STATISTICA, anno LXXVIII, n. 2, 2018

QUANTILE APPROACH OF DYNAMIC GENERALIZED ENTROPY (DIVERGENCE) MEASURE

Vikas Kumar [1]
Department of Applied Sciences, UIET, M. D. University, Rohtak-124001, India
Rekha Rani
Department of Applied Sciences, UIET, M. D. University, Rohtak-124001, India

1. INTRODUCTION

The current literature on information measures has focused mainly on divergence type and entropy-type measure of information. Among the most popular measures of entropy and divergence are the Shannon (1948) concept of information-theoretic entropy and its generalization known as Kullback and Leibler (1951) measure of divergence. In the literature, the potential applications of entropy and divergence measure can be found in econometric estimation and hypothesis testing, income inequality and welfare economics. Shannon entropy has been used to measure dispersion, risk, and volatility. The average amount of uncertainty associated with the nonnegative continuous random variable X can be measured using the differential entropy function

$$H(X) = -\int_0^\infty f(x) \log f(x) dx \,, \tag{1}$$

a continuous counterpart of the Shannon (1948) entropy in the discrete case, where $f(x)$ denotes the probability density function (pdf) of the random variable X. There have been attempts by several authors for the parametric generalizations of this measure. However, most of these extensions are purely mathematical formulations and not much is known about their inter-relationships, inferential properties and their applications in other areas like statistics, thermo-dynamics, accountancy, image processing, reliability, finance, economics and pattern recognition. One of the main extensions of Shannon entropy was defined by Varma (1966). For a nonnegative random variable X; the generalized entropy (see Varma, 1966) of order (α, β) is given by

$$H_\alpha^\beta(X) = \frac{1}{(\beta-\alpha)} \log\left[\int_0^\infty f^{\alpha+\beta-1}(x) dx\right]; \; \beta \neq \alpha, \; \beta - 1 < \alpha < \beta, \; \beta \geq 1 \,. \tag{2}$$

[1] Corresponding Author. E-mail: vikas_iitr82@yahoo.co.in

When $\beta=1, H_{\alpha}^{\beta}(X)$ reduces to $H_{\alpha}(X)=\frac{1}{(1-\alpha)}\log\left[\int_0^{\infty} f^{\alpha}(x)dx\right]$, the Renyi (1961) entropy, and when $\beta=1$ and $\alpha\longrightarrow 1, H_{\alpha}^{\beta}(X)\longrightarrow H(X)$ given in (1). Varma's entropy measure is much more flexible due to the parameters α and β, enabling several measurements of uncertainty within a given distribution and increase the scope of application. In recent years, Varma's entropy has been used by many researchers in the context of information theory, we refer to Kayal and Vellaisamy (2011), and Kayal (2015a).

In survival analysis and life testing, the current age of the system under consideration is also taken into account. The residual lifetime of the system when it is still operating at time t is $X_t=(X-t|X>t)$; Ebrahimi (1996) proposed the entropy of the residual lifetime X_t as

$$H(X;t)=-\int_t^{\infty}\frac{f(x)}{\bar{F}(t)}\log\frac{f(x)}{\bar{F}(t)}dx\,, t>0. \tag{3}$$

In analogy to Ebrahimi (1996), Baig and Dar (2008) have proposed the generalized residual entropy (GRE) of order (α,β), through the relationship

$$H_{\alpha}^{\beta}(X;t)=\frac{1}{(\beta-\alpha)}\log\left\{\frac{\int_t^{\infty} f^{\alpha+\beta-1}(x)dx}{\bar{F}^{\alpha+\beta-1}(t)}\right\};\ \beta\neq\alpha,\ \beta-1<\alpha<\beta,\ \beta\geq 1, \tag{4}$$

and studied its properties. In actuarial science, generalized entropy given in (4) can be presented as the pre-payment entropy of claims (losses) with a deductible t, for more detail refer to, Kumar and Taneja (2011) and Kayal (2014).

In many realistic situations, the random variable is not necessary related to the future, but they can also refer to the past, known as *inactivity time*. Based on this idea, Di Crescenzo and Longobardi (2002) have studied measure on past entropy over $(0,\ t)$ given as

$$\bar{H}(X;t)=-\int_0^{t}\frac{f(x)}{F(t)}\log\frac{f(x)}{F(t)}dx. \tag{5}$$

Similar to (5), *generalized past entropy* (GPE) of order (α,β) was given by

$$\bar{H}_{\alpha}^{\beta}(X;t)=\frac{1}{(\beta-\alpha)}\log\left\{\frac{\int_0^{t} f^{\alpha+\beta-1}(x)dx}{F^{\alpha+\beta-1}(t)}\right\};\ \beta\neq\alpha,\ \beta-1<\alpha<\beta,\ \beta\geq 1. \tag{6}$$

Measures of uncertainty in context with past lifetime distributions have been studied extensively in the literature, refer to Di Crescenzo and Longobardi (2004) and many others.

All these theoretical results and applications are based on the distribution function. But, there are many situations, where distribution functions are not analytically tractable. Quantile functions (QFs) have several properties that are not shared by distribution functions. For example, the sum of two QFs is again a QF. In many cases, QF is more convenient as it is less influenced by extreme observations, and thus provides a straightforward analysis with a limited amount of information. There are explicit general distribution forms for the QF of order statistics. It is easier to generate random numbers from the QF. In reliability analysis, a single long-term survivor can have a marked effect on mean life, especially in the case of heavy-tailed models which are commonly encountered for lifetime data. In such cases, quantile-based estimates are generally found to be more precise and robust against outliers. However, the use of QFs in the place of F provides new models, alternative methodology, easier algebraic manipulations, and methods of analysis in certain cases and some new results that are difficult to derive by using distribution function (Gilchrist, 2000; Nair *et al.*, 2013; Parzen, 1979). Motivated by these, in the present study we consider the dynamic (residual and past) entropy measures based on Varma's entropy in terms of quantile functions. The present paper introduce the generalized entropy of order (α,β) and divergence measure of order (α,β) for residual and reversed residual (past) lifetime using the QFs and proved some characterization results of these.

The rest of the paper is arranged as follows. In Section 2, we introduce the quantile-based generalized entropy of order (α,β) in residual lifetime and various properties of the measure are discussed. Section 3 proves some characterization results based on the measure considered in Section 2. In Section 4, the quantile-based Varma's entropy function in reversed residual (past) lifetime are discussed. In Section 5, we introduce and study the quantile-based dynamic (residual and past both) divergence measure of order (α,β) and their properties.

2. GENERALIZED QUANTILE ENTROPY FOR RESIDUAL LIFETIMES

A probability distribution can be specified either in terms of the distribution function or by the *quantile functions*, de (QF) defined by

$$Q(u)=F^{-1}(u)=\inf\{x \mid F(x)\geq u\},\ \ 0\leq u\leq 1. \tag{7}$$

When the distribution function F is continuous, we have from (7), $FQ(u)=u$, where $FQ(u)$ represents the composite function $F(Q(u))$. Defining the density quantile function by $fQ(u)=f(Q(u))$ and the quantile density function by $q(u)=Q'(u)$, where the prime denotes the differentiation. We have

$$q(u)fQ(u)=1. \tag{8}$$

Sunoj and Sankaran (2012) have considered the quantile version of Shannon entropy and its residual form, defined as

$$\mathcal{H}=\int_0^1 \log q(p)\,dp, \tag{9}$$

and

$$\mathcal{H}(u)=\mathcal{H}(X;Q(u))=\log(1-u)+(1-u)^{-1}\int_u^1 \log q(p)\,dp, \tag{10}$$

respectively. Then the quantile version of generalized entropy of order (α,β) of the nonnegative X becomes

$$\text{H}_{\alpha}^{\beta}=\frac{1}{(\beta-\alpha)}\log\left[\int_{0}^{1}(q(p))^{2-\alpha-\beta}dp\right];\ \beta\neq\alpha,\ \beta-1<\alpha<\beta,\ \beta\geq 1\,. \tag{11}$$

When $\beta=1$ and $\alpha\longrightarrow 1$, the measure (11) reduces to (9). From (4) and (8), generalized residual entropy (GRE) of order (α,β) is denoted by $\text{H}_{\alpha}^{\beta}(X;Q(u))$ is defined as

$$\begin{aligned}
\text{H}_{\alpha}^{\beta}(u)=\text{H}_{\alpha}^{\beta}(X;Q(u)) &= \frac{1}{(\beta-\alpha)}\log\left(\int_{Q(u)}^{\infty}\frac{f^{\alpha+\beta-1}(x)dx}{(1-u)^{\alpha+\beta-1}}\right),\\
&= \frac{1}{(\beta-\alpha)}\log\left(\int_{u}^{1}\frac{(fQ(p))^{\alpha+\beta-1}q(p)dp}{(1-u)^{\alpha+\beta-1}}\right),\\
&= \frac{1}{(\beta-\alpha)}\log\left(\int_{u}^{1}\frac{(q(p))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right).
\end{aligned} \tag{12}$$

The measure (12) may be considered as the *generalized residual quantile entropy of order* (α,β) (GRQE (α,β)) measure. For different values of α and β, GRQE (α,β) measure provides the spectrum of generalized information contained in the conditional density about the predictability of an outcome of X until $100(1-u)$ precent point of distribution. Rewriting the GRQE (α,β) measure (12) as

$$\int_{u}^{1}(q(p))^{2-\alpha-\beta}dp=(1-u)^{\alpha+\beta-1}e^{(\beta-\alpha)\text{H}_{\alpha}^{\beta}(u)}. \tag{13}$$

Differentiating (13) with respect to u both sides and simplify, we obtain

$$q(u)=\frac{e^{\left(\frac{\beta-\alpha}{2-\alpha-\beta}\right)\text{H}_{\alpha}^{\beta}(u)}}{(1-u)}\left\{(\alpha+\beta-1)-(\beta-\alpha)(1-u)\text{H}'^{\beta}_{\alpha}(u)\right\}^{\left(\frac{1}{2-\alpha-\beta}\right)}. \tag{14}$$

Since the GRE $H_{\alpha}^{\beta}(X;t)$ does not determines the pdf $f(x)$, refer to Baig and Dar (2008). However Equation (14) provides a direct relationship between quantile density function $q(u)$ and $\text{H}_{\alpha}^{\beta}(u)$, therefore $\text{H}_{\alpha}^{\beta}(u)$ uniquely determines the underlying distribution. An important quantile measure useful in reliability analysis is the *hazard quantile function*, which is defined as

$$K(u)=h(Q(u))=\frac{fQ(u)}{(1-u)}=\frac{1}{(1-u)q(u)}\,. \tag{15}$$

Next we obtain the upper (lower) bounds of generalized quantile entropy of order $(\alpha,\ \beta)$ in terms of hazard quantile function.

2.1. *An upper bound to GRQE $\text{Ḩ}_{\alpha}^{\beta}(u)$*

To find an upper bound to $\text{Ḩ}_{\alpha}^{\beta}(u)$, we prove the following result.

THEOREM 1. *If generalized residual quantile entropy $\text{Ḩ}_{\alpha}^{\beta}(u)$ is increasing (decreasing) in u, then*

$$\text{Ḩ}_{\alpha}^{\beta}(u) \geq (\leq) \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log K(u) - \frac{\log(\alpha+\beta-1)}{(\beta-\alpha)}, \tag{16}$$

here $K(u)$ is the hazard quantile function.

PROOF. Equation (12) can be written as

$$(\beta-\alpha)\text{Ḩ}_{\alpha}^{\beta}(u) = \log\int_{u}^{1}(q(p))^{2-\alpha-\beta}\,dp - (\alpha+\beta-1)\log(1-u).$$

Differentiating it both sides with respect to u, which gives

$$(\beta-\alpha)\text{Ḩ}'^{\beta}_{\alpha}(u) = \frac{(\alpha+\beta-1)\int_{u}^{1}(q(p))^{2-\alpha-\beta}\,dp - (1-u)(q(u))^{2-\alpha-\beta}}{(1-u)\int_{u}^{1}(q(p))^{2-\alpha-\beta}\,dp}. \tag{17}$$

Case I: Let $0 < \alpha+\beta < 2$. If $\text{Ḩ}_{\alpha}^{\beta}(u)$ is increasing in u, then Equation (17) gives

$$\int_{u}^{1}(q(p))^{2-\alpha-\beta}\,dp \geq \left(\frac{1-u}{\alpha+\beta-1}\right)(q(u))^{2-\alpha-\beta}.$$

Using Equation (12) and simplifying, this gives

$$e^{(\beta-\alpha)\text{Ḩ}_{\alpha}^{\beta}(u)} \geq \frac{\{(1-u)q(u)\}^{2-\alpha-\beta}}{\alpha+\beta-1}.$$

Using Equation (15) in above expression which leads to

$$\text{Ḩ}_{\alpha}^{\beta}(u) \geq \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log K(u) - \frac{\log(\alpha+\beta-1)}{(\beta-\alpha)}.$$

Case II: Let $\alpha+\beta > 1$. If $\text{Ḩ}_{\alpha}^{\beta}(u)$ is increasing in u, then Equation (17) gives

$$\int_{u}^{1}(q(p))^{2-\alpha-\beta}\,dp \leq \left(\frac{1-u}{\alpha+\beta-1}\right)(q(u))^{2-\alpha-\beta}.$$

Using Equation (12) and simplify, we obtain

$$e^{(\beta-\alpha)\text{Ḩ}_{\alpha}^{\beta}(u)} \leq \frac{\{(1-u)q(u)\}^{2-\alpha-\beta}}{\alpha+\beta-1} = \frac{(K(u))^{\alpha+\beta-2}}{\alpha+\beta-1}.$$

which leads to

$$\underset{\smile}{\mathrm{H}}_{\alpha}^{\beta}(u) \geq \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log K(u) - \frac{\log(\alpha+\beta-1)}{(\beta-\alpha)}.$$

Similarly we can prove when $\underset{\smile}{\mathrm{H}}_{\alpha}^{\beta}(u)$ is decreasing in u. Now combining the above two cases, we get the desired result. □

REMARK 2. *When $\beta = 1$ and $\alpha \longrightarrow 1$ then (16) reduces to*

$$\underset{\smile}{H}(u) \geq (\leq)\, 1 - \log K(u)$$

bounds obtained by Sunoj and Sankaran (2012).

The following lemma show that the effect of monotone transformations on generalized quantile entropy defined in (12), which will be used in proving the upcoming theorems of next section.

LEMMA 3. *Let X be a nonnegative and continuous random variables with quantile function $Q_X(.)$ and quantile density function $q_X(.)$. Let $Y = \phi(X)$, with ϕ be a strictly monotonic increasing, continuous and differentiable function, with derivative ϕ'. Then for all $0 < u < 1$, we have*

$$\underset{\smile}{H}_{\alpha}^{\beta}(Y; Q_Y(u)) = \frac{1}{(\beta-\alpha)}\log\left(\int_u^1 \frac{(q_X(p)\phi'(Q_X(p)))^{2-\alpha-\beta}\,dp}{(1-u)^{\alpha+\beta-1}}\right). \tag{18}$$

PROOF. *The proof is similar to Nanda et al. (2014).* □

REMARK 4. *For any absolutely continuous random variable X, define $Y = aX + b$, where $a,\ b > 0$ are constants. Then*

$$\underset{\smile}{H}_{\alpha}^{\beta}(Y; Q_Y(u)) = \underset{\smile}{H}_{\alpha}^{\beta}(X; Q_X(u)) + \left(\frac{2-\alpha-\beta}{\beta-\alpha}\right)\log a.$$

Thus generalized quantile entropy defined in (12) is invariant under location but not under scale transformation. Next, we see how the monotonicity of $\underset{\smile}{\mathrm{H}}_{\alpha}^{\beta}(X, Q(u))$ is affected by increasing transformation. The following lemma help us to prove the results on monotonicity of $\underset{\smile}{\mathrm{H}}_{\alpha}^{\beta}(X, Q(u))$.

LEMMA 5. *Let $f(u,x) : \Re_+^2 \longrightarrow \Re_+$ and $g : \Re_+ \longrightarrow \Re_+$ be any two functions. If $\int_u^\infty f(u,x)dx$ is increasing and $g(u)$ is increasing (decreasing) in u, then $\int_u^\infty f(u,x)g(x)dx$ is increasing (decreasing) in u, provided the integrals exist.*

THEOREM 6. *Let X be a nonnegative and continuous random variable with quantile function $Q_X(.)$ and quantile density function $q_X(.)$. Define $Y = \phi(X)$, where $\phi(.)$ is a nonnegative, increasing and convex (concave) function.*

(i) For $0<\alpha+\beta<2$, $\underset{\cdot}{H}_{\alpha}^{\beta}(Y;Q_Y(u))$ *is increasing (decreasing) in* u *whenever* $\underset{\cdot}{H}_{\alpha}^{\beta}(X;Q_X(u))$ *is increasing (decreasing) in* u.

(ii) For $\alpha+\beta>2$, $\underset{\cdot}{H}_{\alpha}^{\beta}(Y;Q_Y(u))$ *is decreasing (increasing) in* u *whenever* $\underset{\cdot}{H}_{\alpha}^{\beta}(X;Q_X(u))$ *is increasing (decreasing) in* u.

PROOF. (i) The probability density function of $Y=\phi(X)$ is $g(y)=\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$; hence density quantile function is $g(Q_Y(u))=\frac{1}{q_Y(u)}=\frac{f(Q(u))}{\phi' Q(u)}=\frac{1}{q_X(u)\phi'(Q_X(u))}$. Thus we have

$$\begin{aligned}\underset{\cdot}{H}_{\alpha}^{\beta}(Y;Q_Y(u)) &= \frac{1}{\beta-\alpha}\log\left(\int_u^1 \frac{(q_Y(p))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right)\\ &= \frac{1}{\beta-\alpha}\log\left(\int_u^1 \frac{(q_X(p)\phi'(Q_X(p)))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right).\end{aligned}$$

From the given condition we have $\frac{1}{\beta-\alpha}\log\left(\int_u^1 \frac{(q(p))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right)$ is increasing in u, which gives that $\log\left(\int_u^1 \frac{(q(p))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right)$ is increasing in u.

We can rewrite as

$$(\beta-\alpha)\underset{\cdot}{H}_{\alpha}^{\beta}(Y;Q_Y(u))=\left[\log\left\{\int_u^1\left(\frac{(q_X(p))^{2-\alpha-\beta}}{(1-u)^{\alpha+\beta-1}}\right)\left(\phi'(Q_X(p))\right)^{2-\alpha-\beta}dp\right\}\right]. \quad (19)$$

Since $0<\alpha+\beta<2$ and ϕ is non negative, increasing convex (concave), we have $[\phi'(Q(p))]^{2-\alpha-\beta}$ is increasing (decreasing) and is non negative. Hence by Lemma 5, (19) is increasing (decreasing). This prove (i). When $\alpha+\beta>2$, $[\phi'(Q(p))]^{2-\alpha-\beta}=\frac{1}{[\phi'(Q(p))]^{\alpha+\beta-2}}$ is decreasing in p, since ϕ is increasing and convex. Hence we have

$$\underset{\cdot}{H}_{\alpha}^{\beta}(Y;Q_Y(u))=\frac{1}{(\beta-\alpha)}\left[\log\left\{\int_u^1\left(\frac{(q_X(p))^{2-\alpha-\beta}}{(1-u)^{\alpha+\beta-1}}\right)\left(\phi'(Q_X(p))\right)^{2-\alpha-\beta}dp\right\}\right]$$

is decreasing (increasing) in u. Hence prove. □

Let $X_1,X_2,\ldots X_n$ denote the life times of the components in a series system. Assume that the probability density function of life times is $f(x)$ and the survival function is $\bar{F}(x)$. Then $Y=\min(X_1,X_2,\ldots,X_n)$ represents the lifetime of the system, whose components are connected in series with probability density function $g(x)=n[\bar{F}(x)]^{n-1}f(x)$ and with survival function $\bar{G}(x)=[\bar{F}(x)]^n$. Here GRQE measure (12) of the *series system* is independent of u, when X_i's are exponentially distributed.

THEOREM 7. *If* $X_1,X_2,\cdots,X_n$ *be independent random variables having exponential distribution with parameters* θ_i, $i=1,2,\cdots,n$, *then the* GRQE *measure (12), of the random variable* $Y=min(X_1,X_2,\cdots,X_n)$ *is independent of* u.

TABLE 1

Quantile function and generalized residual quantile entropy $H^{\beta}_{\alpha}(u)$ for some specific lifetime distributions.

Distribution	Quantile function $Q(u)$	GRQE $H^{\beta}_{\alpha}(X;Q(u))$
Uniform	$a+(b-a)u$	$=\left(\frac{2-\alpha-\beta}{\beta-\alpha}\right)\log(b-a)(1-u)$
Exponential	$-\lambda^{-1}\log(1-u)$	$=\left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log\lambda-\frac{\log(\alpha+\beta-1)}{(\beta-\alpha)}$
Pareto-I	$b(1-u)^{-\frac{1}{a}}$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{a^{\alpha+\beta-1}(1-u)^{\frac{\alpha+\beta-2}{a}}}{((\alpha+\beta-1)(a+1)-1))b^{\alpha+\beta-2}}\right)$
Lomax	$a[(1-u)^{-\frac{1}{b}}-1]$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{b^{\alpha+\beta-1}(1-u)^{\frac{\alpha+\beta-2}{b}}}{((\alpha+\beta-1)(b+1)-1))a^{\alpha+\beta-2}}\right)$
Generalized Pareto	$\frac{b}{a}\left[(1-u)^{-\frac{a}{a+1}}-1\right]$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{(a+1)^{\alpha+\beta-1}(1-u)^{\frac{(\alpha+\beta-2)a}{a+1}}}{((\alpha+\beta-1)(2a+1)-a)b^{\alpha+\beta-2}}\right)$
Finite range	$b(1-(1-u)^{\frac{1}{a}})$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{a^{\alpha+\beta-1}(1-u)^{\frac{2-\alpha-\beta}{a}}}{((\alpha+\beta-1)(a-1)+1))b^{\alpha+\beta-2}}\right)$
Folded Cramer	$\frac{u}{\theta(1-u)}$	$=\left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log\left(\frac{\theta(1-u)}{2\alpha+2\beta-3}\right)$
Gompertz	$\frac{1}{\log c}\{1-\frac{\log c\log(1-u)}{B}\}$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{B^{\alpha+\beta-2}}{\alpha+\beta-1}\right)$
Inverse Weibull	$\sigma(-\log u)^{-\frac{1}{\lambda}}$	$=\frac{1}{(\beta-\alpha)}\log\left(\left(\frac{\sigma}{\lambda}\right)^{2-\alpha-\beta}\frac{\gamma[(\alpha+\beta-2)(1+\frac{1}{\lambda})+1,(1-\alpha-\beta)\log u]}{(\alpha+\beta-1)^{(\alpha+\beta-2)(1+\frac{1}{\lambda})+1}(1-u)^{\alpha+\beta-2}}\right)$
Half-logistic	$\sigma\log\left(\frac{1+u}{1-u}\right)$	$=\frac{1}{(\beta-\alpha)}\log\left(\sigma\frac{\bar{\beta}_{u^2}[\frac{1}{2},\alpha+\beta-1]}{\{2\sigma(1-u)\}^{\alpha+\beta-1}}\right)$
Inverted exponential	$\frac{-\lambda}{\log u}$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{-\lambda^{2-\alpha-\beta}}{(1-\alpha-\beta)^{2(\alpha+\beta-2)+1}}\frac{\gamma[2(\alpha+\beta-2)+1,(1-\alpha-\beta)\log u]}{(1-u)^{\alpha+\beta-1}}\right)$
Log-logistic	$\frac{1}{a}\left(\frac{u}{(1-u)}\right)^{\frac{1}{b}}$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{\bar{\beta}_u[(\frac{1}{b}-1)(2-\alpha-\beta)+1,(1+\frac{1}{b})(\alpha+\beta-2)]}{(ab)^{2-\alpha-\beta}(1-u)^{\alpha+\beta-1}}\right)$
Govindarajula's	$a\{(b+1)u^b-bu^{b+1}\}$	$=\frac{1}{(\beta-\alpha)}\log\left(\frac{\bar{\beta}_u[(b-1)(2-\alpha-\beta)+1,3-\alpha-\beta]}{\{ab(b+1)\}^{\alpha+\beta-2}(1-u)^{\alpha+\beta-1}}\right)$

REMARK 8. *If $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.) exponential random variables with parameter θ, then*

$$\underset{\sim}{H}_{\alpha}^{\beta}(Y;Q_Y(u)) = \left(\frac{\alpha+\beta-2}{\beta-\alpha}\right)\log n\theta - \left(\frac{1}{\beta-\alpha}\right)\log(\alpha+\beta-1).$$

That is, GRQE (α,β) *of the lifetime of a series system is independent of u and depends only on the parameters α, β and the number of components of the system.*

For some univariate continuous distributions, the expression (12) is evaluated as given in Table 1, where $\beta(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}du, a > 0, b > 0$ and $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ respectively denotes the beta function and incomplete gamma function.

3. CHARACTERIZATION RESULTS

By considering a relationship between the *generalized residual quantile entropy measure* $\underset{\sim}{H}_{\alpha}^{\beta}(u)$ and the *hazard quantile function* $K(u)$, we characterize some lifetime distributions. We give the following theorem.

THEOREM 9. *Let X be a nonnegative continuous random variable with hazard rate quantile function $K(u)$ and generalized residual quantile entropy (GRQE) $\underset{\sim}{H}_{\alpha}^{\beta}(u)$ given by*

$$\underset{\sim}{H}_{\alpha}^{\beta}(u) = \frac{1}{(\beta-\alpha)}\left\{\log c + (\alpha+\beta-2)\log K(u)\right\}, \textit{ for } \alpha+\beta > 2 \tag{20}$$

if, and only if for

(i) $c = \frac{1}{\alpha+\beta-1}$, *$X$ has exponential distribution,*

(ii) $c < \frac{1}{\alpha+\beta-1}$, *$X$ has Pareto distribution with quantile function*

$$q(u) = \frac{b}{a}(1-u)^{-(1+\frac{1}{a})}; a, b > 0, \textit{ and}$$

(iii) $c > \frac{1}{\alpha+\beta-1}$, *$X$ has finite range distribution with quantile function*

$$q(u) = \frac{b}{a}(1-u)^{\frac{1}{a}-1)}; b > 0, a > 1.$$

PROOF. Let (20) be valid, then

$$\frac{1}{(\beta-\alpha)}\log\left(\int_u^1 \frac{(q(p))^{2-\alpha-\beta}dp}{(1-u)^{\alpha+\beta-1}}\right) = \frac{1}{(\beta-\alpha)}\log\left\{c(K(u))^{(\alpha+\beta-2)}\right\}.$$

This gives

$$\int_u^1 (q(p))^{2-\alpha-\beta} dp = c(1-u)^{\alpha+\beta-1}(K(u))^{\alpha-1}.$$

Substituting from (15) and simplifying, it gives

$$\int_u^1 (q(p))^{2-\alpha-\beta} dp = c(1-u)(q(u))^{2-\alpha-\beta}.$$

Differentiating it with respect to u both sides and simplifying, we get

$$\frac{q'(u)}{q(u)} = \left(\frac{1-c}{c(\alpha+\beta-1)-c}\right)\left(\frac{1}{1-u}\right).$$

This gives

$$q(u) = A(1-u)^{\frac{c-1}{c(\alpha+\beta-1)-c}},$$

where A is a constant. Thus the underlying distribution is exponential if $c = \frac{1}{\alpha+\beta-1}$, Pareto distribution if $c < \frac{1}{\alpha+\beta-1}$, and finite range distribution if $c > \frac{1}{\alpha+\beta-1}$. The only if part of the theorem is easy to prove. Hence proved. □

REMARK 10. *If $c = 1$, then Equation (20) is a characterization of uniform distribution.*

4. GENERALIZED QUANTILE ENTROPY FOR PAST LIFETIME

Ruiz and Navarro (1996) defined a new term in reliability analysis known as inactivity time by the conditional random variable ${}_tX = [t - X|X \le t]$ which gives the time elapsed from the failure of a component given that its lifetime is less than or equal to t. The random variable ${}_tX$ is also known as reversed residual (past) life. The past lifetime random variable ${}_tX$ is related with relevant ageing function, *the reversed hazard rate* defined by $\mu_F(x) = \frac{f(x)}{F(x)}$. The quantile version of reversed hazard rate function (see Nair and Sankaran, 2009) is given as

$$\mu(u) = \mu(Q(u)) = u^{-1} f(Q(u)) = [uq(u)]^{-1}. \tag{21}$$

The reversed hazard rate function is quite useful in the forensic science, where exact time of failure (e.g. death in case of human beings) of a unit is of importance. Also the measures of uncertainty in past lifetime distribution plays an important role in the context of information theory, forensic sciences, and other related fields. Sunoj *et al.*

(2013) have considered the quantile version of Shannon past entropy, which is defined as

$$\bar{\text{H}}(u) = \text{H}(X; Q(u) = \log u + u^{-1} \int_0^u \log q(p) dp, \tag{22}$$

From (6) and (8), *generalized past entropy* (GPE) of order (α, β) denoted by $\bar{\text{H}}_\alpha^\beta(X; Q(u))$ is defined as

$$\begin{aligned} \bar{\text{H}}_\alpha^\beta(u) = \bar{\text{H}}_\alpha^\beta(X; Q(u)) &= \frac{1}{(\beta - \alpha)} \log\left(\int_0^u \frac{(f\,Q(p))^{\alpha+\beta-1} q(p) dp}{u^{\alpha+\beta-1}}\right), \\ &= \frac{1}{(\beta - \alpha)} \log\left(\int_0^u \frac{(q(p))^{2-\alpha-\beta} dp}{u^{\alpha+\beta-1}}\right). \end{aligned} \tag{23}$$

The measure (23) may be considered as the *generalized past quantile entropy* (GPQE) of order (α, β). Rewriting the GPQE of order (α, β) (23) as

$$\int_0^u (q(p))^{2-\alpha-\beta} dp = u^{\alpha+\beta-1} e^{(\beta-\alpha)\bar{\text{H}}_\alpha^\beta(u)}. \tag{24}$$

Differentiating (24) with respect to u both sides and simplify, we obtain

$$q(u) = \frac{e^{\left(\frac{\beta-\alpha}{2-\alpha-\beta}\right)\bar{\text{H}}_\alpha^\beta(u)}}{u} \left\{(\alpha + \beta - 1) + (\beta - \alpha) u \bar{\text{H}}'^\beta_\alpha(u)\right\}^{\left(\frac{1}{2-\alpha-\beta}\right)}. \tag{25}$$

Equation (25) provides a direct relationship between quantile density function $q(u)$ and $\bar{\text{H}}_\alpha^\beta(u)$, therefore $\bar{\text{H}}_\alpha^\beta(u)$ uniquely determines the underlying distribution.

To find an upper bound to $\bar{\text{H}}_\alpha^\beta(u)$, we state the following result.

THEOREM 11. *If generalized past quantile entropy of order (α, β) ($\bar{H}_\alpha^\beta(u)$ is increasing (decreasing) in u, then*

$$\bar{H}_\alpha^\beta(u) \leq (\geq) \left(\frac{\alpha + \beta - 2}{\beta - \alpha}\right) \log \mu(u) - \frac{\log(\alpha + \beta - 1)}{(\beta - \alpha)}, \tag{26}$$

here $\mu(u)$ is the reversed hazard quantile function.

The proof is similar to that of Theorem 1 and hence is omitted.

In the following theorem we characterize the power distribution, when GPQE (α, β) is expressed in terms of quantile reversed hazard rate function. We give the following result.

THEOREM 12. *Let X be a non-negative continuous random variable with distribution function $F(x)$ and the quantile reversed hazard rate $\mu(u)$, then the* generalized past quantile entropy (GPQE (α,β)) *is expressed as*

$$\bar{H}_{\alpha}^{\beta}(u)=\frac{1}{(\beta-\alpha)}\left\{\log c+(\alpha+\beta-2)\log\mu(u)\right\}, \tag{27}$$

if and only if X has power distribution function.

PROOF. The quantile reversed hazard rate of power distribution is $\mu(u)=\frac{bu^{-\frac{1}{b}}}{a}$. Taking $c=\left(\frac{b}{(\alpha+\beta-1)(b-1)+1}\right)$ gives the if part of the theorem. To prove the only if part, consider (27) to be valid. Using (21) and (23) , it gives

$$\int_0^u (q(p))^{2-\alpha-\beta}\,dp=cu(q(u))^{2-\alpha-\beta}. \tag{28}$$

Differentiating both sides with respect to u, we obtain

$$\frac{q'(u)}{q(u)}=\left(\frac{c-1}{c(\alpha+\beta-1)-c}\right)\frac{1}{u}.$$

This gives

$$q(u)=Au^{\frac{c-1}{c(\alpha+\beta-1)-c}}=Au^{\frac{1}{b}-1},$$

which characterizes the power distribution function. □

REMARK 13. *If $c=1$, then Equation (27) is a characterization of the uniform distribution.*

5. QUANTILE-BASED GENERALIZED DIVERGENCE MEASURE OF ORDER (α,β)

Discrimination or divergence measures play an important role in measuring the distance between two probability distribution functions. They have great importance in information theory, reliability theory, genetics, economics, approximations of probability distributions, signal processing and pattern recognition. Let X and Y be two non-negative random variables with density functions f and g, and survival functions $\bar{F}$ and $\bar{G}$ respectively. Several divergence measures have been proposed for this purpose which the most fundamental one is Kullback-Leibler (1951). The information divergence of order (α,β) (Varma, 1966) between two distributions is defined by

$$D_{\alpha}^{\beta}(X,Y)=\frac{1}{\alpha-\beta}\log\int_0^{\infty} f(x)\left[\frac{f(x)}{g(x)}\right]^{\alpha+\beta-2}dx\,;\ \alpha\neq\beta,\beta\geq 1,\beta-1<\alpha<\beta. \tag{29}$$

When $\beta = 1$, $D_\alpha^\beta(X,Y)$ reduces to

$$D_\alpha(X,Y) = \frac{1}{\alpha-1}\log\int_0^\infty f(x)\left(\frac{f(x)}{g(x)}\right)^{\alpha-1} dx,$$

the Renyi divergence measure, and when $\beta = 1$ and $\alpha \longrightarrow 1$,

$$D_\alpha^\beta(X,Y) \longrightarrow D(f,g) = \int_0^\infty f(x)\log\frac{f(x)}{g(x)}dx$$

is the Kullback-Leibler information between f and g.

Recently, Sankaran *et al.* (2016) and Sunoj *et al.* (2017) respectively introduced quantile versions of the Kullback-Leibler and Renyi divergence measures and studied its properties. Following with Sankaran *et al.* (2016), the quantile-based divergence measure of order (α,β) is defined as

$$\begin{aligned}
\c{D}_\alpha^\beta(X,Y) &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\int_0^1\left(\frac{f(Q_1(p))}{g(Q_1(p))}\right)^{\alpha+\beta-2} f(Q_1(p))d(Q_1(p))\right\}\right], \\
&= \frac{1}{(\alpha-\beta)}\left[\log\left\{\int_0^1 \{q_1(p)g(Q_1(p))\}^{2-\alpha-\beta}\,dp\right\}\right]. \qquad (30)
\end{aligned}$$

Now using the relationship $F((Q_1(u)) = u$, we have $F^{-1}(u) = Q_1(u)$, which gives

$$G(F^{-1}(u)) = G(Q_1(u)). \qquad (31)$$

Differentiating (31) both sides with respect to u, we obtain

$$\frac{d}{du}(G(F^{-1}(u))) = \frac{d}{du}(G(Q_1(u))) = g(Q_1(u))q_1(u),$$

which is equivalent to

$$\frac{d}{du}(Q_2^{-1}(Q_1(u)) = g(Q_1(u))q_1(u). \qquad (32)$$

Using (32) in (30), we get

$$\c{D}_\alpha^\beta(X,Y) = \frac{1}{(\alpha-\beta)}\left[\log\left\{\int_0^1\left(\frac{d}{dp}(Q_2^{-1}(Q_1(p)))\right)^{2-\alpha-\beta} dp\right\}\right]. \qquad (33)$$

The measure (33) may be considered as the *quantile divergence measure of order* (α,β) between the two random variable X and Y in terms of their quantile functions. Denote $Q_3(u) = Q_2^{-1}(Q_1(u))$, which is the quantile function of $F(G^{-1})$. Then (33) simplifies to

$$\c{D}_\alpha^\beta(X,Y) = \frac{1}{(\alpha-\beta)}\left[\log\left\{\int_0^1 (q_3(p))^{2-\alpha-\beta}\,dp\right\}\right], \qquad (34)$$

where $q_3(u) = Q_3'(u)$ is the quantile density function of $Q_3(u)$. Recently, Sunoj *et al*. (2017) have introduced the quantile versions of the cumulative Kullback-Leibler divergence measures.

EXAMPLE 14. *Let X and Y be two nonnegative exponential random variables with quantile functions respectively by* $Q_1(u) = -\frac{1}{\lambda_1}\log(1-u)$, $\lambda_1 > 0$ *and* $Q_2(u) = -\frac{1}{\lambda_2}\log(1-u)$, $\lambda_2 > 0$. *Then* $Q_3(u) = Q_2^{-1}(Q_1(u)) = 1-(1-u)^{\frac{\lambda_2}{\lambda_1}}$ *and* $q_3(u) = \frac{\lambda_2}{\lambda_1}(1-u)^{\frac{\lambda_2}{\lambda_1}-1}$. *The quantile divergence measure of order* (α,β) *(34) is given by*

$$\mathcal{D}_\alpha^\beta(X,Y) = \frac{1}{(\alpha-\beta)}\left[\log\left(\frac{\left(\frac{\lambda_2}{\lambda_1}\right)^{2-\alpha-\beta}}{(\frac{\lambda_2}{\lambda_1}-1)(2-\alpha-\beta)+1}\right)\right].$$

EXAMPLE 15. *Suppose X and Y be follow Pareto II random variables with QFs respectively by* $Q_1(u) = (1-u)^{-\frac{1}{p_1}} - 1$, $p_1 > 0$ *and* $Q_2(u) = (1-u)^{-\frac{1}{p_2}} - 1$, $p_2 > 0$. *Then* $Q_3(u) = Q_2^{-1}(Q_1(u)) = 1-(1-u)^{\frac{p_2}{p_1}}$ *and* $q_3(u) = \frac{p_2}{p_1}(1-u)^{\frac{p_2}{p_1}-1}$. *Hence*

$$\mathcal{D}_\alpha^\beta(X,Y) = \frac{1}{(\alpha-\beta)}\left[\log\left(\frac{\left(\frac{p_2}{p_1}\right)^{2-\alpha-\beta}}{(\frac{p_2}{p_1}-1)(2-\alpha-\beta)+1}\right)\right].$$

EXAMPLE 16. *Assume that there are two distributions with QFs* $Q_1(u) = c_1 u^{\lambda_1}(1-u)^{-\lambda_2}, c_1, \lambda_2 > 0$ *and* $Q_2(u) = c_2 u^{\frac{1}{\lambda_3}}, c_2, \lambda_3 > 0$. *Here,* $Q_1(u)$ *and* $Q_2(u)$ *are the QFs of Davies and power distributions, respectively. Simple calculation then shows that*

$$q_3(u) = \lambda_3\left(\frac{c_1}{c_2}\right)^{\lambda_3} u^{\lambda_1\lambda_3}(1-u)^{-\lambda_1\lambda_3}\left[\frac{\lambda_1}{u} + \frac{\lambda_2}{1-u}\right]. \tag{35}$$

Substituting (35) in (34), one can easily obtain $\mathcal{D}_\alpha^\beta(X,Y)$. *In particular, we assume that* $\lambda_1 = \lambda_2 = 1$. *Then the quantile divergence measure of order* (α,β) *is given by*

$$\begin{aligned}\mathcal{D}_\alpha^\beta(X,Y) &= \frac{(2-\alpha-\beta)}{(\alpha-\beta)}\log\lambda_3 + \lambda_3\frac{(2-\alpha-\beta)}{(\alpha-\beta)}\log(\frac{c_1}{c_2}) \\ &+ \frac{1}{(\alpha-\beta)}\log\beta[(\lambda_3-1)(2-\alpha-\beta)+1,(\lambda_3+1)(\alpha+\beta-2)+1].\end{aligned}$$

EXAMPLE 17. *Consider the QFs of Govindarajulu and inverted (reciprocal) exponential distribution as* $Q_1(u) = 2u - u^2$ *and* $Q_2(u) = -\frac{\lambda}{\log u}$, *where* $\lambda > 0$. *Then,* $q_3(u)$ *can be obtained as*

$$q_3(u) = \left(\frac{2\lambda(1-u)}{(2u-u^2)^2}\right)e^{-\left(\frac{\lambda}{2u-u^2}\right)}.$$

Thus, from (34), we have

$$D_{\alpha}^{\beta}(X,Y)=\frac{(2-\alpha-\beta)}{(\alpha-\beta)}\log(2\lambda)+\frac{1}{(\alpha-\beta)} \times\log\int_0^1\left(\frac{(1-u)}{(2u-u^2)^2}\right)^{2-\alpha-\beta}e^{-\left(\frac{\lambda(\alpha+\beta-2)}{2u-u^2}\right)}du, \tag{36}$$

which is not easy to evaluate analytically. In this case, one may use Mathematica software to compute (36) numerically.

EXAMPLE 18. *Let X_1 and X_2 be the two random variables with QFs $Q_1(u)=\lambda_1+\frac{1}{\lambda_2}(u^{\lambda_3}-(1-u)^{\lambda_4})$ and $Q_2(u)=\frac{2u}{\lambda_2}+(\lambda_1-\frac{1}{\lambda_2})$, respectively, where $\lambda_i>0,\ i=1,2,3,4$. Note that X_1 and X_2 follow generalized lambda distribution and uniform distribution, respectively, with support $(\lambda_1+\frac{1}{\lambda_2},\lambda_1-\frac{1}{\lambda_2})$, simple calculations lead to*

$$Q_3(u)=\frac{1}{2}\left(u^{\lambda_3}-(1-u)^{\lambda_4}+1\right).$$

Thus, we have

$$q_3(u)=\frac{1}{2}\left[\lambda_3u^{\lambda_3-1}+\lambda_4(1-u)^{\lambda_4-1}\right].$$

From (34), we obtain

$$D_{\alpha}^{\beta}(X,Y)=\frac{(\alpha+\beta-2)}{(\alpha-\beta)}\log 2+\frac{1}{(\alpha-\beta)}\log\left[\int_0^1(\lambda_3u^{\lambda_3-1}+\lambda_4(1-u)^{\lambda_4-1})^{(2-\alpha-\beta)}du\right],$$

which can be easily computed numerically.

EXAMPLE 19. *Let us consider van Staden- Loots and uniform distributions with quantile functions $Q_1(u)=\lambda_1+\lambda_2[(\frac{(1-\lambda_3}{\lambda_4})(u^{\lambda_4}-1)-(\frac{\lambda_3}{\lambda_4})\{(1-u)^{\lambda_4}-1\}]$ and $Q_2(u)=u$, respectively, where $\lambda_i>0$ for i= 1,2,3,4. Here,*

$$Q_3(u)=\lambda_1+\lambda_2[(\frac{(1-\lambda_3}{\lambda_4})(u^{\lambda_4}-1)-(\frac{\lambda_3}{\lambda_4})\{(1-u)^{\lambda_4}-1\}]$$

and

$$q_3(u)=\lambda_2[(1-\lambda_3)u^{\lambda_4-1}+\lambda_3(1-u)^{\lambda_4-1}].$$

Thus, from (34), we have

$$D_{\alpha}^{\beta}(X,Y)=\frac{(2-\alpha-\beta)}{(\alpha-\beta)}\log\lambda_2+\frac{1}{(\alpha-\beta)} \times\log\left[\int_0^1\left((1-\lambda_3)u^{\lambda_4-1}+\lambda_3(1-u)^{\lambda_4-1}\right)^{(2-\alpha-\beta)}du\right],$$

which can be easily computed numerically.

EXAMPLE 20. *Let X_1 and X_2 be two random variable with respectively QFs $Q_1(u) = \frac{c_1 u}{(1-u)}$ and $Q_2(u) = c_2 u^{\frac{1}{c_3}}$, $c_i > 0$, $i = 1, 2, 3$. Then, it is easy to be obtain that*

$$Q_3(u) = \left(\frac{c_1}{c_2}\right)^{c_3} u^{c_3}(1-u)^{-c_3}$$

and

$$q_3(u) = c_3\left(\frac{c_1}{c_2}\right)^{c_3} u^{c_3-1}(1-u)^{-(c_3+1)}.$$

Using these in (34), we obtain

$$\begin{aligned} \underset{\sim}{D}_\alpha^\beta(X,Y) &= \frac{(2-\alpha-\beta)}{(\alpha-\beta)}\log c_3 + \frac{c_3(2-\alpha-\beta)}{(\alpha-\beta)}\log\left(\frac{c_1}{c_2}\right) \\ &+ \frac{1}{(1-\alpha)}\log\beta[(c_3-1)(2-\alpha-\beta)+1,(c_3+1)(\alpha+\beta-2)+1]. \end{aligned}$$

The discrimination measure (29) is not appropriate in reliability and life-testing studies as the current age of a system needs to be included. The Varma's divergence between two residual lifetime $X_t = (X-t|X>t)$ and $Y_t = (Y-t|Y>t)$ can be defined by

$$D_\alpha^\beta(X,Y;t) = \frac{1}{(\alpha-\beta)}\left[\log\left(\int_t^\infty \frac{f(x)}{\bar{F}(t)}\left(\frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)}\right)^{\alpha+\beta-2} dx\right)\right]. \tag{37}$$

For more details we refer to Maya and Sunoj (2008), Sunoj and Linu (2012) and Kayal (2015b). The quantile version of Varma's redidual divergence measure of order (α,β) between two random variables X_t and Y_t is defined by

$$\begin{aligned} \underset{\sim}{D}_\alpha^\beta(X,Y,u) &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\int_u^1\left(\frac{f(Q_1(p))}{1-u}\right)^{\alpha+\beta-1}\left(\frac{\bar{G}(Q(u))}{g(Q_1(p))}\right)^{\alpha+\beta-2} q_1(p)dp\right\}\right] \\ &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(1-G(Q_1(u)))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}\int_u^1\{q_1(p)g(Q_1(p))\}^{2-\alpha-\beta}dp\right\}\right] \\ &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(1-Q_2^{-1}(Q_1(u)))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}\int_u^1(q_3(p))^{2-\alpha-\beta}dp\right\}\right] \\ &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(1-Q_3(u))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}\int_u^1(q_3(p))^{2-\alpha-\beta}dp\right\}\right]. \end{aligned} \tag{38}$$

Cox (1972) introduced and studied a dependence structure among two distributions, which is referred to as the *proportional hazard* (PH) model. This model is extensively used in biomedical applications, reliability engineering and survival analysis. In the following we obtain a characterization result of proportional hazard (PH) model through the quantile generalized residual divergence measure of order (α,β) given in (38). Under this model, the survival functions of the random variables X and Y are related by

$$\bar{G}(x) = [\bar{F}(x)]^{\theta},\ \theta > 0, \tag{39}$$

where θ is a positive constant.

THEOREM 21. *The quantile-based generalized residual divergence measure of order* (α,β) $(\mathcal{D}_{\alpha}^{\beta}(X,Y,u))$ *is independent of* u, *for* $(\theta-1)(2-\alpha-\beta)+1>0$, *if and only if* X *and* Y *satisfy the Cox's PH model.*

PROOF. Assume that Cox's PH model is satisfy. In case of Cox PH model, $Q_2(u) = Q_1(1-(1-u)^{\frac{1}{\theta}})$, for some constant $\theta > 0$, we obtain $Q_3(u) = 1-(1-u)^{\theta}$ and $q_3(u) = \theta(1-u)^{\theta-1}$, refer to Nair *et al.* (2013). On using this condition, we have

$$\begin{aligned}
\mathcal{D}_{\alpha}^{\beta}(X,Y,u) &= \frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(1-Q_3(u))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}\int_u^1 (q_3(p))^{2-\alpha-\beta}dp\right\}\right]\\
&= \frac{1}{(\alpha-\beta)}\left[\log\left\{\theta^{2-\alpha-\beta}(1-u)^{(\theta-1)(\alpha+\beta-1)-\theta}\int_u^1 (1-p)^{(\theta-1)(2-\alpha-\beta)}dp\right\}\right]\\
&= \frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{\theta^{2-\alpha-\beta}}{(\theta-1)(2-\alpha-\beta)+1}\right\}\right],
\end{aligned}$$

which is independent of u.

Conversely, suppose $\mathcal{D}_{\alpha}^{\beta}(X,Y,u) = \text{constant}$. Then using (38), we have

$$\log\left\{\frac{(1-Q_3(u))^{\alpha+\beta-2}}{(1-u)^{\alpha+\beta-1}}\int_u^1 (q_3(p))^{2-\alpha-\beta}dp\right\} = C(\alpha-\beta),$$

this gives

$$\int_u^1 (q_3(p))^{2-\alpha-\beta}dp = e^{c(\alpha-\beta)}(1-u)^{\alpha+\beta-1}(1-Q_3(u))^{2-\alpha-\beta}. \tag{40}$$

Differentiating (40) with respect to u, we get, after simplification

$$(\alpha+\beta-1)\left(\frac{(1-u)q_3(u)}{1-Q_3(u)}\right)^{\alpha+\beta-2} + (2-\alpha-\beta)\left(\frac{(1-u)q_3(u)}{1-Q_3(u)}\right)^{\alpha+\beta-1} = e^{-c(\alpha-\beta)}. \tag{41}$$

Substituting $p(u) = \frac{(1-u)q_3(u)}{1-Q_3(u)}$ and $e^{-c(\alpha-\beta)} = C_1$ in (41), this gives

$$(\alpha+\beta-1)(p(u))^{\alpha+\beta-2} + (2-\alpha-\beta)(p(u))^{\alpha+\beta-1} = C_1,$$

which leads to $p(u)=\theta$, a constant. This means that

$$-\frac{d}{du}\log(1-Q_3(u))=\frac{\theta}{1-u}. \tag{42}$$

Integrating (42) with respect to u between the limits 0 to u, we get $\log(1-Q_3(u))=\theta\log(1-u)$, and hence $Q_3(u)=Q_2^{-1}(Q_1(u))=1-(1-u)^{\theta}$, which completes the proof of the theorem. □

Due to duality it is natural to study the dynamic divergence measure of order (α,β) between past lifetimes. Thus the generalized divergence measure of order (α,β) between the past lives ${}_tX=[t-X|X\leq t]$ and ${}_tY=[t-Y|Y\leq t]$ is given by

$$\bar{D}_{\alpha}^{\beta}(X,Y;t)=\frac{1}{(\alpha-\beta)}\left[\log\left\{\int_0^t\frac{f(x)}{F(t)}\left(\frac{f(x)/F(t)}{g(x)/G(t)}\right)^{\alpha+\beta-2}dx\right\}\right]. \tag{43}$$

Given that at time t units are found to be down, $\bar{D}_{\alpha}^{\beta}(X,Y;t)$ measures the information distance between two past lifetimes ${}_tX$ and ${}_tY$. Similar to $Ḑ_{\alpha}^{\beta}(X,Y,u)$, a quantile version of divergence measure of order (α,β) between two past lifetime random variables ${}_tX$ and ${}_tY$ is obtained as

$$\begin{aligned}\bar{Ḑ}_{\alpha}^{\beta}(X,Y,u)&=\frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(G(Q_1(u)))^{\alpha+\beta-2}}{u^{\alpha+\beta-1}}\int_0^u f(Q_1(p))\left(\frac{f(Q_1(p))}{g(Q_1(p))}\right)^{\alpha+\beta-2}\right.\right.\\&\qquad\left.\left.\times\, q_1(p)dp\right\}\right]\\&=\frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(Q_2^{-1}(Q_1(u)))^{\alpha+\beta-2}}{u^{\alpha+\beta-1}}\int_0^u\{q_1(p)g(Q_1(p))\}^{2-\alpha-\beta}dp\right\}\right]\\&=\frac{1}{(\alpha-\beta)}\left[\log\left\{\frac{(Q_3(u))^{\alpha+\beta-2}}{u^{\alpha+\beta-1}}\int_0^u(q_3(p))^{2-\alpha-\beta}dp\right\}\right].\end{aligned} \tag{44}$$

EXAMPLE 22. *We consider a parallel system of n components with lifetimes* $X_i, i=1,\cdots n$, *which are independent and identically distributed (i.i.d.) each with distribution function* $F(x)$. *The lifetime of the parallel system is given by* $Y=\max(X_1,X_2,.......X)$ *with distribution function is given by*

$$G(x)=[F(x)]^n.$$

Hence X_i *and* Y *satisfy the proportional reversed hazard (PRH) model. For PRH model, we have* $Q_2(u)=Q_1(u^{\frac{1}{n}})$, *for some positive integer* n, *we obtain* $Q_3(u)=u^n$ *and* $q_3(u)=$

nu^{n-1}. On using this condition, the corresponding quantile-based divergence measure of order (α,β) for two past lifetimes is given by

$$\bar{D}_{\alpha}^{\beta}(X,Y,u)=\frac{1}{(\alpha-\beta)}\log\left[\frac{n^{2-\alpha-\beta}}{(n-1)(2-\alpha-\beta)+1}\right].$$

EXAMPLE 23. *Let X and Y be two random variables with power function having QFs respectively by $Q_1(u)=u^{\frac{1}{b_1}}$; $b_1>0$ and $Q_2(u)=u^{\frac{1}{b_2}}$; $b_2>0$. Then $Q_3(u)=Q_2^{-1}(Q_1(u))=u^{\frac{b_2}{b_1}}$ and $q_3(u)=(\frac{b_2}{b_1})u^{\frac{b_2}{b_1}-1}$. Hence the quantile divergence measure of order (α,β) between two past lifetime random variables (44), is given by*

$$\bar{D}_{\alpha}^{\beta}(X,Y,u)=\frac{1}{(\alpha-\beta)}\left[\log\left(\frac{\left(\frac{b_2}{b_1}\right)^{2-\alpha-\beta}}{(\frac{b_2}{b_1}-1)(2-\alpha-\beta)+1}\right)\right].$$

Next, we give a characterization problem for proportional reversed hazard (PRH) model.

THEOREM 24. *A necessary and sufficient condition for $\bar{D}_{\alpha}^{\beta}(X,Y,u)$ to be independent of u, for $(\theta-1)(2-\alpha-\beta)+1>0$, is that X and Y satisfy the PRH model.*

The proof is similar to that of Theorem 21 and hence omitted.

REMARK 25. *For $\beta=1$ and $\alpha\to 1$, (38) reduces to*

$$D_{X/Y}(u)=D(X/Y;Q(u))=\log\left(\frac{1-Q_3(u)}{1-u}\right)-(1-u)^{-1}\int_u^1\log q_3(p)dp, \tag{45}$$

the quanlite-based residual Kullback-Leibler relative entropy, refer to Sankaran et al. (2016).

REMARK 26. *The results in the present paper also hold for Renyi's entropy (1961) and Renyi's divergence information measure, when $\beta=1$.*

6. CONCLUSION

Residual and past lifetime is an important concepts in many discipline. The residual lifetime is defined as the remaining time to an event given that the survival time X of a patient is at least t. In several clinical studies, particularly when the associated diseases are chronic or/and incurable, it is great concern to patients to know residual lifetime. The present work introduced an alternative approach to generalized dynamic (residual and past both) entropy and divergence measure of order (α,β) using quantile functions. The results obtained in this article are general in the sense that they reduce to some of the results for quantile-based Shannon entropy and K-L divergence information measure obtained by Sunoj and Sankaran (2012) and Sankaran *et al.* (2016), when $\beta=1$ and α tends to 1. Also, we have studied some characterization results based on proposed measures and study their certain properties and application in reliability engineering.

ACKNOWLEDGEMENTS

The authors would like to express their gratitude to the reviewers and the editor in chief for their valuable comments, which have considerably improved the earlier version of the article. The first author wishes to acknowledge the Science and Engineering Research Board (SERB), Government of India, for the financial assistance (Ref. No. ECR/2017/001987) for carrying out this research work.

REFERENCES

M. A. K. BAIG, J. G. DAR(2008). *Generalized residual entropy function and its applications*. European Journal of Pure and Applied Mathematics, 4, pp. 30–40.

D. R. COX (1972). *Regression models and life tables*. Journal of the Royal Statistical Society, Series B, 34, pp. 187–220.

A. DI CRESCENZO, M. LONGOBARDI (2002). *Entropy-based measure of uncertainty in past lifetime distributions*. Journal of Applied Probability, 39, pp. 434–440.

A. DI CRESCENZO, M. LONGOBARDI (2004). *A measure of discrimination between past lifetime distributions*. Statistics and Probability Letters, 67, pp. 173–182.

N. EBRAHIMI (1996). *How to measure uncertainty in the residual lifetime distribution*. Sankhya, Series A, 58, pp. 48–56.

W. GILCHRIST (2000). *Statistical Modelling with Quantile Functions*. Chapman and Hall/CRC, Boca Raton, FL.

S. KAYAL (2014). *Some results on a generalized residual entropy based on order statistics*. Statistica, 74, no. 4., pp. 383–402.

S. KAYAL (2015a). *On generalized dynamic survival and failure entropies of order (α, β)*. Statistics and Probability Letters, 96, pp. 123–132.

S. KAYAL (2015b). *Some results on dynamic discrimination measures of order* (α, β). Hacettepe Journal of Mathematics and Statistics, 44, pp. 179–188.

S. KAYAL, P. VELLAISAMY (2011). *Generalized entropy properties of records*. Journal of Analysis, 19, pp. 25–40.

S. KULLBACK, R. A. LEIBLER (1951). *On information and sufficiency*. The Annals of Mathematical Statistics, 22, no. 1, pp. 79–86.

V. KUMAR, H. C. TANEJA (2011). *Some characterization results on generalized cumulative residual entropy measure*. Statistics and Probability Letters, 81, no. 8, pp. 72–77.

S. S. MAYA, S. M. SUNOJ (2008). *Some dynamic generalized information measures in the context of weighted models*. Statistica, 68, no.1, pp. 71–84.

N. U. NAIR, P. G. SANKARAN, N. BALAKRISHNAN (2013). *Quantile-Based Reliability Analysis*. Springer, New York.

N. U. NAIR, P. G. SANKARAN (2009). *Quantile based reliability analysis*. Communication in Statistics - Theory and Methods, 38, pp. 222–232.

A. K. NANDA, P. G. SANKARAN, S. M. SUNOJ (2014). *Residual Renyi entropy: A Quantile approach*. Statistics and Probability Letters, 85, pp. 114-121.

E. PARZEN (1979). *Non parametric statistical data modelling*. Journal of the American Statistical Association, 74, pp. 105–22.

A. RENYI (1961). *On measure of entropy and information*. Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1, pp. 547–561.

J. M. RUIZ, J. NAVARRO (1996). *Characterizations based on conditional expectations of the doubled truncated distribution*. Annals of the Institute of Statistical Mathematics , 48, no. 3, pp. 563–572.

P. G. SANKARAN, S. M. SUNOJ, N. U. NAIR (2016). *Kullback-Leibler divergence: A quantile approach*. Statistics and Probability Letters, 111, pp. 72–79.

C. E. SHANNON (1948). *A mathematical theory of communication*. Bell System Technical Journal, 27, pp.379–423.

S. M. SUNOJ, M. N. LINU (2012). *On bounds of some dynamic information divergence measures*. Statistica, 72, no. 1, pp. 23–36.

S. M. SUNOJ, P. G. SANKARAN (2012). *Quantile based entropy function*. Statistics and Probability Letters, 82, pp. 1049–1053.

S. M. SUNOJ, P. G. SANKARAN, A. K. NANDA (2013). *Quantile based entropy function in past lifetime*. Statistics and Probability Letters, 83, pp. 366–372.

S. M. SUNOJ, P. G. SANKARAN, N. U. NAIR (2017). *Quantile-based cumulative Kullaback-Leibler divergence*. Statistics: A Journal of Theoretical and Applied Statistics, 52, no. 1, pp. 1–17.

R. S. VARMA (1966). *Generalization of Renyi's entropy of order* α. Journal of Mathematical Sciences, 1, pp. 34–48.

SUMMARY

In the present paper, we propose a quantile version of generalized entropy measure for residual and past lifetimes and study their properties. Lower and upper bounds of the proposed measures are derived. Some of the quantile lifetime distributions have been characterized. We also introduce quantile versions of the generalized divergence measure of Varma between two residual and two past lifetime random variables. Some properties of this measure are studied and a characterization of the proportional (reversed) hazards model is given.

Keywords: Quantile function; Varma's entropy; Divergence measure; Hazard quantile function; PH Model.