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ON SOME ASPECTS OF A GENERALIZED ASYMMETRIC NORMAL DISTRIBUTION

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1. INTRODUCTION

The normal distribution is the basis of many statistical works and it enjoys a unique position in probability theory. It is an unavoidable tool for the analysis and interpretation of data. In many practical applications it has been observed that real life data sets are not symmetric. They exhibit some skewness, therefore do not conform to the normal distribution, which is popular and easy to be handled. Azzalini (1985) introduced a new class of distributions namely "the skew normal distribution", which is mathematically tractable and includes the normal distribution as a special case. This family of distributions is well known for modeling and analyzing skewed data. This distribution has been developed via standard normal probability density function (p.d.f) and cumulative distribution function (c.d.f) through adding a shape parameter to regulate skewness, so as to have more flexibility in fitting real life data sets.

Let $f(.)$ and $F(.)$ be the p.d.f and c.d.f of a standard normal variate. Then a random variable X is said to follow the skew normal distribution with parameter $\lambda \in R = (-\infty, \infty)$ if its probability density function (p.d.f.) $h(x;\lambda)$ is of the following form. For $x \in R$,

$$h(x;\lambda) = 2f(x)F(\lambda x), \tag{1}$$

hereafter, we denoted a distribution with p.d.f. (1) as $SND(\lambda)$. This distribution has been studied by several authors such as Azzalini (1986), Henze (1986), Liseo (1990), Azzalini and Dalla Valle (1996), Branco and Dey (2001), Genton *et al.* (2001), Loperfido (2001), Gupta and Kollo (2003), Loperfido (2004), Genton (2004), Genton and Loperfido (2005), Lachos *et al.* (2007), Gupta *et al.* (2007), Kim (2008), Wang *et al.* (2009) and Kumar and Anusree (2011, 2013, 2014a,b).

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The normal and skew normal models are not adequate to describe the situations of plurimodality. To overcome this drawback Kumar and Anusree (2011) considered a new class of generalized skew normal distribution as a generalized mixture of standard normal and skew normal distributions through the following p.d.f in which $x \in R$, $\lambda \in R$ and $\alpha > -1$.

$$h_1(x;\lambda,\alpha) = \frac{2}{\alpha+2} f(x)[1+\alpha F(\lambda x)]. \tag{2}$$

The distribution given in (2) they termed as generalized mixture of standard normal and skew normal distributions ($GMNSN(\alpha,\lambda)$). Clearly $GMNSN(-1, \lambda)$ is $SN(-\lambda)$. In order to develop a more flexible plurimodal asymmetric normal distribution, through the present paper we consider a generalized version of the skew normal distribution of Kumar and Anusree (2011) which we call "the generalized asymmetric normal distribution (GAND)".

The organization of the paper is as follows. In Section 2 we present the definition and some properties of the GAND. In Section 3 certain reliability measures such as reliability function, failure rate, and mean residual life function are derived and condition for unimodal and plurimodal situations are obtained. In Section 4 a location scale extension of the GAND is proposed and derive its important properties such as characteristic function, mean, variance, measure of skewness and kurtosis, reliability measures etc. Further in Section 5 we discuss the maximum likelihood estimation of the parameters of extended GAND and a real life application of the distribution is considered in Section 6.

2. The generalized asymmetric normal distribution

Here we define a new class of generalized skew normal distribution and derive some of its important properties.

Definition 1. *A random variable X is said to have a generalized asymmetric normal distribution if its p.d.f is of the following form, in which* $x \in R$, $\lambda, \beta, \in R$ *and* $\alpha > -1$.

$$g(x;\alpha,\lambda,\beta) = \frac{f(x)}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})\right]. \tag{3}$$

Here f(.) and F(.) are p.d.f and c.d.f of standard normal variate. A distribution with p.d.f (3) we denoted as GAND(α,λ,β)*. Note that when* $\beta = 0$ *GAND*(α,λ,β) *reduces to skew normal distribution of Kumar and Anusree (2011).*

For some particular choices of α, λ and β the p.d.f. given in (3) of GAND(α, λ, β) is plotted in Figures 1 and 2.

Result 1. *If X has GAND*(α,λ,β) *then* $Y_1 = -X$ *has GAND*$(\alpha,-\lambda,\beta)$.

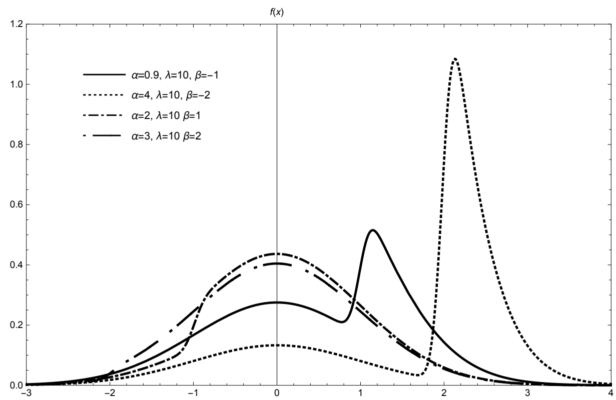

Figure 1 – Probability plots of GAND(α, λ, β) for fixed values of λ and various values of α and β.

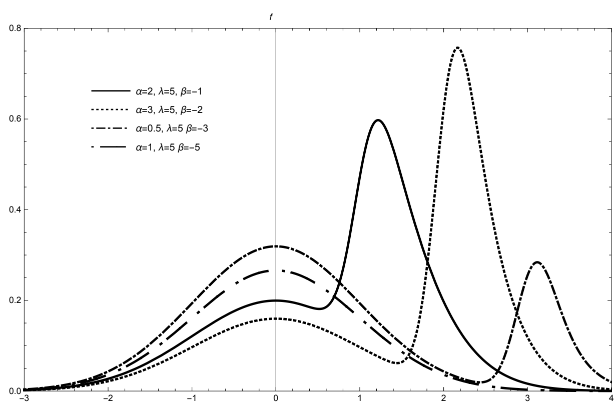

Figure 2 – Probability plots of GAND(α, λ, β) for fixed values of λ and various values of α and β.

PROOF. The p.d.f $g_1(y_1)$ of Y_1 is

$$\begin{aligned} g_1(y_1) &= g(-y_1;\alpha,\lambda,\beta)|\frac{dx}{dy_1}| \\ &= \frac{f(-y_1)}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(-\lambda y_1+\beta\sqrt{1+\lambda^2})\right] \\ &= g(y_1;\alpha,-\lambda,\beta), \end{aligned}$$

since $f(.)$ is the p.d.f. of standard normal variate. Hence Y_1 follows GAND$(\alpha,-\lambda,\beta)$. □

RESULT 2. *If X has GAND(α,λ,β) then $Y_2 = X^2$ has a p.d.f (4) in which* $\Delta(y) = F(\lambda y+\beta\sqrt{1+\lambda^2})+F(-\lambda y+\beta\sqrt{1+\lambda^2})$.

PROOF. The p.d.f. $g_2(y_2)$ of $Y_2 = X^2$ is the following, for $y_2 > 0$.

$$\begin{aligned} g_2(y_2) &= g(\sqrt{y_2},\alpha,\lambda,\beta)|\frac{dx}{dy_2}| + g(-\sqrt{y_2},\alpha,\lambda,\beta)|\frac{dx}{dy_2}| \\ &= \frac{f(\sqrt{y_2})}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(\lambda\sqrt{y_2}+\beta\sqrt{1+\lambda^2})\right]\frac{1}{2\sqrt{y_2}} + \\ &\quad \frac{f(-\sqrt{y_2})}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(-\lambda\sqrt{y_2}+\beta\sqrt{1+\lambda^2})\right]\frac{1}{2\sqrt{y_2}} \\ &= \frac{f(\sqrt{y_2})}{2(\alpha+2)\sqrt{y_2}}\Big[4+\alpha[F(\beta)]^{-1}\Big\{F(\lambda\sqrt{y_2}+\beta\sqrt{1+\lambda^2}) \\ &\quad +F(-\lambda\sqrt{y_2}+\beta\sqrt{1+\lambda^2})\Big\}\Big] \\ &= \left(\frac{f(\sqrt{y_2})}{2\sqrt{y_2}}\right)\frac{1}{(\alpha+2)}\left[4+\alpha[F(\beta)]^{-1}\Delta(\sqrt{y_2})\right]. \qquad (4) \end{aligned}$$

□

RESULT 3. *If X has GAND(α,λ,β) then $Y_3 = |X|$ has a p.d.f (5) in which $\Delta(y)$ as defined in Result 2.*

PROOF. For $x > 0$, the p.d.f of $g_3(x)$ of Y_3 is

$$\begin{aligned} g_3(y_3) &= g(y_3;\alpha,\lambda,\beta)|\frac{dx}{dy_3}| + g(-y_3;\alpha,\lambda,\beta)|\frac{dx}{dy_3}| \\ &= \frac{f(y_3)}{\alpha+2}\Big[2+\alpha[F(\beta)]^{-1}F(\lambda y_3+\beta\sqrt{1+\lambda^2})\Big]+ \\ &\quad \frac{f(-y_3)}{\alpha+2}\Big[2+\alpha[F(\beta)]^{-1}F(-\lambda y_3+\beta\sqrt{1+\lambda^2})\Big] \\ &= \frac{f(y_3)}{\alpha+2}\Big[4+\alpha[F(\beta)]^{-1}\Big\{F(\lambda y_3+\beta\sqrt{1+\lambda^2})+F(-\lambda y_3+\beta\sqrt{1+\lambda^2})\Big\}\Big] \\ &= \frac{f(y_3)}{\alpha+2}\big[4+\alpha[F(\beta)]^{-1}\Delta(y_3)\big]. \end{aligned} \tag{5}$$

□

RESULT 4. *The cumulative distribution function (c.d.f) $G(x)$ of GAND(α,λ,β) with p.d.f (3) is the following, for $x \in R$*

$$G(x) = \frac{F(x)}{\alpha+2}\Big[2+\frac{\alpha}{2}[F(\beta)]^{-1}\Big] - \frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\xi_\beta(x,\lambda), \tag{6}$$

where $\xi_\beta(x,\lambda) = \int_x^\infty \int_0^{\lambda x+\beta\sqrt{(1+\lambda^2)}} f(t)f(u)\,du\,dt$, which can be evaluated using the software MATHCAD.

PROOF.

$$\begin{aligned} G(x) &= \int_{-\infty}^{x} g(t;\alpha,\lambda,\beta)dt \\ &= \frac{2}{\alpha+2}F(x) + \frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\left[\frac{F(x)}{2} - \int_x^\infty \int_0^{\lambda x+\beta\sqrt{1+\lambda^2}} f(t)f(u)du\,dt\right] \\ &= \frac{F(x)}{\alpha+2}\Big[2+\frac{\alpha}{2}[F(\beta)]^{-1}\Big] - \frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\xi_\beta(x,\lambda). \end{aligned}$$

□

Now we derive the characteristic function of GAND(α,λ,β) and we need the following lemma.

LEMMA 2. *Ellison (1964). For a standard normal random variable X with distribution function F we have the following for all $a, b \in R$*

$$E\{F(aX+b)\} = F\left\{\frac{b}{\sqrt{1+a^2}}\right\}.$$

RESULT 5. *The characteristic function $\phi_X(t)$ of GAND(α,λ,β) with p.d.f (3) is the following, for $t \in R$ and $i = \sqrt{-1}$*

$$\phi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(\delta i t+\beta)\right], \tag{7}$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.

PROOF. Let X follows GAND(α,λ,β) with p.d.f (3). Then by the definition of characteristic function, we have the following for any $t \in R$ and $i = \sqrt{-1}$

$$\begin{aligned} \phi_X(t) &= E(e^{itX}) \\ &= \frac{2}{\alpha+2}\int_{-\infty}^{\infty} e^{itx} f(x)dx + \frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\int_{-\infty}^{\infty} e^{itx} f(x)F(\lambda x+\beta\sqrt{1+\lambda^2})dx \\ &= \frac{1}{\alpha+2}e^{\frac{-t^2}{2}}\left\{2+\alpha[F(\beta)]^{-1}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{\frac{-(x-it)^2}{2}}F(\lambda x+\beta\sqrt{1+\lambda^2})dx\right\}. \end{aligned} \tag{8}$$

On substituting $x - it = u$, in (8) we obtain

$$\phi_X(t) = \frac{e^{\frac{-t^2}{2}}}{\alpha+2}\left[2+\alpha[F(\beta)]^{-1}F(\delta i t+\beta)\right],$$

which implies (7) in the light of Lemma 2. □

RESULT 6. *The nth raw moment μ_n' of $GAND(\alpha,\lambda,\beta)$ with p.d.f (3) is the following, for $n \geq 0$*

$$\mu_n' = \frac{1}{\alpha+2}\xi_r + \frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\sum_{r=0}^{n}\binom{n}{r}\xi_r\varphi_{n-r}, \tag{9}$$

where for $r = 0, 1, 2, ..., n$

$$\xi_r = \begin{cases} 0, & \text{if } r \text{ is odd} \\ \frac{(-1)^{\frac{r}{2}} r!}{(\frac{r}{2})! 2^{\frac{r}{2}}}, & \text{if } r \text{ is even} \end{cases}$$

$$\varphi_r = \begin{cases} \frac{(\delta i)^r(-1)^{\frac{r-1}{2}}(r-1)!\beta^{r-1}f(\beta)}{(\frac{r-1}{2})! 2^{\frac{r-1}{2}}}, & \text{if } r \text{ is odd} \\ \frac{(\delta i)^r(-1)^{\frac{r}{2}}(r-1)!\beta^{r-1}f(\beta)}{2^{\frac{r}{2}-1}}, & \text{if } r \text{ is even.} \end{cases}$$

PROOF. The characteristic function of $GAND(\alpha,\lambda,\beta)$ can be written as

$$\phi_X(t)=\frac{2}{\alpha+2}I(t)+\frac{\alpha[F(\beta)]^{-1}}{\alpha+2}I(t)J(t), \tag{10}$$

in which $I(t)=e^{-\frac{t^2}{2}}$ and $J(t)=F(\delta it+\beta)$ on differentiating (10) with respect to t, n times and putting $t=0$ we get the nth moment of X as

$$\mu_n^{'}=\left[\frac{2}{\alpha+2}I^r(t)+\frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\sum_{r=0}^{n}I^r(t)J^{n-r}(t)\right]_{t=0}, \tag{11}$$

in which $I^r(t)$ and $J^{n-r}(t)$ respectively denote the rth and $(n-r)th$ derivative of $I(t)$ and $J(t)$ which are obtained as

$$I^{(r)}(t)=\sum_{j=0}^{[\frac{r}{2}]}\frac{(-1)^{r-j}t^{r-2j}r!e^{-\frac{t^2}{2}}}{j!2^j(r-2j)!} \tag{12}$$

and

$$J^{(r)}(t)=\sum_{k=0}^{[\frac{r-1}{2}]}\frac{(\delta it+\beta)^{r-2k-1}(-1)^{r-k-1}(r-1)!f(\delta it+\beta)(\delta i)^r}{k!(r-2k-1)!2^k}. \tag{13}$$

If we put $t=0$ in (12) and (13) and using the notation $\xi_r=I^{(r)}(0)$ and $\varphi_{n-r}=J^{(n-r)}(0)$ we get (9) from (11). □

Using Result 6 we prove the following.

RESULT 7. *The mean and variance of GAND(α,λ,β) with p.d.f (3) is given by*

$$Mean=\frac{\alpha}{\alpha+2}.\frac{\delta f(\beta)}{F(\beta)},$$

$$Variance=\frac{-\alpha\delta\beta f(\beta)}{(\alpha+2)F(\beta)}+1-\frac{\alpha^2\delta^2[f(\beta)]^2}{(\alpha+2)^2[F(\beta)]^2}.$$

RESULT 8. *The measure of skewness(γ_1) and measure of kurtosis (γ_2) of GAND(α,λ,β) with p.d.f (3) are respectively given by*

$$\gamma_1=\frac{(-\delta^2 d+\delta^2 d\beta^2+3\beta d^2\delta+2d^3)^2}{(-\beta d+1-d^2)^3}$$

and

$$\gamma_2=\frac{3+\delta^3\beta d-\delta^3\beta^3 d-6\beta\delta d-4\delta^2\beta^2 d^2+4\delta^2 d^2-6d^2-6\delta\beta d^3-3d^4}{(-\beta d+1-d^2)^3},$$

where $d=\frac{\alpha}{\alpha+2}\frac{\delta f(\beta)}{F(\beta)}$.

3. Reliability measures and mode

Here we investigate some properties of GAND(α,λ,β) with p.d.f. (3) useful in reliability studies.

Let X follows GAND(α,λ,β) with p.d.f (3). Now from the definition of reliability function $R(t)$, failure rate $r(t)$ and mean residual life function $\mu(t)$ of X we obtain the following results.

RESULT 9. *The reliability function $R(t)$ of X is the following, in which $\xi_\beta(x,\lambda)=\int_x^\infty\int_0^{\lambda x+\beta\sqrt{(1+\lambda^2)}} f(t)f(u)\,du\,dt$ is as defined in Result 4*

$$R(t)=\frac{1}{\alpha+2}[1-F(t)]\left\{2+\frac{\alpha[F(\beta)]^{-1}}{2}\right\}+\frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\xi_\beta(t,\lambda).$$

RESULT 10. *The failure rate $r(t)$ of X is given by*

$$r(t)=\frac{f(t)\left[2+\alpha[F(\beta)]^{-1}F(\lambda t+\beta\sqrt{1+\lambda^2})\right]}{(1-F(t))\left[2+\frac{\alpha[F(\beta)]^{-1}}{2}\right]+\alpha[F(\beta)]^{-1}\xi_\beta(t,\lambda)}.$$

RESULT 11. *The mean residual life function of GAND(α,λ,β) is*

$$\begin{aligned}\mu(t) &= \frac{1}{(\alpha+2)R(t)}\left\{\frac{1}{\sqrt{2\pi}}\left(2e^{\frac{-t^2}{2}}+\frac{\alpha\lambda[F(\beta)]^{-1}e^{\frac{-\beta^2}{2}}}{\sqrt{1+\lambda^2}}\right)\right.\\ &\quad+\alpha[F(\beta)]^{-1}F(\lambda t+\beta\sqrt{1+\lambda^2})f(t)\\ &\quad\left.-\frac{\alpha\lambda[F(\beta)]^{-1}}{\sqrt{2\pi}\sqrt{1+\lambda^2}}e^{\frac{-\beta^2}{2}}F\left(\sqrt{1+\lambda^2}(t+\frac{\lambda\beta}{\sqrt{1+\lambda^2}})\right)\right\}-t.\end{aligned}\tag{14}$$

PROOF. By definition, the mean residual life function (MRLF) of X is given by

$$\begin{aligned}\mu(t) &= E(X-t/X>t)\\ &= E(X/X>t)-t,\end{aligned}$$

where

$$\begin{aligned}E(X/X>t) &= \frac{1}{(\alpha+2)R(t)}\int_t^\infty xf(x)dx+\frac{\alpha[F(\beta)]^{-1}}{(\alpha+2)R(t)}\\ &\quad\int_t^\infty xf(x)F(\lambda x+\beta\sqrt{1+\lambda^2})dx\\ &= \frac{1}{(\alpha+2)R(t)}[I_1+\alpha[F(\beta)]^{-1}I_2],\end{aligned}\tag{15}$$

where

$$
\begin{aligned}
I_1 &= \int_t^\infty x f(x) dx \\
&= \frac{e^{\frac{-t^2}{2}}}{\sqrt{2\pi}} \qquad (16) \\
I_2 &= \int_t^\infty x f(x) F(\lambda x + \beta\sqrt{1+\lambda^2}) dx \\
&= -\int_t^\infty f'(x) F(\lambda x + \beta\sqrt{1+\lambda^2}) dx \\
&= F(\lambda t + \beta\sqrt{1+\lambda^2}) f(t) + \lambda \int_t^\infty f(\lambda x + \beta\sqrt{1+\lambda^2}) f(x) dx \\
&= F\left(\lambda t + \beta\sqrt{1+\lambda^2}\right) f(t) + \frac{\lambda}{\sqrt{2\pi}\sqrt{1+\lambda^2}} e^{\frac{-\beta^2}{2}} \\
&\quad \left\{1 - F\left[\sqrt{1+\lambda^2}\left(t + \frac{\lambda\beta}{\sqrt{1+\lambda^2}}\right)\right]\right\}. \qquad (17)
\end{aligned}
$$

Now by applying (16) and (17) in (15), we get (14). The functions $R(t), r(t)$, and $\mu(t)$ are equivalent in the sense that if one of them is given the other two can be uniquely determined. □

REMARK 3. *$GAND(\alpha, \lambda, \beta)$ has increasing failure rate for all α and λ and hence decreasing mean residual life.*

RESULT 12. *Case 1: For $x > 0$ the p.d.f of $GAND(\alpha, \lambda, \beta)$ is log concave*

(i) if $\lambda > 0$ provided either $\alpha \geq 0$ and $\beta \geq 0$ or $\alpha < 0$ and $\beta < 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| < |1 + A_3|$,

where A_1, A_2 and A_3 are as defined in (18), (19) *and* (20).

Case 2: For $x < 0$ the p.d.f of $GAND(\alpha, \lambda, \beta)$ is log concave

(i) if $\lambda < 0$ provided either $\alpha \geq 0$ and $\beta \geq 0$ or $\alpha < 0$ and $\beta < 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| < |1 + A_3|$.

PROOF. To establish $\log[g(x; \alpha, \lambda, \beta)]$ is a concave function of x, it is enough to show that its second derivative is negative for all x. Then

$$\frac{d}{dx}\{log[g(x; \alpha, \lambda, \beta)]\} = -x + \frac{\alpha\lambda[F(\beta)]^{-1} f(\lambda x + \beta\sqrt{1+\lambda^2})}{2 + \alpha[F(\beta)]^{-1} F(\lambda x + \beta\sqrt{1+\lambda^2})}$$

and

$$\frac{d^2}{dx^2}\{\log[g(x;\alpha,\lambda,\beta)]\} = -1 - \Delta(x;\alpha,\lambda,\beta),$$

in which

$$\begin{aligned}\Delta(x;\alpha,\lambda,\beta) &= \frac{\alpha\lambda^2[F(\beta)]^{-1}f(\lambda x+\beta\sqrt{1+\lambda^2})}{2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})}\{\lambda x+\beta\sqrt{1+\lambda^2}+\\ &\quad \frac{\alpha[F(\beta)]^{-1}f(\lambda x+\beta\sqrt{1+\lambda^2})}{2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})}\}\\ &= A_1+A_2+A_3,\end{aligned}$$

where

$$A_1 = \frac{\lambda^3 x\alpha[F(\beta)]^{-1}f(\lambda x+\beta\sqrt{1+\lambda^2})}{2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})} \tag{18}$$

$$A_2 = \frac{\alpha\beta\sqrt{1+\lambda^2}-\lambda^2[F(\beta)]^{-1}f(\lambda x+\beta\sqrt{1+\lambda^2})}{2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})} \tag{19}$$

$$A_3 = \frac{\alpha^2\lambda^2[F(\beta)]^{-2}[f(\lambda x+\beta\sqrt{1+\lambda^2})]^2}{[2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})]^2}. \tag{20}$$

Note that $f(\lambda x+\beta\sqrt{1+\lambda^2})$ and $F(\lambda x+\beta\sqrt{1+\lambda^2})$ are positive for all $x \in R$ and hence $A_1 > 0$ for $x > 0$, $\alpha > 0$ or $x < 0$, $\alpha < 0$ and $A_2 > 0$ for $\alpha, \beta > 0$ or < 0. Clearly $A_3 > 0$ for all values of $\alpha, \beta, \lambda > 0$. Also $2+\alpha[F(\beta)]^{-1}F(\lambda x+\beta\sqrt{1+\lambda^2})$ is positive for all values of α, β and λ . Further, if $A_1 > 0, A_2, A_3 > 0$ then $\Delta(x;\alpha,\lambda,\beta) > 0$. □

As a consequence of Result 12, we have the following results regarding the unimodality and plurimodality of the $GAND(\alpha,\lambda,\beta)$.

RESULT 13. *GAND(α,λ,β) density is strongly unimodal under the following two cases.*
Case 1: For $x > 0$

(i) if $\lambda > 0$ provided either $\alpha \geq 0$ and $\beta \geq 0$ or $\alpha < 0$ and $\beta < 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| < |1 + A_3|$.

Case 2: For $x < 0$

(i) if $\lambda < 0$ provided either $\alpha \geq 0$ and $\beta \geq 0$ or $\alpha < 0$ and $\beta < 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| < |1 + A_3|$.

REMARK 4. *GAND(α, λ, β) density is plurimodal under the following two cases.*
Case 1: For $x > 0$

(i) if $\lambda > 0$ provided either $\alpha \geq 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta > 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| > |1 + A_3|$.

Case 2: For $x < 0$

(i) if $\lambda < 0$ provided either $\alpha \geq 0$ and $\beta < 0$ or $\alpha < 0$ and $\beta > 0$ and

(ii) if $\lambda < 0$ provided $|A_1 + A_2| > |1 + A_3|$.

4. LOCATION SCALE EXTENSION

In this section we discuss an extended form of GAND(α, λ, β) by introducing the location parameter μ and scale parameter σ.

DEFINITION 5. *Let $X \sim GAND(\alpha, \lambda, \beta)$ with p.d.f given in (3). Then $Y = \mu + \sigma X$ is said to have an extended GAND with $\mu, \sigma, \lambda, \beta$ and α with the following p.d.f*

$$g^*(y, \mu, \sigma; \alpha, \lambda, \beta) = \frac{1}{\sigma(\alpha+2)} f\left(\frac{y-\mu}{\sigma}\right) \left[2 + \alpha[F(\beta)]^{-1} F\left(\lambda\left(\frac{y-\mu}{\sigma}\right) + \beta\sqrt{1+\lambda^2}\right)\right], \quad (21)$$

in which $y \in R$, $\mu \in R$, $\lambda \in R$, $\beta \in R$, $\sigma > 0$ and $\alpha > -1$. A distribution with p.d.f (21) is denoted as EGAND($\mu, \sigma; \alpha, \lambda, \beta$). Clearly when $\alpha = 0$ and/ or when $\lambda = 0$ and $\beta = 0$, EGAND($\mu, \sigma; \alpha, \lambda, \beta$) reduces to $N(\mu, \sigma^2)$.

Now we have the following results. The proof of these results are similar to the results given in $GAND(\alpha, \lambda, \beta)$ and hence omitted.

RESULT 14. *The cumulative distribution function (c.d.f) $G(x)$ of $EGAND(\mu, \sigma; \alpha, \lambda, \beta)$ with p.d.f (21) is the following, for $y \in R$*

$$G^*(y) = \frac{F\left(\frac{x-\mu}{\sigma}\right)}{\sigma(\alpha+2)}\left[2 + \frac{\alpha}{2}[F(\beta)]^{-1}\right] - \frac{\alpha[F(\beta)]^{-1}}{\sigma(\alpha+2)}\xi^*_\beta(y, \lambda),$$

where $\xi^*_\beta(y, \lambda)$ is as defined in Result 4.

RESULT 15. *The characteristic function of EGAND$(\mu,\sigma;\alpha,\lambda,\beta)$ is given by*

$$\psi_Y(t)=\frac{1}{\sigma(\alpha+2)}e^{it\mu-\frac{t^2\sigma^2}{2}}\left\{2+\alpha[F(\beta)]^{-1}F\left(\delta^{'}it+\frac{\sigma\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)\right\}.$$

RESULT 16. *Mean and variance of EGAND$(\mu,\sigma;\alpha,\lambda,\beta)$ is given by*

$$Mean=\mu+a_1,$$

where

$$a_1=\frac{\sigma\alpha[F(\beta)]^{-1}\delta' f\left(\frac{\sigma\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)}{2+\alpha[F(\beta)]^{-1}F\left(\frac{\sigma\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)}$$

and

$$Variance=\sigma^2-\delta^{'}\sigma^2\beta\left(\frac{\sqrt{1+\lambda^2}}{\sqrt{1+\lambda^2}}\right)a_1-a_1^2.$$

RESULT 17. *The coefficient of skewness of $EGAND(\mu,\sigma;\alpha,\lambda,\beta)$ is*

$$\gamma_1^*=\frac{\left[\delta^{'}\beta^2(\frac{1+\lambda^2}{\sigma^2+\lambda^2})a_1-\delta^{'2}\sigma^2a_1+3\delta^{'}\sigma^2(\frac{\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}})a_1+2a_1^3\right]^2}{\left[\sigma^2-\delta'\sigma^2\beta(\frac{\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}})a_1-a_1^2\right]^3}$$

and the coefficient of kurtosis is

$$\begin{aligned}\gamma_2^* \;=\; & \frac{-\delta'^3\sigma^6\left(\frac{\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)^3a_1-6\sigma^4\delta'\left(\frac{\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)a_1+3\sigma^4}{\left[\sigma^2-\delta'\sigma^2\left(\frac{\beta\sqrt{1+\lambda^2}}{\sigma^2+\lambda^2}\right)a_1-a_1^2\right]^2}\\
&+\frac{\delta'^3\sigma^4\left(\frac{\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)a_1-4\delta'^2\sigma^4a_1^2\left(\frac{\beta^2(1+\lambda^2)}{\sigma^2+\lambda^2}\right)}{\left[\sigma^2-\delta'\sigma^2\left(\frac{\beta\sqrt{1+\lambda^2}}{\sigma^2+\lambda^2}\right)a_1-a_1^2\right]^2}\\
&+\frac{4\delta'^2\sigma^2a_1^2-6\sigma^2a_1^2}{\left[\sigma^2-\delta'\sigma^2\left(\frac{\beta\sqrt{1+\lambda^2}}{\sigma^2+\lambda^2}\right)a_1-a_1^2\right]^2}\\
&-\frac{6\delta'\sigma^2\left(\frac{\beta\sqrt{1+\lambda^2}}{\sqrt{\sigma^2+\lambda^2}}\right)a_1^3+3a_1^4}{\left[\sigma^2-\delta'\sigma^2\left(\frac{\beta\sqrt{1+\lambda^2}}{\sigma^2+\lambda^2}\right)a_1-a_1^2\right]^2}.\end{aligned}$$

RESULT 18. *If* Y *follows EGAND*$(\mu,\sigma;\alpha,\lambda,\beta)$ *then* $X_1=-Y$ *follows EGAND*$(\mu, \sigma;\alpha,-\lambda,\beta)$.

RESULT 19. *The reliability function* $R^*(t)$ *of* Y *is the following, in which* $\xi^*_\beta(t,\lambda)=\int_t^\infty\int_0^{\lambda(\frac{t-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}} f(\frac{y-\mu}{\sigma})f(v)dvdy$ *is as defined in Result 4*

$$R^*(t)=\frac{1}{\alpha+2}\left[1-F\left(\frac{t-\mu}{\sigma}\right)\right]\left\{2+\frac{\alpha[F(\beta)]^{-1}}{2}\right\}+\frac{\alpha[F(\beta)]^{-1}}{\alpha+2}\xi^*_\beta(t,\lambda).$$

RESULT 20. *The failure rate* $r^*(t)$ *of* Y *is given by*

$$r^*(t)=\frac{f\left(\frac{t-\mu}{\sigma}\right)\left[2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{t-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)\right]}{\left[1-F(\frac{t-\mu}{\sigma})\right]\left\{2+\frac{\alpha[F(\beta)]^{-1}}{2}\right\}+\alpha[F(\beta)]^{-1}\xi^*_\beta(t,\lambda)}.$$

RESULT 21. *The mean residual life function of EGAND*$(\mu,\sigma;\alpha,\lambda,\beta)$ *is*

$$\begin{aligned}\mu^*(t) \;=\; & \frac{1}{(\alpha+2)R(t)}\left\{f(\frac{t-\mu}{\sigma})\left[2+\alpha[F(\beta)]^{-1}F(\lambda(\frac{t-\mu}{\sigma})+\beta\sqrt{1+\lambda^2})\right]\right.\\ & +\frac{\alpha\lambda[F(\beta)]^{-1}e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}\sqrt{1+\lambda^2}}\left[1-F\left(\sqrt{1+\lambda^2}\left(\frac{t-\mu}{\sigma}+\frac{\beta\lambda}{\sqrt{1+\lambda^2}}\right)\right)\right]+\\ & \left.\frac{\mu}{\sigma}\left[2\left(1-F(\frac{t-\mu}{\sigma})\right)+M(t;\mu,\sigma,\lambda,\beta)\right]\right\},\end{aligned}$$

where $M(t;\mu,\sigma,\lambda,\beta)=\int_t^\infty\int_{-\infty}^{\lambda u+\beta\sqrt{1+\lambda^2}} f(u)f(v)dvdu$ *which can be evaluated using the software MATHCAD.*

5. MAXIMUM LIKELIHOOD ESTIMATION

The log likelihood function, $\ln L$ of the random sample of size n from a population following EGAND$(\mu,\sigma;\alpha,\lambda,\beta)$ is the following in which $c=-\frac{n}{2}\ln 2\pi$

$$\begin{aligned}\ln L \;=\; & c-n\ln(\alpha+2)-\frac{n}{2}\ln\sigma^2-\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^2}\\ & +\sum_{i=1}^{n}\log\left(2+\alpha[F(\beta)]^{-1}F\left(\lambda\left(\frac{y_i-\mu}{\sigma}\right)+\beta\sqrt{1+\lambda^2}\right)\right).\end{aligned} \tag{22}$$

On differentiating (22) with respect to parameters $\mu, \sigma, \lambda, \beta$ and α and then equating to zero, we obtain the following normal equations

$$\sum_{i=1}^{n}\frac{(y_i-\mu)}{\sigma^2}-\frac{\alpha\lambda}{\sigma}\sum_{i=1}^{n}\frac{[F(\beta)]^{-1}f\left(\lambda\left(\frac{y_i-\mu}{\sigma}\right)+\beta\sqrt{1+\lambda^2}\right)}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}=0, \tag{23}$$

$$-\frac{n}{2\sigma^2}+\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^4}-\frac{\alpha\lambda}{\sigma^2}\sum_{i=1}^{n}\frac{[F(\beta)]^{-1}f\left(\lambda\left(\frac{y_i-\mu}{\sigma}\right)+\beta\sqrt{1+\lambda^2}\right)(y_i-\mu)}{2+\alpha[F(\beta)]^{-1}F\left(\lambda\left(\frac{y-\mu}{\sigma}\right)+\beta\sqrt{1+\lambda^2}\right)}=0, \tag{24}$$

$$\alpha\sum_{i=1}^{n}\frac{[F(\beta)]^{-1}f\left(\frac{\lambda(y_i-\mu)}{\sigma}+\beta\sqrt{1+\lambda^2}\right)\left[(\frac{y_i-\mu}{\sigma})+\frac{\beta\lambda}{\sqrt{1+\lambda^2}}\right]}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}=0, \tag{25}$$

$$\begin{aligned}&\alpha\sum_{i=1}^{n}\frac{[F(\beta)]^{-1}f\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)\sqrt{1+\lambda^2}}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}-\\&\alpha\sum_{i=1}^{n}\frac{F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)f(\beta)[F(\beta)]^{-2}}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}=0,\end{aligned} \tag{26}$$

$$-\frac{n}{(\alpha+2)}+\sum_{i=1}^{n}\frac{[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}=0. \tag{27}$$

Let

$$\Delta(y_i)=\frac{[F(\beta)]^{-1}f\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}{2+\alpha[F(\beta)]^{-1}F\left(\lambda(\frac{y_i-\mu}{\sigma})+\beta\sqrt{1+\lambda^2}\right)}.$$

Then Equations from (23) to (27) become

$$\frac{\alpha\lambda}{\sigma}\sum_{i=1}^{n}\Delta(y_i)=\sum_{i=1}^{n}\frac{(y_i-\mu)}{\sigma^2}, \tag{28}$$

$$\frac{n}{2\sigma^2}=\frac{1}{2}\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{\sigma^4}-\frac{\alpha\lambda}{\sigma^2}\sum_{i=1}^{n}\Delta(y_i)(y_i-\mu), \tag{29}$$

$$\alpha\sum_{i=1}^{n}\Delta(y_i)\left[(\frac{y_i-\mu}{\sigma})+\frac{\beta\lambda}{\sqrt{1+\lambda^2}}\right]=0, \tag{30}$$

$$\alpha \sum_{i=1}^{n} \Delta(y_i)\sqrt{1+\lambda^2} = \frac{[F(\beta)]^{-2} f(\beta) F(\lambda(\frac{y_i-\mu}{\sigma}) + \beta\sqrt{1+\lambda^2})}{2+\alpha[F(\beta)]^{-1} F\left(\lambda(\frac{y_i-\mu}{\sigma}) + \beta\sqrt{1+\lambda^2}\right)}, \tag{31}$$

$$\frac{-n}{\alpha+2} + \sum_{i=1}^{n} \frac{\Delta(y_i) F\left(\lambda(\frac{y_i-\mu}{\sigma}) + \beta\sqrt{1+\lambda^2}\right)}{f\left(\lambda(\frac{y_i-\mu}{\sigma}) + \beta\sqrt{1+\lambda^2}\right)} = 0. \tag{32}$$

On solving the equations (28) to (32) we get the maximum likelihood estimate (MLE) of the parameters of EGAND$(\mu, \sigma; \alpha, \lambda, \beta)$.

6. Applications

In this section we consider three real life data applications of the EGAND. The first data is related to the milk production of 28 cows in which the variable under study is the daily milk production in kilogram and the variable recorded for three times milking cows. This data set is taken from (Bhuyan, 2005,pp. 77). The data are given below.

Data set 1: (34.6 27.7 29.2 25.3 27.6 37.9 32.6 32 30.7 29.6 38.3 32.9 30.8 32.2 32.9 28.1 33.9 28.6 28.1 35.9 34.8 40.3 30.9 34.4 19.8 25.8 37.3 32.4).

The second data set is taken from Australian Institute of Sport data by Cook and Weisberg (1994). The data include 100 females and 102 males with 13 variables such as height, weight, body mass index (BMI) etc. We choose for the variable under study the BMI values for the second 50 females. The data are given below.

Data set 2:(24.47 23.99 26.24 20.04 25.72 25.64 19.87 23.35 22.42 20.42 22.13 25.17 23.72 21.28 20.87 19.00 22.04 20.12 21.35 28.57 26.95 28.13 26.85 25.27 31.93 16.75 19.54 20.42 22.76 20.12 22.35 19.16 20.77 19.37 22.37 17.54 19.06 20.30 20.15 25.36 22.12 21.25 20.53 17.06 18.29 18.37 18.93 17.79 17.05 20.31).

The third data set is from (Deshmukh and Purohit, 2007,pp. 368) which was collected in connection with a study for determining the undesirable side effect of a pill for reducing the blood pressure of the user. The study involves recording the initial blood pressure of 15 women. After they use the pill regularly for six months, their blood pressures are again recorded. Here both before and after blood pressure are studied. The variable under study is before and after blood pressures of 15 women. The data sets are as given below.

Data set 3 (Initial blood pressure of 15 women): (70 80 72 76 76 76 72 78 82 64 74 92 74 68 84).

Data set 4 (Blood pressure of 15 women after taking the pill): (68 72 62 70 58 66 68 52 64 72 74 60 74 72 74).

We have fitted the EGAND$(\mu,\ \sigma;\ \alpha, \lambda, \beta)$ all these four data sets. For illustrating the suitability of the model, we have fitted EGMNSND$(\mu, \sigma^2; \lambda, \alpha)$ to each of the data sets and computed the Kolmogorov Smirnov statistic (KSS) values, Akaike's information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike's information criterion (AICc) in respective cases for comparing the fitted models. All

TABLE 1
Estimated values of the parameters for the model: EGMNSND($\mu, \sigma^2; \lambda, \alpha$) and EGAND($\mu, \sigma; \alpha, \lambda, \beta$) with respective values of KSS, AIC, BIC and AICc in case of data sets 1, 2, 3 and 4.

Data set	Estimates of the parameters	EGMNSND($\mu, \sigma^2; \lambda, \alpha$)	EGAND($\mu, \sigma; \alpha, \lambda, \beta$)
1	$\hat{\mu}$	31.468	31.482
	$\hat{\sigma}$	4.425	4.425
	$\hat{\lambda}$	31.246	4.065
	$\hat{\beta}$	-	8.683
	$\hat{\alpha}$	1.353	4.567
	KSS	0.363	0.083
	AIC	684.588	172.249
	BIC	689.917	178.910
	AICc	686.327	174.976
2	$\hat{\mu}$	20.715	21.812
	$\hat{\sigma}$	3.489	3.313
	$\hat{\lambda}$	26.844	0.264
	$\hat{\beta}$	-	8.452
	$\hat{\alpha}$	0.102	4.468
	KSS	0.464	0.116
	AIC	337.163	271.313
	BIC	344.811	280.873
	AICc	338.052	272.677
3	$\hat{\mu}$	72.527	76.286
	$\hat{\sigma}$	7.656	6.670
	$\hat{\lambda}$	5.925	0.281
	$\hat{\beta}$	-	8.249
	$\hat{\alpha}$	3.186	4.409
	KSS	0.869	0.187
	AIC	912.938	99.385
	BIC	918.354	104.801
	AICc	916.938	106.052
4	$\hat{\mu}$	63.987	67.000
	$\hat{\sigma}$	7.315	6.666
	$\hat{\lambda}$	11.656	0.281
	$\hat{\beta}$	-	8.429
	$\hat{\alpha}$	1.257	4.409
	KSS	0.863	0.173
	AIC	849.824	98.4989
	BIC	855.240	103.915
	AICc	853.824	105.166

these numerical results obtained are presented in Table 1. From Table 1, it is clear that the EGAND($\mu, \sigma; \alpha, \lambda, \beta$) is a more appropriate model to all the data sets considered in this paper compared to the existing model due to Kumar and Anusree (2011) (ie., EGMNSND(μ, σ^2; λ, α)). Thus, the model discussed in this paper provides more flexibility in modeling perspectives due to the presence of extra parameter.

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SUMMARY

The normal and skew normal distributions are not adequate enough for modeling plurimodal data situations. In order to overcome this drawback of normal and skew normal distribution, Kumar and Anusree (2011) proposed a new class of distribution namely "the generalized mixture of standard normal and skew normal distributions (GMNSND)". In this paper we consider an extended version of the GMNSND as a wide class of plurimodal asymmetric normal distribution and investigate some of its important distributional properties. Location-scale extension of the proposed model is also defined and discussed the estimation of its parameters by method of maximum likelihood. Further, four real life data sets are considered for illustrating the usefulness of this model.

Keywords: Asymmetric distributions; Characteristic function; Maximum likelihood estimation; Plurimodality; Reliability measures.