \left(X_1,X_2,\ldots,X_p\right)^T$ is said to have an elliptical distribution if the characteristic function $\psi_{\mathbf{X}-\mu}(t)$ of $\mathbf{X}-\mu$ is a function of the quadratic form $\mathbf{t}^T\mathbf{\Sigma}\mathbf{t}$, that is, $\psi_{\mathbf{X}-\mu}(t) = \phi\left(\mathbf{t}^T \Sigma \mathbf{t}\right)$ (Cambanis *et al.*, 1981; Fang *et al.*, 1990). Elliptical distributions can also be defined by the joint density function of $\mathbf{X}$ taking the form

$$f(\mathbf{x}) = C\, g\left((\mathbf{x}-\mu)^T \Sigma^{-1} (\mathbf{x}-\mu)\right),$$

where C is a normalizing constant and g is a scaling function. If $\mathbf{X}$ is an elliptical random vector then the joint distribution of $\left(F_1(X_1), F_2(X_2), \ldots, F_p\left(X_p\right)\right)$ is said to be an elliptical copula. Kendall's tau rank correlation coefficient for elliptical copulas is $\frac{2}{\pi} \arcsin\left(\rho_{i,j}\right)$, where $\rho_{i,j} = \sigma_{i,j} / \left(\sigma_i \sigma_j\right)$ (Frahm *et al.*, 2003). Suppose that $1 - F_i(x) = \lambda_i(x) x^{-\eta}$ for $i = 1, 2, \ldots, p$, where $\lambda_i(x)$ are slowly varying functions and η is a tail index. Then the tail dependence coefficient for elliptical copulas (Frahm *et al.*, 2003) is

$$\frac{\displaystyle\int_0^{f(\rho_{i,j})} \frac{u^\eta}{\sqrt{u^2-1}} du}{\displaystyle\int_0^1 \frac{u^\eta}{\sqrt{u^2-1}} du},$$

where

$$f\left(\rho_{i,j}\right) = \sqrt{\frac{1+\rho_{i,j}}{2}}.$$

The Spearman's correlation coefficient for elliptical copulas (Fang *et al.*, 2002) is

$$12 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q_g(x) Q_g(y) f(x,y) dx dy - 3,$$

where

$$Q_g(x) = \frac{1}{2} + \frac{\pi^{(p-1)/2}}{\Gamma((p-1)/2)} \int_0^x \int_{u^2}^{\infty} \left(y-u^2\right)^{\frac{p-1}{2}-1} g(y) dy du.$$

The Gaussian copula in Section 3.1 and the t copula in Section 3.2 are particular cases of elliptical copulas. Other particular cases include Kotz type copula and Pearson type VII copula.

Meta elliptical copulas are extensions due to Fang *et al.* (2002). If $\mathbf{X}$ is an elliptical random vector then $\left(F_1^{-1}\left(Q_g(X_1)\right), F_2^{-1}\left(Q_g(X_2)\right), \ldots, F_p^{-1}\left(Q_g\left(X_p\right)\right)\right)$ is said to be a meta elliptical random vector. Kendall's tau rank correlation coefficient for meta elliptical copulas is the same as those for elliptical copulas. Applications of meta elliptical copulas have included frequency analysis of multivariate hydrological data (Genest *et al.*, 2007).

3.1. Gaussian copula

Let $\Phi(\cdot)$ denote the cumulative distribution function of a standard normal random variable and let $\Phi^{-1}(\cdot)$ denote its inverse. The Gaussian copula with correlation matrix Σ is defined by

$$C\left(u_1,\ldots,u_p\right)=\Phi_{\Sigma}\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}\left(u_p\right)\right),$$

where Φ_{Σ} denotes the joint cumulative distribution function of a p-variate normal random vector with zero means and correlation matrix Σ. If $p=2$ and the correlation coefficient is ρ then the tail dependence coefficient is zero. The Kendall tau rank correlation coefficient is $\frac{2}{\pi}\arcsin\rho$. A perturbed version of the Gaussian copula is presented in Fouque and Zhou (2008). A bivariate normal copula is discussed in Meyer (2013).

Because of the popularity of the normal distribution, Gaussian copula has been the most applied copula. Two oldest references are: Frees and Valdez (1998) on the "broken heart" phenomenon in insurance; Li (2000) giving an application to credit derivative pricing. Some recent applications have included: quantitative trait linkage analysis (Li *et al.*, 2006); reliability-based design optimization of problems with correlated input variables (Noh *et al.*, 2009); modeling of functional disability data (Dobra and Lenkoski, 2011); analysis of secondary phenotypes in case-control genetic association studies (He and Li, 2012); stochastic modeling of power demand (Lojowska *et al.*, 2012); long-term wind speed prediction (Yu *et al.*, 2013).

3.2. t copula

Let $t_\nu(\cdot)$ denote the cumulative distribution function of a Student's t random variable with degree of freedom ν and let $t_\nu^{-1}(\cdot)$ denote its inverse. Let $G_\nu(\cdot)$ denote the cumulative distribution function of $\sqrt{\nu/\chi_\nu^2}$ and let $G_\nu^{-1}(\cdot)$ denote its inverse. Let $z_i(u_i,s)=t_{\nu_i}^{-1}(u_i)/G_{\nu_i}^{-1}(s)$ for $i=1,\ldots,p$. Luo and Shevchenko (2012) have defined the t copula with degrees of freedom $\left(\nu_1,\ldots,\nu_p\right)$ and correlation matrix Σ as

$$C\left(u_1,\ldots,u_p\right)=\int_0^1\Phi_{\Sigma}\left(z_1(u_1,s),\ldots,z_p\left(u_p,s\right)\right)ds, \tag{1}$$

where Φ_{Σ} denotes the joint cumulative distribution function of a p-variate normal random vector with zero means and correlation matrix Σ. Copulas of several multivariate t distributions are particular cases of (1). If $p=2$, $\nu_1=\nu_2=\nu$ and the correlation coefficient is ρ then the tail dependence coefficient is $2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$. The Kendall tau rank correlation coefficient is $\frac{2}{\pi}\arcsin\rho$. Note that the Kendall tau rank correlation coefficient is independent of ν and is the same as that for Gaussian copula. Estimation of t copulas is difficult. An open problem here is to develop efficient algorithms for estimation of t copulas.

Marginal tails of financial data are heavy tailed and hence they should be fitted by a distribution like the Student's t distribution, not the Gaussian distribution. Also dependence in joint extremes of multivariate financial data suggests a dependence structure allowing for tail-dependence. Hence, t copulas have become popular for modeling dependencies in financial data. Some recent applications have been: analysis of nonlinear and asymmetric dependence in the German equity market (Sun *et al.*, 2008); estimation of large portfolio loss probabilities (Chan and Kroese, 2010); risk modeling for future cash flow (Pettere and Kollo, 2011). See also Dakovic and Czado (2011).

4. EFGM COPULAS

Eyraud-Farlie-Gumbel-Morgenstern copula (Eyraud, 1936; Farlie, 1960; Gumbel, 1958, 1960; Morgenstern, 1956; Nelsen, 2006) is defined by

$$C(u_1,u_2)=u_1u_2[1+\phi(1-u_1)(1-u_2)] \tag{2}$$

for $-1\leq\phi\leq 1$. The Pearson correlation is $\phi/3$. Independence corresponds to $\phi=0$. The copula exhibits positively quadrant dependence, likelihood ratio dependence and positively regression dependence if $0\leq\phi\leq 1$. The p-variate version of (2) is

$$C\left(u_1,\ldots,u_p\right)=u_1\cdots u_p\left[1+\sum_{k=2}^{p}\sum_{1\leq j_1<\cdots<j_k\leq p}\theta_{j_1,\ldots,j_k}\left(1-u_{j_1}\right)\cdots\left(1-u_{j_k}\right)\right]$$

for $-1\leq\theta_{j_1,\ldots,j_k}\leq 1$ for all $j_1,\ldots,j_k$. Independence corresponds to $\theta_{j_1,\ldots,j_k}=0$ for all $j_1,\ldots,j_k$.

Several extensions of the Eyraud-Farlie-Gumbel-Morgenstern copula exist in the literature. An extension proposed by Ibragimov (2009) is

$$\begin{aligned} & C\left(u_1,\ldots,u_p\right) \\ = & \prod_{i=1}^{p}u_i\left[1+\sum_{c=2}^{p}\sum_{1\leq i_2<\cdots<i_c\leq p}a_{i_1,\ldots,i_c}\left(u_{i_1}^{\ell}-u_{i_1}^{\ell+1}\right)\cdots\left(u_{i_c}^{\ell}-u_{i_c}^{\ell+1}\right)\right]\end{aligned}$$

for $-\infty<a_{i_1,\ldots,i_c}<\infty$ such that

$$\sum_{c=2}^{p}\sum_{1\leq i_2<\cdots<i_c\leq p}\left|a_{i_1,\ldots,i_c}\right|\leq 1.$$

Independence corresponds to $a_{i_1,\ldots,i_c}=0$ for all $i_1,\ldots,i_c$.

Two extensions proposed by Bekrizadeh *et al.* (2012) are

$$C(u_1,u_2)=u_1u_2[1+\theta(1-u_1^{\alpha})(1-u_2^{\alpha})]^{n}$$

and

$$
\begin{aligned}
& C\left(u_1, \ldots, u_p\right) \\
= & \prod_{i=1}^p u_i\left\{1+\sum_{k=2}^p \sum_{1 \leq j_1<\cdots j_k \leq p} \theta_{j_1, \ldots, j_k}\left(1-u_{j_1}^{\theta_{j_1, \ldots, j_k}}\right) \cdots\left(1-u_{j_k}^{\theta_{j_1, \ldots, j_k}}\right)\right\}^n
\end{aligned}
$$

for $\alpha>0$, $\theta_{j_1, \ldots, j_k}>0$ and $n \geq 0$. Independence corresponds to $\theta=0$ and $\theta_{j_1, \ldots, j_k}=0$, respectively, for all $j_1, \ldots, j_k$. The Spearman's rank correlation coefficient of the former is

$$
12 \sum_{r=1}^n\binom{n}{r} \theta^r\left[\frac{\Gamma(r+1) \Gamma\left(\frac{2}{\alpha}\right)}{\alpha \Gamma\left(r+1+\frac{2}{\alpha}\right)}\right]^2 .
$$

EFGM copulas are some of the most popular copulas. Recent applications have included: modeling of directional dependence in exchange markets (Jung *et al.*, 2008); modeling of directional dependence of genes (Kim *et al.*, 2009); risk models with constant dividend barriers (Cossette *et al.*, 2011); modeling of Brazilian HIV data (Louzada *et al.*, 2012).

4.1. Rodríguez-Lallena and Úbeda-Flores's copula

Sarmanov (1966), Kim and Sungur (2004) and Rodríguez-Lallena and Úbeda Flores (2004) have defined the copula

$$
C\left(u_1, u_2\right)=u_1 u_2+\theta f\left(u_1\right) g\left(u_2\right) \tag{3}
$$

for $0 \leq \theta \leq 1$ and $f, g:[0,1] \rightarrow \mathbb{R}$ such that

i) $f(0)=f(1)=g(0)=g(1)=0$;

ii) f and g are absolutely continuous;

iii) $\min(\alpha\delta, \beta\gamma) \geq -1$, where $\alpha=\inf\left\{f^{\prime}(u): u \in A\right\}<0$, $\beta=\sup\left\{f^{\prime}(u): u \in A\right\}>0$, $\gamma=\inf\left\{g^{\prime}(v): v \in B\right\}<0$, $\delta=\sup\left\{g^{\prime}(v): v \in B\right\}>0$, $A=\left\{0 \leq u \leq 1: f^{\prime}(u) \text{ exists}\right\}$ and $B=\left\{0 \leq v \leq 1: g^{\prime}(v) \text{ exists}\right\}$.

The Kendall tau rank and Spearman's rank correlation coefficients are

$$
8 \int_0^1 f(t) d t \int_0^1 g(t) d t \tag{4}
$$

and

$$
12 \int_0^1 f(t) d t \int_0^1 g(t) d t, \tag{5}
$$

respectively. The copula can exhibit a variety of dependence structures, including left tail decreasing, right tail increasing, stochastically increasing, left corner set decreasing, right corner set increasing and positively likelihood ratio dependence. A recent application of (3) is to the Bayes premium in a collective risk model (Hernández-Bastida and Pilar Fernández-Sánchez, 2012).

Many authors have studied special cases of (3) in detail. A special case given in Rodríguez-Lallena and Úbeda Flores (2004) is

$$C\left(u_1, u_2\right) = u_1 u_2 + \theta u_1^a u_2^b \left(1-u_1\right)^c \left(1-u_2\right)^d$$

for $a, b, c, d \geq 1$. Rodríguez-Lallena and Úbeda Flores (2004) have shown that this is a copula if and only if

$$-\frac{1}{\max\left(\nu\gamma, \omega\delta\right)} \leq \theta \leq -\frac{1}{\min\left(\nu\delta, \omega\gamma\right)},$$

where $\omega = -\nu = 1$ if $a = c = 1$, $\delta = -\gamma = 1$ if $b = d = 1$ and

$$\begin{aligned}
\nu = & -\left(\frac{a}{a+c}\right)^{a-1} \left[1 + \sqrt{\frac{c}{a(a+c-1)}}\right]^{a-1} \left(\frac{c}{a+c}\right)^{c-1} \\
& \cdot \left[1 - \sqrt{\frac{a}{c(a+c-1)}}\right]^{c-1} \sqrt{\frac{ac}{a+c-1}}, \\
\omega = & \left(\frac{a}{a+c}\right)^{a-1} \left[1 - \sqrt{\frac{c}{a(a+c-1)}}\right]^{a-1} \left(\frac{c}{a+c}\right)^{c-1} \\
& \cdot \left[1 + \sqrt{\frac{a}{c(a+c-1)}}\right]^{c-1} \sqrt{\frac{ac}{a+c-1}}, \\
\gamma = & -\left(\frac{b}{b+d}\right)^{b-1} \left[1 + \sqrt{\frac{d}{b(b+d-1)}}\right]^{b-1} \left(\frac{d}{b+d}\right)^{d-1} \\
& \cdot \left[1 - \sqrt{\frac{b}{d(b+d-1)}}\right]^{d-1} \sqrt{\frac{bd}{b+d-1}}, \\
\delta = & \left(\frac{b}{b+d}\right)^{b-1} \left[1 - \sqrt{\frac{d}{b(b+d-1)}}\right]^{b-1} \left(\frac{d}{b+d}\right)^{d-1} \\
& \cdot \left[1 + \sqrt{\frac{b}{d(b+d-1)}}\right]^{d-1} \sqrt{\frac{bd}{b+d-1}}.
\end{aligned}$$

For this special case, (4) and (5) simplify to $8\theta B(a+1, c+1) B(b+1, d+1)$ and $12\theta B(a+1, c+1) B(b+1, d+1)$, respectively. Independence corresponds to $\theta = 0$.

Special cases also include the EFGM copulas. Two special cases due to Huang and Kotz (1999) are

$$C(u_1,u_2) = u_1u_2 + \theta u_1u_2(1-u_1)^{\gamma}(1-u_2)^{\gamma}$$

for $0 \le \theta \le 1$ and $\gamma \ge 1$; and

$$C(u_1,u_2) = u_1u_2 + \theta u_1u_2\left(1-u_1^{\gamma}\right)\left(1-u_2^{\gamma}\right)$$

for $0 \le \theta \le 1$ and $\gamma \ge 1/2$. The Pearson correlation coefficients are $12\theta(\gamma+2)^{-2}(\gamma+1)^{-2}$ and $3\theta\gamma^2(\gamma+2)^{-2}$, respectively. A special case due to Lai and Xie (2000) is

$$C(u_1,u_2) = u_1u_2 + \theta u_1^p u_2^p (1-u_1)^q (1-u_2)^q$$

for $0 \le \theta \le 1$. Its Pearson correlation coefficient is $12\theta B^2(p,q)$. Two special cases due to Jung *et al.* (2007) are

$$C(u_1,u_2) = u_1u_2 + \theta u_1^{\alpha} u_2^{\beta} (1-u_1)(1-u_2)$$

and

$$C(u_1,u_2) = u_1u_2 + \theta u_1u_2(1-u_1)^{\alpha}(1-u_2)^{\beta}$$

for $\alpha \ge 1$, $\beta \ge 1$ and $-1 \le \theta \le 1$. Independence for all these special cases corresponds to $\theta = 0$.

Based on (3), Kim *et al.* (2009) have introduced the three-dimensional EFGM copula

$$\begin{aligned} C(u_1,u_2,u_3) &= u_1u_2u_3\left[1+\theta_{13}(1-u_1)(1-u_3)\right]\left[1+\theta_{23}(1-u_2)(1-u_3)\right] \\ &\quad \cdot\left[1+\theta_{12}(1-u_1)(1-u_2)\right]\left[1+\theta_{13}u_1(1-u_3)\right] \\ &\quad \cdot\left[1+\theta_{23}u_2(1-u_3)\right]. \end{aligned}$$

Independence corresponds to $\theta_{13} = 0$, $\theta_{23} = 0$ and $\theta_{12} = 0$.

4.2. NQR copulas

Nelsen *et al.* (1997) have shown that

$$\begin{aligned} C(u_1,u_2) &= u_1u_2 + A_1u_1u_2^2(1-u_1)^2(1-u_2) + A_2u_1u_2(1-u_1)^2(1-u_2)^2 \\ &\quad + B_1u_1^2u_2^2(1-u_1)(1-u_2) + B_2u_1^2u_2(1-u_1)(1-u_2)^2 \end{aligned} \tag{6}$$

is a copula if A_1, A_2, B_1 and B_2 are in $\left\{(u_1,u_2) \in \mathbb{R}^2 : u_1^2 - u_1u_2 + u_2^2 - 3u_1 + 3u_2 = 0\right\}$ or $\left\{(u_1,u_2) \in \mathbb{R}^2 : -1 \le u_1 \le 2, -2 \le u_2 \le 1\right\}$. In particular,

$$C(u_1,u_2) = u_1u_2\left\{1 + (1-u_1)(1-u_2)\left[a + b(1-2u_1)(1-2u_2)\right]\right\} \tag{7}$$

is a copula if either $-1 \leq b \leq 1/2$ and $| a | \leq b+1$ or $1/2 \leq b \leq 2$ and $| a | \leq \sqrt{6b-3b^2}$. The Kendall tau rank and Spearman's rank correlation coefficients of (6) are $\frac{A_1+A_2+B_1+B_2}{18} + \frac{A_2B_1-A_1B_2}{450}$ and $\frac{A_1+A_2+B_1+B_2}{12}$, respectively. The copula can exhibit positively quadrant dependence, left tail dependence, right tail dependence and stochastically increasing dependence. Independence in (6) corresponds to $A_1 = 0$, $A_2 = 0$, $B_1 = 0$ and $B_2 = 0$. Independence in (7) corresponds to $a = 0$ and $b = 0$. Some of the EFGM copulas are particular cases of (6).

4.3. *Cubic copula*

Durrleman *et al.* (2000) have defined a copula referred to as a cubic copula by

$$C(u_1, u_2) = u_1 u_2 [1 + \alpha (u_1 - 1)(u_2 - 1)(2u_1 - 1)(2u_2 - 1)]$$

for $-1 \leq \alpha \leq 2$. Independence corresponds to $\alpha = 0$. The Kendall tau rank and Spearman's rank correlation coefficients are both equal to zero for any α. This copula is actually a particular case of Rodríguez-Lallena and Úbeda-Flores's copula in Section 4.1. Cubic copulas have been used to model rainfall data from Belgium and the USA (Evin and Favre, 2008).

4.4. *Polynomial copula*

A polynomial copula of degree m due to Drouet-Mari and Kotz (2001) is defined by

$$C(u_1, u_2) = u_1 u_2 \left[1 + \sum_{k \geq 1, q \geq 1, k+q \leq m-2} \frac{\theta_{k,q}}{(k+1)(q+1)} \left(u_1^k - 1 \right) \left(u_2^q - 1 \right) \right] \quad (8)$$

for $k \geq 1$ and $q \geq 1$, where

$$0 \leq \min \left[\sum_{k \geq 1, q \geq 1} \frac{q \theta_{k,q}}{(k+1)(q+1)}, \sum_{k \geq 1, q \geq 1} \frac{k \theta_{k,q}}{(k+1)(q+1)} \right] \leq 1.$$

Independence corresponds to $\theta_{k,q} = 0$ for all k and q. Some of the EFGM copulas are particular cases of (8).

4.5. *Bernstein copulas*

Let $\alpha \left(k_1/m_1, \ldots, k_p/m_p \right)$ be real constants for $1 \leq k_i \leq m_i$, $i = 1, \ldots, p$. Also let

$$P_{k_i, m_i}(u_i) = \binom{m_i}{k_i} u_i^{k_i} (1 - u_i)^{m_i - k_i}$$

for $i=1,\ldots,p$. Sancetta and Satchell (2004) have defined Bernstein copula as

$$\sum_{k_1=1}^{m_1}\cdots\sum_{k_p=1}^{m_p}\alpha\left(\frac{k_1}{m_1},\ldots,\frac{k_p}{m_p}\right)P_{k_1,m_1}(u_1)\cdots P_{k_p,m_p}\left(u_p\right) \tag{9}$$

provided that

$$\sum_{\ell_1=1}^{1}\cdots\sum_{\ell_p=0}^{1}(-1)^{\ell_1+\cdots+\ell_p}\alpha\left(\frac{k_1+\ell_1}{m_1},\ldots,\frac{k_p+\ell_p}{m_p}\right)<1$$

for $0\le k_i\le m_i$, $i=1,\ldots,p$ and

$$\min\left(\frac{k_1}{m_1}+\cdots+\frac{k_p}{m_p}-p+1\right)\le\alpha\left(\frac{k_1}{m_1},\ldots,\frac{k_p}{m_p}\right)\le\min\left(\frac{k_1}{m_1},\ldots,\frac{k_p}{m_p}\right).$$

But Bernstein copulas seem to have been known at least as early as Li *et al.* (1997). If $p=2$ and $m_1=m_2=m$ then Spearman's rank correlation coefficient is

$$12\sum_{p=0}^{m}\sum_{q=0}^{m}\gamma\left(\frac{p}{m},\frac{q}{m},1,\ldots,1\right)\binom{m}{p}\binom{m}{q}B(p+1,m+1-p)B(q+1,m+1-q),$$

where

$$\gamma\left(\frac{k_1}{m},\ldots,\frac{k_p}{m}\right)=\alpha\left(\frac{k_1}{m},\ldots,\frac{k_p}{m}\right)-\frac{k_1\cdots k_p}{m^p}.$$

Some of the EFGM copulas are particular cases of (9).

Sancetta and Satchell (2004) describe applications of this copula as approximations for multivariate distributions. Some recent applications have included: dependence modeling in non-life insurance (Diers *et al.*, 2012); modeling of nonlinear dependence structures between petrophysical properties (Hernández-Maldonado *et al.*, 2012); joint distribution of wind direction and quantity of rainfall in the North of Spain and the joint distribution of wind directions in two nearby buoys at the Atlantic ocean (Carnicero *et al.*, 2013).

4.6. Fischer and Köck's copulas

Fischer and Köck (2012) have proposed three further extensions of the EFGM copula. The first of them is defined by

$$C(u_1,u_2)=u_1u_2\left[1+\theta\left(1-u_1^{\frac{1}{r}}\right)\left(1-u_2^{\frac{1}{r}}\right)\right]^r$$

for $r \geq 1$ and $-1 \leq \theta \leq 1$. Independence corresponds to $\theta = 0$. The second is defined by

$$\begin{aligned} C(u_1, u_2) &= 2^{-r}\left\{u_1^{\frac{\alpha}{r}}\left(2u_2^{\frac{1}{r}} - u_2^{\frac{\alpha}{r}}\right)\left[1+\theta\left(1-u_1^{\frac{\alpha}{r}}\right)\left(1-2u_2^{\frac{1}{r}}+u_2^{\frac{\alpha}{r}}\right)\right]\right. \\ &\quad \left. +u_1^{\frac{\beta}{r}}\left(2u_2^{\frac{1}{r}} - u_2^{\frac{\beta}{r}}\right)\left[1+\theta\left(1-u_1^{\frac{\beta}{r}}\right)\left(1-2u_2^{\frac{1}{r}}+u_2^{\frac{\beta}{r}}\right)\right]\right\}^{r} \end{aligned}$$

for $1 \leq \alpha, \beta \leq 2$, $-1 \leq \theta \leq 1$ and $r \geq 1$. Independence corresponds to $\alpha = \beta = 1$ and $\theta = 0$. The third and the final one is defined by

$$\begin{aligned} C(u_1, u_2) &= u_1 u_2 \left[1+\theta_1\left(1-u_1^{\frac{1}{r}}\right)\left(1-u_2^{\frac{1}{r}}\right)\right]^{r} \\ &\quad +\left[1+\theta_2\left(1-u_1^{\frac{1}{r}}\right)\left(1-u_2^{\frac{1}{r}}\right)\right]^{r} \end{aligned}$$

for $r \geq 1$ and $-1 \leq \theta_1, \theta_2 \leq 1$. Independence corresponds to $\theta_1 = 0$ and $\theta_2 = 0$. Some of the EFGM copulas are particular cases of these copulas for $r = 1$.

4.7. *Bozkurt's copulas*

Bozkurt (2013) has proposed four extensions of the EFGM copula. The first of these is defined by

$$\begin{aligned} C(u_1, u_2) &= \beta u_1 u_2 \left[1+\alpha(1-u_1)^2(1-u_2)\right] \\ &\quad +(1-\beta)u_1 u_2 [1+\alpha(1-u_1)(1-u_2)] \end{aligned}$$

for $0 \leq \beta \leq 1$ and

$$\max\left(-\frac{3\beta}{\beta^2-\beta+1}, -1\right) \leq \alpha \leq \min\left(\frac{3\beta}{\beta^2-\beta+1}, 1\right).$$

The particular case for $\beta = 0$ is the EFGM copula. The second is defined by

$$\begin{aligned} C(u_1, u_2) &= \beta u_1 u_2 \left[1+\alpha\left(1-u_1^2\right)(1-u_2)\right] \\ &\quad +(1-\beta)u_1 u_2 [1+\alpha(1-u_1)(1-u_2)] \end{aligned}$$

for $0 \leq \beta \leq 1$ and

$$\max\left(-\frac{3\beta}{\beta^2-\beta+1}, -\frac{1}{\beta+1}\right) \leq \alpha \leq \min\left(\frac{3\beta}{\beta^2-\beta+1}, \frac{1}{\beta+1}\right).$$

The third one is defined by

$$\begin{aligned} C(u_1,u_2) &= \beta u_1 u_2 \left[1+\alpha\left(1-u_1^p\right)\left(1-u_2^p\right)\right] \\ &\quad +(1-\beta) u_1 u_2 \left[1+\alpha\left(1-u_1^q\right)\left(1-u_2^q\right)\right] \end{aligned}$$

for $p>0$, $q>0$, $0\leq\beta\leq 1$ and

$$\max\left(-1,-\frac{1}{p^2},-\frac{1}{q^2}\right)\leq\alpha\leq\min\left(\frac{1}{p},\frac{1}{q}\right).$$

The fourth and the final one is defined by

$$\begin{aligned} C(u_1,u_2) &= \beta u_1 u_2 \left[1+\alpha(1-u_1)^p(1-u_2)^p\right] \\ &\quad +(1-\beta) u_1 u_2 \left[1+\alpha(1-u_1)^q(1-u_2)^q\right] \end{aligned}$$

for $p>1$, $q>1$, $0\leq\beta\leq 1$ and

$$-1\leq\alpha\leq\min\left[\left(\frac{p+1}{p-1}\right)^{p-1},\left(\frac{q+1}{q-1}\right)^{q-1}\right].$$

For all four copulas independence corresponds to $\alpha=0$. The Pearson correlation coefficients for the four copulas in the given order are

$$\frac{\alpha(2-\beta)}{6},$$

$$\frac{\alpha(2+\beta)}{6},$$

$$\frac{3\alpha\left[4\beta p^2(q+1)+4(1-\beta)q^2(p+1)+p^2q^2\right]}{(p+2)^2(q+2)^2}$$

and

$$12\alpha\left[\frac{1-\beta}{(q^2+3q+2)^2}+\frac{\beta}{(p^2+3p+2)^2}\right],$$

respectively.

5. Extreme value copulas

Let $A:[0,1]\rightarrow[1/2,1]$ be a convex function satisfying $\max(w,1-w)\leq A(w)\leq 1$ for all $w\in[0,1]$. The extreme value copula due to Pickands (1981) is defined by

$$C(u_1,u_2)=\exp\left[\log(u_1u_2)A\left(\frac{\log u_2}{\log(u_1u_2)}\right)\right]. \tag{10}$$

Independence corresponds to $A(w) = 1$ for all $w \in [0,1]$. Complete dependence corresponds to $A(w) = \max(w, 1-w)$. The Kendall tau rank correlation coefficient, Spearman's rank correlation coefficient and the tail dependence coefficient are

$$\int_0^1 \frac{t(1-t)A'(t)}{A(t)} dt,$$

$$12 \int_0^1 \frac{1}{[1+A(t)]^2} dt$$

and

$$2[1-A(1/2)],$$

respectively.

Some popular models for $A(\cdot)$ are

$$A(w) = \left[w^\theta + (1-w)^\theta\right]^{1/\theta}$$

due to Gumbel (1960), where $\theta \geq 1$ (see Section 5.1);

$$A(w) = 1 - \left[w^{-\theta} + (1-w)^{-\theta}\right]^{-1/\theta}$$

due to Galambos (1975), where $\theta \geq 0$;

$$A(w) = 1 - (\theta + \phi)w + \theta w^2 + \phi w^3$$

due to Tawn (1988), where $\theta \geq 0$, $\theta + 3\phi \geq 0$, $\theta + \phi \leq 1$ and $\theta + 2\phi \leq 1$;

$$A(w) = (1-\phi_1)(1-w) + (1-\phi_2)w + \left[(\phi_1 w)^{1/\theta} + (\phi_2(1-w))^{1/\theta}\right]^\theta$$

due to Tawn (1988), where $0 < \theta \leq 1$ and $0 \leq \phi_1, \phi_2 \leq 1$;

$$A(w) = w\Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\log\frac{w}{1-w}\right) + (1-w)\Phi\left(\frac{1}{\theta} - \frac{\theta}{2}\log\frac{w}{1-w}\right)$$

due to Hüsler and Reiss (1989), where $\theta \geq 0$ and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable;

$$A(w) = 1 - \left[(\phi_1(1-w))^{-1/\theta} + (\phi_2 w)^{-1/\theta}\right]^{-\theta}$$

due to Joe (1990), where $\theta > 0$ and $0 \leq \phi_1, \phi_2 \leq 1$;

$$A(w) = \int_0^1 \max\left[(1-\beta)(1-w)t^{-\beta}, (1-\delta)w(1-t)^{-\delta}\right] dt$$

due to Joe *et al.* (1992) and Coles and Tawn (1994), where $(\beta,\delta) \in (0,1)^2 \cup (-\infty,0)^2$; and

$$
\begin{aligned}
A(w) &= w t_{\xi+1}\left(\sqrt{\frac{1+\xi}{1-\rho^2}}\left[\left(\frac{w}{1-w}\right)^{1/\xi}-\rho\right]\right) \\
&\quad +(1-w)t_{\xi+1}\left(\sqrt{\frac{1+\xi}{1-\rho^2}}\left[\left(\frac{1-w}{w}\right)^{1/\xi}-\rho\right]\right)
\end{aligned}
$$

due to Demarta and McNeil (2005), where $-1<\rho<1$, $\xi>0$ and $t_\nu(\cdot)$ denotes the cumulative distribution function of a Student's t random variable with ν degrees of freedom. Others include Marshall and Olkin's copula in Section 5.2, Gumbel's copula in Section 5.1 and Galambos's copula in Section 5.3.

The p-variate generalization of (10) is

$$
C\left(u_1,\ldots,u_p\right)=\exp\left[\sum_{i=1}^{p}\log u_i A\left(\frac{\log u_1}{\sum_{i=1}^{p}\log u_i},\ldots,\frac{\log u_{p-1}}{\sum_{i=1}^{p}\log u_i}\right)\right],
$$

where $A(\cdot)$ is a convex function on the $(p-1)$-dimensional simplex satisfying the condition $\max\left(w_1,\ldots,w_p\right)\le A\left(w_1,\ldots,w_p\right)\le 1$ for all $\left(w_1,\ldots,w_p\right)$ in the $(p-1)$-dimensional simplex. p-variate versions for most of the given bivariate models for $A(\cdot)$ can be easily deduced. A p-variate version of the model for $A(\cdot)$ due to Demarta and McNeil (2005) is given in Nikoloulopoulos *et al.* (2009).

Other important contributions to extreme value copulas include: Ressel (2013), explaining how the stable tail dependence function of extreme value copulas is characterized in higher dimensions; Mai and Scherer (2011) on bivariate extreme value copulas having a Pickands dependence measure with two atoms; Chapter 1 in Joe (1997) giving an excellent introduction on extreme value copulas.

Extreme value copulas are very popular. Some recent applications have included: analysis of the risk dependence for foreign exchange data (Lu *et al.*, 2008); modeling of maxima sampled via a network of non-independent gauge stations (Durante and Salvadori, 2010); multivariate extreme value models for floods (Salvadori and de Michele, 2010); empirical evidence from Asian emerging markets (Hsu *et al.*, 2012); multivariate assessment of droughts (de Michele *et al.*, 2013); multivariate return period calculation (Salvadori *et al.*, 2013); modeling of oil and gas supply disruption risks (Gülpinar and Katata, 2014); analysis of dependencies between exchange rates and exports of Thailand (Praprom and Sriboonchitta, 2014); multivariate analysis and design in coastal and offshore engineering (Salvadori *et al.*, 2014); multivariate assessment of the structural risk in coastal and offshore engineering (Salvadori *et al.*, 2015); multivariate real-time assessment of droughts (Salvadori and de Michele, 2015).

5.1. Gumbel's copula

The Gumbel-Barnett copula due to Gumbel (1960) and Barnett (1980) is defined by

$$C(u_1,u_2) = u_1 + u_2 - 1 + (1-u_1)(1-u_2)\exp\left[-\phi \log(1-u_1)\log(1-u_2)\right]$$

for $0 \leq \phi \leq 1$. Independence corresponds to $\phi = 0$. Another copula also due to Gumbel (1960) and Hougaard (1984) is

$$C\left(u_1,\ldots,u_p\right) = \exp\left\{-\left[\sum_{i=1}^{p}(-\log u_i)^{\phi}\right]^{\frac{1}{\phi}}\right\} \tag{11}$$

for $\phi \geq 1$. Now independence corresponds to $\phi = 1$. The Pearson correlation coefficient and tail dependence coefficient in the bivariate case are $(\phi-1)/\phi$ and $2-2^{1/\phi}$, respectively. Furthermore, (11) is the only copula that is an Archimedean copula as well as an extreme value copula, see Sections 2 and 5. This is shown in Genest and Rivest (1989).

Some recent applications of these copulas have included: frequency analysis (Salvadori and de Michele, 2004b); checking adequacy of dam spillway (de Michele *et al.*, 2005); trivariate flood frequency analysis (Zhang and Singh, 2007b); cost analysis of complex system under preemptive-repeat repair discipline (Ram and Singh, 2010); estimation of return period and design (Salvadori *et al.*, 2011).

5.2. Marshall and Olkin's copula

Marshall and Olkin (1967) have defined the copula

$$C(u_1,u_2) = \begin{cases} u_1^{1-\alpha} u_2, & \text{if } u_1^{\alpha} \geq u_2^{\beta}, \\ u_1 u_2^{1-\beta}, & \text{if } u_1^{\alpha} < u_2^{\beta} \end{cases} \tag{12}$$

for $0 \leq \alpha, \beta \leq 1$. Independence corresponds to $\alpha = \beta = 0$. Complete dependence corresponds to $\alpha = \beta = 1$. The Kendall tau rank and Spearman's rank correlation coefficients are $\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$ and $\frac{3\alpha\beta}{2\alpha+2\beta-\alpha\beta}$, respectively. The tail dependence coefficient is $\min(\alpha,\beta)$. A multivariate version of (12) is also presented in Marshall and Olkin (1967). An excellent reference on Marshall and Olkin's copula is Chapter 3 of Mai and Scherer (2012).

Some recent applications of Marshall and Olkin's copula have included: pricing of CDO contracts (Bernhart *et al.*, 2013); modeling of cross-border bank contagion (Osmetti and Calabrese, 2013). A wide range of other applications can be found in Cherubini *et al.* (2015).

5.3. *Galambos's copula*

Galambos (1975) has defined the copula

$$C(u_1, u_2) = u_1 u_2 \exp\left\{\left[(1-u_1)^{-\theta} + (1-u_2)^{-\theta}\right]^{-1/\theta}\right\}$$

for $\theta \geq 0$. A p-variate version is

$$C\left(u_1, \ldots, u_p\right) = \exp\left\{\sum_{S \in \mathscr{S}} (-1)^{|S|} \left[\sum_{i \in S} (1-u_i)^{-\theta}\right]^{-1/\theta}\right\}$$

for $\theta \geq 0$, where $\mathscr{S}$ is the set of all nonempty subsets of $\{1, 2, \ldots, p\}$. Independence corresponds to $\theta = 0$. Complete dependence corresponds to $\theta \to \infty$. A recent application of Galambos' copula is the analysis of the meteorological drought characteristics of the Sharafkhaneh gauge station, located in the northwest of Iran (Mirabbasi *et al.*, 2012).

5.4. *Cuadras and Augé's copula*

Cuadras and Augé (1981) have defined the copula

$$C(u_1, u_2) = [\min(u_1, u_2)]^{\theta} (u_1 u_2)^{1-\theta} \tag{13}$$

for $0 \leq \theta \leq 1$. Independence corresponds to $\theta = 0$. Complete dependence corresponds to $\theta = 1$. The Pearson, Kendall tau rank and Spearman's rank correlation coefficients are $\frac{3\alpha}{4-\alpha}$, $\frac{\alpha}{2-\alpha}$ and $\frac{3\alpha}{4-\alpha}$, respectively. This copula is a particular case of Marshall and Olkin's copula in (12) for $\alpha = \beta = \theta$.

A p-variate version of (13) due to Cuadras (2009) is

$$C\left(u_1, \ldots, u_p\right) = \min\left(u_1, \ldots, u_p\right) \prod_{i=2}^{p} u_{(i)}^{\prod_{j=1}^{i-1}(1-\theta_{ij})}$$

for $0 \leq \theta_{ij} \leq 1$, where $u_{(1)} \leq \cdots \leq u_{(p)}$ are the sorted values of $u_1, \ldots, u_p$. Independence corresponds to $\theta_{ij} = 0$ for all i and j. Complete dependence corresponds to $\theta_{ij} = 1$ for all i and j.

The copula due to Cuadras and Augé (1981) has been used for plant-specific dynamic failure assessment (Meel and Seider, 2006).

5.5. *Lévy-frailty copulas*

Let $u_{(1)} \leq u_{(2)} \leq \cdots \leq u_{(p)}$ denote sorted values of $u_1, u_2, \ldots, u_p$ and let a_i, $i = 0, 1, \ldots, p-1$ denote some real numbers. Mai and Scherer (2009) have shown that

$$C\left(u_1, \ldots, u_p\right) = \prod_{i=1}^{p} u_{(i)}^{a_{i-1}} \tag{14}$$

is a valid copula if and only if $a_0 = 1$ and $\{a_i\}$ are p-monotone; that is, $\Delta^{j-1} a_k \geq 0$ for all $k = 0, 1, \ldots, p-1$ and $j = 1, 2, \ldots, p-k$, where

$$\Delta^j a_k = \sum_{i=0}^{j} (-1)^i \binom{j}{i} a_{k+i}$$

for $j \geq 0$ and $k \geq 0$. Copulas defined by (14) are referred to as Lévy-frailty copulas. Independence corresponds to $a_i = 1$ for all i. Complete dependence corresponds to $a_0 = 1$ and $a_i = 0$ for all $i > 0$. The copulas in Sections 5.4 and 5.2 are particular cases of (14). An interesting analytical and probabilistic extension of (14) is provided in Mai *et al.* (2016).

5.6. *Durante and Salvadori's copulas*

Let $0 \leq \lambda_{i,j} \leq 1$, $\lambda_{i,j} = \lambda_{j,i}$ and $\sum_{j=1, j \neq i}^{p} \lambda_{i,j} \leq 1$. Durante and Salvadori (2010) have shown that

$$C\left(u_1, \ldots, u_p\right) = \left(\prod_{i=1}^{p} u_i\right)^{1-\sum_{j=1,j\neq i}^{p} \lambda_{i,j}} \prod_{i<j} \left[\min\left(u_i, u_j\right)\right]^{\lambda_{i,j}}$$

are valid copulas. Independence corresponds to $\lambda_{i,j} = 0$ for all i and j. These copulas contain Cuadras and Augé's copula in Section 5.4 and Marshall and Olkin's copula in Section 5.2 as particular cases.

6. Other copulas

6.1. *Fréchet's copula*

Fréchet copula due to Fréchet (1958) is defined by

$$C(u_1, u_2) = a \min(u_1, u_2) + (1-a-b) u_1 u_2 + b \max(u_1 + u_2 - 1, 0)$$

for $0 \leq a, b \leq 1$ and $a + b \leq 1$. Independence corresponds to $a = b = 0$. Complete dependence corresponds to $a = 1$. The Kendall tau rank and Spearman's rank correlation coefficients are $(a-b)(2+a+b)/3$ and $a-b$, respectively.

An equivalent copula due to Mardia (1970) is defined by

$$\begin{aligned} C(u_1, u_2) &= \frac{\theta^2(1+\theta)}{2} \min(u_1, u_2) + \left(1-\theta^2\right) u_1 u_2 \\ &\quad + \frac{\theta^2(1-\theta)}{2} \max(u_1 + u_2 - 1, 0) \end{aligned}$$

for $-1 \leq \theta \leq 1$. Another equivalent copula considered by Gijbels *et al.* (2010) is

$$\begin{aligned} C(u_1,u_2) &= \frac{\gamma\theta^2(1+\theta)}{2}\min(u_1,u_2)+\left(1-\gamma\theta^2\right)u_1u_2 \\ &\quad +\frac{\gamma\theta^2(1-\theta)}{2}\max(u_1+u_2-1,0) \end{aligned}$$

for $-1 \leq \theta \leq 1$ and $\gamma \leq 1/\theta^2$. For the former, independence corresponds to $\theta = 0$ and complete dependence corresponds to $\theta = 1$. For the latter, independence corresponds to $\theta = 0$ and complete dependence corresponds to $\theta = 1$ and $\gamma = 1$.

Recent applications of the Fréchet copula have included: modeling of floods (Durante and Salvadori, 2010); computation of the rainbow option prices and stop-loss premiums (Zheng *et al.*, 2011). The copulas in Section 6.16 are multivariate generalizations of the Fréchet's copula.

6.2. *Raftery copula*

Raftery (1984) introduced the copula defined by

$$C(u_1,u_2)=\begin{cases} u_1-\dfrac{1-\theta}{1+\theta}u_1^{\frac{1}{1-\theta}}\left(u_2^{-\frac{\theta}{1-\theta}}-u_2^{\frac{1}{1-\theta}}\right), & \text{if } u_1 \leq u_2, \\ u_2-\dfrac{1-\theta}{1+\theta}u_2^{\frac{1}{1-\theta}}\left(u_1^{-\frac{\theta}{1-\theta}}-u_1^{\frac{1}{1-\theta}}\right), & \text{if } u_1 > u_2 \end{cases}$$

for $0 \leq \theta < 1$. See also Nelsen (1991). Complete dependence corresponds to $\theta = 0$. The Kendall tau rank and Spearman's rank correlation coefficients are $\frac{2\theta}{3-\theta}$ and $\frac{\theta(4-3\theta)}{(2-\theta)^2}$, respectively. Applications of this copula have included semiparametric density estimation (Liebscher, 2005).

6.3. *Brownian motion copula*

A Brownian motion copula due to Darsow *et al.* (1992) is defined by

$$C(u_1,u_2)=\int_0^{u_1}\Phi\left(\frac{\sqrt{t}\Phi^{-1}(u_2)-\sqrt{s}\Phi^{-1}(x)}{\sqrt{t-s}}\right)dx$$

for $t > s$, where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable. Independence corresponds to $t-s \rightarrow \infty$. Complete dependence corresponds to $t-s \rightarrow 0$. Recent applications of this copula have included in option pricing (Cherubini and Romagnoli, 2009). A survey of copulas and processes related to the Brownian motion can be found in Sempi (2016) and references therein.

6.4. *Koehler and Symanowski's copula*

Let $V = \{1,2,\ldots,p\}$ denote an index set, let $\mathcal{V}$ denote the power set of V and let $\mathcal{I}$ denote the set of all $I \in \mathcal{V}$ with $|I| \geq 2$. For all subsets $I \in \mathcal{I}$, let $\alpha_I \geq 0$ and $\alpha_i \geq 0$ for all $i \in V$ be such that $\alpha_{i+} = \alpha_i + \sum_{I \in \mathcal{I}} \alpha_I > 0$ for all $i \in I$. Koehler and Symanowski (1995) have shown that

$$C\left(u_1,\ldots,u_p\right) = \frac{\prod_{i \in V} u_i}{\prod_{I \in \mathcal{I}} \left[\sum_{i \in I} \prod_{j \in I, j \neq i} u_j^{\alpha_{j+}} - (|I|-1) \prod_{i \in I} u_i^{\alpha_{i+}} \right]^{\alpha_I}}$$

is a valid copula. Independence corresponds to $\alpha_I = 0$ for all I. An application is described to the joint distribution of the times for the occurrence of first eye opening, eruption of incisor teeth and testes decent for male pubs (Koehler and Symanowski, 1995).

6.5. *Shih and Louis's copula*

Shih and Louis (1995) have defined the copula

$$C(u_1,u_2) = \begin{cases} (1-\rho)u_1u_2 + \rho \min(u_1,u_2), & \text{if } \rho > 0, \\ (1+\rho)u_1u_2 + \rho(u_1 - 1 + u_2)\Theta(u_1 - 1 + u_2), & \text{if } \rho \leq 0, \end{cases}$$

where $\Theta(a) = 1$ if $a \geq 0$ and $\Theta(a) = 0$ if $a < 0$. Independence corresponds to $\rho = 0$. Complete dependence corresponds to $\rho = 1$. Shih and Louis (1995) describe an application to modeling of AIDS data.

6.6. *Joe's copulas*

Joe and Hu (1996) have proposed several copulas. Here, we discuss some of them. The first of these is defined by

$$C(u_1,u_2) = \left\{ 1 + \left[\left(u_1^{-a} - 1\right)^b + \left(u_2^{-a} - 1\right)^b \right]^{\frac{1}{b}} \right\}^{-\frac{1}{a}}$$

for $a > 0$ and $b \geq 1$. The second is defined by

$$C(u_1,u_2) = \left\{ u_1^{-a} + u_2^{-a} - 1 - \left[\left(u_1^{-a} - 1\right)^{-b} + \left(u_2^{-a} - 1\right)^{-b} \right]^{-\frac{1}{b}} \right\}^{-\frac{1}{a}}$$

for $a \geq 0$ and $b > 0$. The third (Joe, 1997, see also page 153) is defined by

$$C(u_1, u_2) = 1 - \left\{1 - \left[\left(1 - u_1^{-a}\right)^{-b} + \left(1 - u_2^{-a}\right)^{-b} - 1\right]^{-\frac{1}{b}}\right\}^{\frac{1}{a}}$$

for $a \geq 1$ and $b > 0$. Another copula presented in Joe (1997) is

$$C(u_1, u_2) = \exp\left\{-\left[\theta_2^{-1} \log\left(\exp\left(-\theta_2 (\log u_1)^{\theta_1}\right) + \exp\left(-\theta_2 (\log u_2)^{\theta_1}\right) - 1\right)\right]^{\frac{1}{\theta_1}}\right\}$$

for $\theta_1 \geq 1$ and $\theta_2 \geq 1$. These copulas allow for positive dependence only. The first two have tail dependence coefficients equal to $2 - 2^{1/b}$ and $2^{-1/b}$, respectively.

Further copulas due to Joe (1993) are given in Section 2.6. A recent application of copulas due to Joe is portfolio risk analysis with Asian equity markets (Ozun and Cifter, 2007).

6.7. *Linear Spearman copula*

Joe (1997), page 148 has defined the linear Spearman copula as that given by

$$C(u_1, u_2) = \begin{cases} [u_1 + \theta(1 - u_1)]\, u_2, & \text{if } u_2 \leq u_1,\ 0 \leq \theta \leq 1, \\ [u_2 + \theta(1 - u_2)]\, u_1, & \text{if } u_2 > u_1,\ 0 \leq \theta \leq 1, \\ (1 + \theta) u_1 u_2, & \text{if } u_1 + u_2 < 1,\ -1 \leq \theta \leq 0, \\ u_1 u_2 + \theta(1 - u_1)(1 - u_2), & \text{if } u_1 + u_2 \geq 1,\ -1 \leq \theta \leq 0. \end{cases}$$

Independence corresponds to $\theta = 0$. Complete dependence corresponds to $\theta = 1$. The Kendall tau rank and Spearman's rank correlation coefficients are $\theta[2 + \theta \text{sign}(\theta)]/3$ and θ, respectively. The tail dependence coefficient is θ. This copula has been used for covariance estimation of the six popular stocks: Credit Suisse Group, UBS, Nestle, Novartis, Sulzer, Swisscom (Hürlimann, 2004a).

6.8. *Burr copulas*

Frees and Valdez (1998) have defined what is referred to as a Burr copula (Burr, 1942) as

$$C(u_1, u_2) = u_1 + u_2 - 1 + \left[(1 - u_1)^{-1/\alpha} + (1 - u_2)^{-1/\alpha} - 1\right]^{-\alpha}$$

for $\alpha > 0$, which is the survival copula associated with a Clayton copula. The Kendall tau rank correlation coefficient is $1/(2\alpha + 1)$. An extension of this copula provided by de Waal and van Gelder (2005) is

$$\begin{aligned} C(u_1,u_2) &= u_1+u_2-1+\left[(1-u_1)^{-1/\alpha}+(1-u_2)^{-1/\alpha}-1\right]^{-\alpha} \\ &\quad +\beta\left\{\left[(1-u_1)^{-1/\alpha}+(1-u_2)^{-1/\alpha}-1\right]^{-\alpha}\right. \\ &\quad +\left[2(1-u_1)^{-1/\alpha}+2(1-u_2)^{-1/\alpha}-3\right]^{-\alpha} \\ &\quad +\left[2(1-u_1)^{-1/\alpha}+(1-u_2)^{-1/\alpha}-2\right]^{-\alpha} \\ &\quad \left.-\left[(1-u_1)^{-1/\alpha}+2(1-u_2)^{-1/\alpha}-2\right]^{-\alpha}\right\} \end{aligned}$$

for $\alpha > 0$ and $-1 \leq \beta \leq 1$. Frees and Valdez (1998)'s copula is the particular case for $\beta = 0$. In both copulas, complete dependence corresponds to $\alpha \to \infty$. Frees and Valdez (1998) discuss a range of application areas (including stochastic ordering, fuzzy logic, and insurance pricing) of the Burr copula. de Waal and van Gelder (2005) use the generalization for joint modeling of wave heights and wave periods measured at White Rose, Canada from severe storms.

6.9. *Archimax copulas*

Let $\phi(\cdot)$ be as defined in Section 2 and let $A(\cdot)$ be as defined in Section 5. Capéraà *et al.* (2000) have shown that

$$C(u_1,u_2)=\phi^{-1}\left(\min\left(\phi(0),[\phi(u_1)+\phi(u_2)]A\left(\frac{\phi(u_1)}{\phi(u_1)+\phi(u_2)}\right)\right)\right)$$

is a copula. It is referred to as the Archimax copula. Extreme value copulas in Section 5 and Archimedean copulas in Section 2 are particular cases of Archimax copulas. Multivariate Archimax copulas have developed in Charpentier *et al.* (2014). Bacigal *et al.* (2011) have used this copula to model the joint distribution of the flow rates of two rivers in Hungary as well as the joint distribution of flow rates and the corresponding flow volumes.

6.10. *Knockaert's copula*

Knockaert (2002) has defined a copula by

$$\begin{aligned} C(u_1,u_2) &= u_1u_2+\frac{\epsilon}{4\pi^2mn}\left\{\cos[2\pi(mu_2-\Delta)]+\cos[2\pi(nu_1-\Delta)]\right. \\ &\quad \left.-\cos[2\pi(nu_1+mu_2-\Delta)]-\cos[2\pi\Delta]\right\} \end{aligned}$$

for $\epsilon = -1, 1$, $0 \le \Delta \le 2\pi$ and $m, n = \ldots, -2, -1, 0, 1, 2, \ldots$. Independence corresponds to $m \to \infty$ or $n \to \infty$. Applications of this copula to signal processing are discussed in Knockaert (2002) and Davy and Doucet (2003).

6.11. Hürlimann's copula

Let $0 \le \theta_{ij} \le 1$ for $i = 1, \ldots, p$ and $j = 1, \ldots, p$. Hürlimann (2004b) has proposed the copula

$$\begin{aligned} & C\left(u_1, \ldots, u_p\right) \\ = & \frac{1}{c_p}\left\{p \prod_{i=1}^{p} u_i + \sum_{r=2}^{p} \sum_{i_1 \neq \cdots = i_r} \left[\prod_{j=2}^{r} \frac{\theta_{i_1 i_j}}{1-\theta_{i_1 i_j}}\right] \min_{1 \le j \le r} u_{i_j} \left[\prod_{k \in \{i_1, \ldots, i_r\}} u_k\right]\right\}, \end{aligned}$$

where

$$c_p = \sum_{i=1}^{p} \prod_{j \neq i} \frac{1}{1-\theta_{ij}}.$$

Independence corresponds to $\theta_{ij} = 0$ for all i and j. Complete dependence corresponds to $\theta_{ij} \to 1$ for all i and j. An application is given to the evaluation of the economic risk capital for a portfolio of risks using conditional value-at-risk measures (Hürlimann, 2004b).

6.12. Power variance copula

Andersen (2005) and Massonnet *et al.* (2009) have introduced the power variance copula as that defined by

$$C(u_1, u_2) = \exp\left[\frac{v}{\theta(1-v)}\left[1 - \left\{\sum_{j=1}^{2}\left(1 + \theta\left(1 - \frac{1}{v}\right)\log u_j\right)^{\frac{1}{1-v}} - 3\right\}^{1-v}\right]\right]$$

for $\theta \ge 0$ and $0 \le v \ge 1$. The particular case for $v \to 1$ is the inverse Gaussian copula defined by

$$C(u_1, u_2) = \exp\left[\frac{1}{\theta} - \left[\frac{1}{\theta} + \sum_{j=1}^{2} \log u_j \left\{\log u_j - \frac{2}{\theta}\right\}\right]^{1/2}\right]$$

for $\theta \ge 0$. Independence corresponds to $\theta \to \infty$. An application discussed in Massonnet *et al.* (2009) is the joint distribution of infection times of the four udder quarters of a cow.

6.13. *Fischer and Hinzmann's copulas*

Fischer and Hinzmann (2006) have defined a copula by

$$C(u_1, u_2) = \{\alpha [\min(u_1, u_2)]^m + (1-\alpha)[u_1 u_2]^m\}^{1/m}$$

for $0 \leq \alpha \leq 1$ and $-\infty < m < \infty$. Independence corresponds to $\alpha = 0$ and $m = 1$. Complete dependence corresponds to $\alpha = 1$ and $m = 1$. The particular case for $m = -1$ is known as the harmonic mean copula. The Kendall tau rank correlation coefficient, Spearman's rank correlation coefficient and the tail dependence coefficient for this particular case are

$$\alpha^{-4}\left[18\alpha^2 - 12\alpha - 4\alpha^3 - \alpha^4 + \left(24\alpha - 12\alpha^2 - 12\right)\log(1-\alpha)\right],$$

$$\alpha^{-4}\left[12\alpha - 30\alpha^2 + 22\alpha^3 - 3\alpha^4 + \left(12 - 36\alpha + 36\alpha^2 - 12\alpha^3\right)\log(1-\alpha)\right]$$

and α, respectively.

6.14. *Roch and Alegre's copula*

Roch and Alegre (2006) have proposed the copula

$$\begin{aligned} & C(u_1, u_2) \\ = \quad & \exp\left\{1 - \left[\left(((1-\log u_1)^\alpha - 1)^\delta + ((1-\log u_2)^\alpha - 1)^\delta\right)^{1/\delta} + 1\right]^{1/\alpha}\right\} \end{aligned}$$

for $\alpha > 0$ and $\delta \geq 1$. Independence corresponds to $\alpha = 1$ and $\delta = 1$. An application is given to the pairwise joint distribution of daily equity returns for sixteen companies of the Spanish stock market (Abertis, Acciona, Acerinox, ACS, Aguas de Barcelona, Altadis, Banco Popular, Bankinter, BBVA, Corp Alba, Endesa, FCC, NHHoteles, Repsol, Santander and Telefónica).

The copulas in this section and Section 6.13 are based on power transformations of a copula. This method is discussed in detail in Theorem 3.3.3 in Nelsen (2006).

6.15. *Fourier copulas*

Ibragimov (2009) has defined Fourier copulas as

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} [1 + g(x, y)]\, dx dy,$$

where

$$g(x, y) = \sum_{j=1}^{N} \left[\alpha_j \sin\left(2\pi\left(\beta_1^j x + \beta_2^j y\right)\right) + \gamma_j \cos\left(2\pi\left(\beta_1^j x + \beta_2^j y\right)\right)\right]$$

for $N \geq 1$, $-\infty < \alpha_j, \gamma_j < \infty$ and $\beta_1^j, \beta_2^j \in \{\ldots, -1, 0, 1, \ldots\}$ arbitrary numbers such that $\beta_1^j + \beta_2^j \neq 0$ for all $j_1, j_2 \in \{1, \ldots, N\}$ and

$$1 + \sum_{j=1}^{N} \left(\alpha_j \delta_j + \gamma_j \delta_{j+N} \right) \geq 0$$

for $-1 \leq \delta_1, \ldots, \delta_{2N} \leq 1$. Independence corresponds to $\alpha_j = 0$ and $\gamma_j = 0$ for all j. An alternative form for $g(x, y)$ due to Lowin (2010) is

$$g(x, y) = \sum_{i=-N}^{N} \sum_{j=-N}^{N} \left\{ \alpha_{i,j} \sin[2\pi(ix + jy)] + \gamma_{i,j} \cos[2\pi(ix + jy)] \right\}.$$

In this case, the Kendall tau rank and Spearman's rank correlation coefficients are

$$-\frac{1}{2\pi^2} \sum_{i=-N}^{N} \sum_{j=-N}^{N} \frac{4\gamma_{i,j} + \gamma_{i,j}^2 + 2\alpha_{i,j}^2}{ij}$$

and

$$-\frac{3}{\pi^2} \sum_{i=-N}^{N} \sum_{j=-N}^{N} \frac{\gamma_{i,j}}{ij},$$

respectively. Fourier copulas have been used to provide characterizations for higher order Markov processes (Ibragimov, 2009).

6.16. Yang et al.'s copula

Let U_i, $i = 1, \ldots, n$ be uniform $[0, 1]$ random variables. Suppose there exists a uniform $[0, 1]$ random variable U such that U_i, $i = 1, \ldots, n$ are independent conditionally on U. Suppose also that U_i and U have the joint cumulative distribution function

$$a_{i,1} \min(u_i, u) + a_{i,3} u_i u + a_{i,2} \max(u_i + u - 1, 0)$$

for $0 \leq a_{i,1}, a_{i,2}, a_{i,3} \leq 1$ and $a_{i,1} + a_{i,2} + a_{i,3} = 1$. For $(j_1, \ldots, j_n)$, where $j_i \in \{1, 2, 3\}$, write

$$C^{(j_1, \ldots, j_n)}(u_1, \ldots, u_n) = \max\left[\min_{1 \leq i \leq n, j_i = 1} u_i + \min_{1 \leq i \leq n, j_i = 3} u_i - 1, 0 \right] \prod_{1 \leq i \leq n, j_i = 2} u_i.$$

Yang *et al.* (2009) have shown that the joint distribution of $(U_1, \ldots, U_n)$ can be expressed as

$$C(u_1, \ldots, u_n) = \sum_{j_1=1}^{3} \cdots \sum_{j_n=1}^{3} \left(\prod_{i=1}^{n} a_{i,j_i} \right) C^{(j_1, \ldots, j_n)}(u_1, \ldots, u_n),$$

which is a copula. Two applications to actuarial science are given: one to the joint-life status where the future lifetimes of the individuals in the group are correlated by the copula; the other to the individual risk models with the individual risks' dependency modeled by the copula (Yang *et al.*, 2009).

6.17. *Zhang's copula*

Zhang (2009) has proposed a copula defined by

$$C\left(u_1,\ldots,u_p\right)=\prod_{j=1}^{p}\min_{1\le d\le D}\left(u_d^{a_{j,d}}\right)$$

for $a_{j,d}\geq 0$ and $a_{1,d}+\cdots+a_{p,d}=1$ for all $d=1,\ldots,D$. Independence corresponds to $a_{j,j}=1$ for all j. Complete dependence corresponds to $a_{j,d}=1/p$ for all j and d. Two applications are provided: the joint distribution of indemnity payment and allocated loss adjustment expense for insurance claims; the joint distribution of daily average temperatures at three different locations in the United States (Raleigh/Durham, Saint Louis and Madison).

6.18. *Andronov's copula*

Let $x=q(u)$ denote the root of the equation

$$u=1-\frac{1}{p}\sum_{j=0}^{p-1}\frac{p-j}{j!}x^j\exp(-x).$$

If for example $p=2$ then $q(u)=-W(-2(1-u)\exp(-2))-2$, where $W(\cdot)$ denotes Lambert's W function (Corless *et al.*, 1996). Andronov (2010) has shown that

$$\begin{aligned}C\left(u_1,\ldots,u_p\right) &= 1-\sum_{j=0}^{p-1}\frac{1}{j!}q^j\left(u_1\right)\exp\left[-q\left(u_1\right)\right]+\sum_{i=0}^{p}\frac{q\left(u_1\right)\cdots q\left(u_{i-1}\right)}{p(p-1)\cdots(p-i+2)}\\ &\cdot\sum_{j=0}^{p-i}\frac{1}{j!}\left\{q^j\left(u_{i-1}\right)\exp\left[-q\left(u_{i-1}\right)\right]-q^j\left(u_i\right)\exp\left[-q\left(u_i\right)\right]\right\}\end{aligned}$$

is a valid copula. This copula was motivated by the joint distribution of failure times of the elements of a system (Andronov, 2010).

6.19. *Cube copula*

Holman and Ritter (2010) have defined what is referred to as a Cube copula as

$$C(u_1,u_2) = \begin{cases} q_2 u_1 u_2, & \text{if } u_1 \le a, u_2 \le a, \\ u_1\left[q_2 a + q_1(u_2 - a)\right], & \text{if } u_1 \le a < u_2, \\ u_2\left[q_2 a + q_1(u_1 - a)\right], & \text{if } u_2 \le a < u_1, \\ q_2 a^2 + q_1 a(u_1 + u_2 - 2a) + q_0(u_1 - a)(u_2 - a), & \text{if } u_1 > a, u_2 > a \end{cases}$$

for some suitable constants q_0, q_1, q_2 and a. This copula is related to the linear Spearman copula in Section 6.7. The Spearman's rank correlation coefficient is

$$3(a-1)^4 q_0 + 3(a-2)^2 a^2 q_2 - 12a(a-1)^3(a+1)q_1 - 3.$$

An application to hedge fund returns is provided (Holman and Ritter, 2010).

6.20. *Generalized beta copula*

Let $\gamma(a,x)$ and $I_x(a,b)$ denote the incomplete gamma function ratio and the incomplete beta function ratio defined by

$$\gamma(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} \exp(-t) dt$$

and

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

respectively, where

$$\Gamma(a) = \int_0^\infty t^{a-1} \exp(-t) dt$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

denote the gamma and beta functions, respectively. Let $\gamma^{-1}(a,x)$ and $I_x^{-1}(a,b)$ denote the inverse functions of $\gamma(a,x)$ and $I_x(a,b)$ with respect to x. With these notation, Yang *et al.* (2011) have defined the generalized beta copula as

$$C\left(u_1,\ldots,u_p\right)=\int_0^{\infty}\prod_{i=1}^{p}\gamma^{-1}\left(\frac{I_{u_i}^{-1}(p_i,q)}{\theta-\theta I_{u_i}^{-1}(p_i,q)},p_i\right)\frac{\theta^{-q-1}\exp(-1/\theta)}{\Gamma(q)}d\theta.$$

The particular case for $p=2$ has the tail dependence coefficient

$$\begin{aligned}&I_{B^{1/q}(p_1,q)/\left[B^{1/q}(p_1,q)+B^{1/q}(p_2,q)\right]}(q+p_2,p_1)\\&\quad+I_{B^{1/q}(p_2,q)/\left[B^{1/q}(p_1,q)+B^{1/q}(p_2,q)\right]}(q+p_1,p_2).\end{aligned}$$

The particular case for $p_i=1$ for all i is

$$\begin{aligned}C\left(u_1,\ldots,u_p\right) &= \left[\sum_{i=1}^{p}u_i-p+1\right]\\&\quad+\sum_{i_1<i_2}\left[\left(1-u_{i_1}\right)^{-\frac{1}{q}}+\left(1-u_{i_2}\right)^{-\frac{1}{q}}-1\right]^{-q}\\&\quad+\cdots+(-1)^p\left[\sum_{i=1}^{p}(1-u_i)^{-\frac{1}{q}}-p+1\right]^{-q},\end{aligned}\tag{15}$$

a copula due to Al-Hussaini and Ateya (2006). One of the Burr copulas in Section 6.8 is a particular case of (15) for $p=2$.

An application is given to the joint distribution of bodily injury (BI) liability payments and the time-to-settlement using auto injury data from the Insurance Research Council's (IRC) Closed Claim Survey (Yang *et al.*, 2011).

6.21. Sanfins and Valle' copula

Let $x=\psi_m(u)$ denote the root of the equation

$$u=x\sum_{j=0}^{m-1}(-1)^j\frac{(\log x)^j}{j!}$$

and let

$$r_{m-1}(u_1,\ldots,u_m)=\psi_\ell(u_\ell)\sum_{j=0}^{m-1}(-1)^j\frac{\left[\log\psi_\ell(u_\ell)\right]^j}{j!}$$

if $\psi_\ell(u_\ell) = \min[\psi_1(u_1), \ldots, \psi_m(u_m)]$. For every $\left(u_1, \ldots, u_p\right)$ such that $u = \psi_1(u_1) \geq \cdots \geq \psi_p\left(u_p\right)$, let

$$\mathscr{H}_p\left(u_1, \ldots, u_p\right) = u_p - \psi_p\left(u_p\right) \sum_{j=1}^{p-1} \frac{\left[-\log \psi_j\left(u_j\right)\right]^j}{j!} J_{p-j}\left(-\log \psi_{j+1}\left(u_{j+1}\right), \ldots, -\log \psi_p\left(u_p\right)\right),$$

where J_m is given by the recurrence relation

$$J_m(x_1, \ldots, x_m) = \sum_{j=0}^{m-1} \frac{x_m^j}{j!} - \sum_{j=0}^{m-1} \frac{x_j^j}{j!} J_{m-j}\left(x_{j+1}, \ldots, x_m\right)$$

for $m \geq 1$ with $J_1 \equiv 1$. Under these assumptions, Sanfins and Valle (2012) have shown that

$$C\left(u_1, \ldots, u_p\right) = \mathscr{H}_p\left(u_1, r_1(u_1, u_2), r_2(u_1, u_2, u_3), \ldots, r_{p-1}\left(u_1, \ldots, u_p\right)\right)$$

is a valid copula. The Kendall tau rank and Spearman's rank correlation coefficients of this copula converge to zero as $p \to \infty$.

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SUMMARY

Copulas are used to specify dependence between two or more random variables. The last few years have seen a surge of developments of parametric models for copulas. Here, we provide an up-to-date and a comprehensive review of known parametric copulas as well as applications and open problems. This review is believed to be the first of its kind.

Keywords: Bivariate distributions; Dependence; Independence; Multivariate distributions; Trivariate distributions.