STATISTICA, anno LXXVII, n. 3, 2017

ESTIMATION AND TESTING PROCEDURES FOR THE RELIABILITY FUNCTIONS OF THREE PARAMETER BURR DISTRIBUTION UNDER CENSORINGS

Ajit Chaturvedi
Department of Statistics, University of Delhi, Delhi, India
Shantanu Vyas[1]
Department of Statistics, University of Delhi, Delhi, India

1. INTRODUCTION

The reliability function $R(t)$ is defined as the probability of failure-free operation until time t. Thus, if the random variable (rv) X denotes lifetime of an item or system, then $R(t) = P(X > t)$. Another measure of reliability under stress-strength set-up is the probability $P = P(X > Y)$, which represents the reliability of item or system of random strength X subject to random stress Y. Various author have considered the problem of point estimation of $R(t)$ and P under censoring and complete sample cases for various distributions. For a brief review, one may refer to Pugh (1963), Basu (1964), Bartholomew (1957, 1963), Tong (1974, 1975), Johnson (1975), Kelley *et al.* (1976), Sathe and Shah (1981), Chao (1982), Constantine *et al.* (1986), Awad and Gharraf (1986), Tyagi and Bhattacharya (1989a,b), Chaturvedi and Rani (1997, 1998), Chaturvedi and Surinder (1999), Chaturvedi and Tomer (2002); Chaturvedi *et al.* (2002); Chaturvedi and Tomer (2003),Chaturvedi and Singh (2006, 2008),Chaturvedi and Vyas (2017) and others.

The two parameter Burr XII distribution was firstly introduced by Burr (1942). Its cumulative distribution function (cdf) and probability distribution function (pdf) with parameters c and k are given by

$$F(x;k,c) = 1-(1+x^c)^{-k}; x > 0, c, k > 0 \tag{1}$$

and

$$f(x;k,c) = kcx^{c-1}(1+x^c)^{-(k+1)}; x > 0, c, k > 0. \tag{2}$$

[1] Corresponding Author. E-mail: shanstatsdu@gmail.com

The Burr XII distribution (Burr, 1942) is very useful in reliability analysis as a more flexible alternative to Weibull distribution. In the last few years various authors have developed inferential procedures for the parameters of this distribution. To cite a few one may refer to Wingo (1983, 1993), Wang *et al.* (1996), Zimmer *et al.* (1998), Wang and Shi (2010). Wingo (1983) introduced the maximum likelihood methods for fitting the Burr XII distribution to complete data. He also obtained the ML estimators of Burr XII distribution under type II censoring (Wingo, 1993). Wang *et al.* (1996) obtained the ML estimators for the parameters of Burr XII distribution for censored as well as uncensored data. Wang and Shi (2010) developed the empirical Bayes estimators for the parameters of Burr XII distribution based on records.

In the present article we propose a three parameter Burr distribution by introducing a scale parameter say α in (1) and (2). Three-parameter form of Burr XII distribution is a generalization of the log-logistic distribution (Shao, 2004). Shao (2004) investigated for the asymptotic properties of distribution function of three parameter Burr distribution. He also discussed various properties of ML estimators of the parameters of three parameter Burr distribution. Chaturvedi and Malhotra (2017) have recently proposed point and interval estimators of parameters and reliability functions of three parameter Burr distribution based on records. They also developed testing procedures for different statistical hypotheses.

The present article is devoted to the development of inferential procedures for the parameters as well as reliability measures of three parameter Burr distribution under type II and type I censoring schemes. The structure of article is as follows. In Section 2, we discuss the model of a three parameter Burr distribution. In Section 3 and 4, we provide point estimators of parameters and reliability function under type II and type I censoring, respectively. As far as the estimation procedures are concerned, UMVU and ML estimators are derived. Some confidence intervals are also provided for the estimators of parameters. In Section 5, testing procedures are developed for various statistical hypotheses. Finally in Section 6 we present some numerical findings and in Section 7 we give some remarks and conclusions.

2. THE MODEL

A random variable X is said to have a three parameter Burr distribution if its pdf and cdf are given by

$$f(x;k,c,\alpha)=\frac{kcx^{c-1}}{\alpha}\left(1+\frac{x^c}{\alpha}\right)^{-(k+1)};x>0,k,c,\alpha>0 \tag{3}$$

and

$$F(x;k,c,\alpha)=1-\left(1+\frac{x^c}{\alpha}\right)^{-k};x>0,k,c,\alpha>0. \tag{4}$$

From (4), the reliability function $R(t)$ is given by

$$R(t) = \left(1 + \frac{t^c}{\alpha}\right)^{-k}. \tag{5}$$

From (3) and (5), the hazard rate is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{k c t^{c-1}}{\alpha\left(1 + \frac{t^c}{\alpha}\right)}. \tag{6}$$

Hazard rate is plotted in Figure 1 for different values of parameter k, other two parameters c and α remaining fixed.

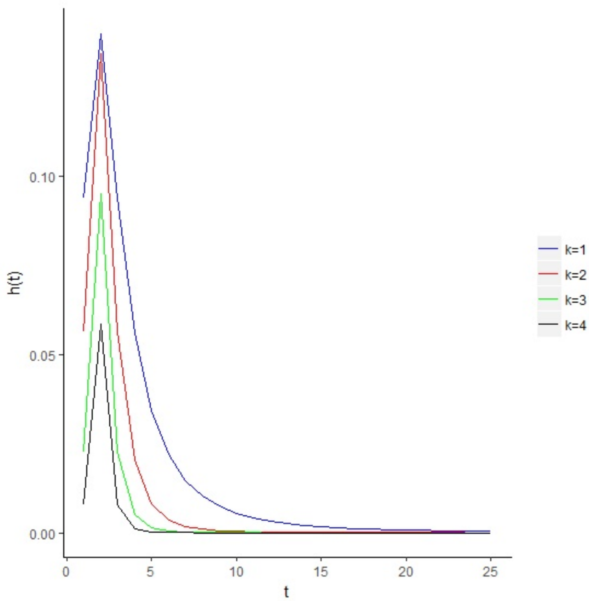

Figure 1 – Hazard rate for $c = 2$ and $\alpha = 3$.

3. Estimation under type-II censoring

Let n items are put an a life test and failure times of first r units are observed. Let $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq ... \leq X_{(r)}, (0 < r \leq n)$ be the lifetimes of first r units. Obviously, $(n - r)$ items survived until $X_{(r)}$. The joint pdf of n order statistics $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq ... \leq X_{(n)}$ is

$$f(x_{(1)}, x_{(2)}, ..., x_{(n)}; k, c, \alpha) = n! \prod_{i=1}^{n} f(x_{(i)}, k, c, \alpha). \tag{7}$$

Rewrite (3) as follows

$$f(x;k,c,\alpha)=\frac{kcx^{c-1}}{\alpha\left(1+\frac{x^c}{\alpha}\right)}\exp\left(-k\log\left(1+\frac{x^c}{\alpha}\right)\right);x>0,k,c,\alpha>0. \tag{8}$$

Using (8) in (7) we have

$$f(x_{(1)},x_{(2)},...,x_{(n)};k,c,\alpha)=n!\left(\frac{kc}{\alpha}\right)^n\prod_{i=1}^{n}\frac{x_{(i)}^{c-1}}{\left(1+\frac{x_{(i)}^c}{\alpha}\right)}\exp\left(-k\log\left(1+\frac{x_{(i)}^c}{\alpha}\right)\right). \tag{9}$$

The joint pdf of $X_{(1)}\leq X_{(2)}\leq X_{(3)}\leq ...\leq X_{(r)}$ is obtained by integrating out $X_{(r+1)}\leq X_{(r+2)}\leq ...\leq X_{(n)}$ from (9), which leads us to

$$f(x_{(1)},x_{(2)},...,x_{(r)};k,c,\alpha)=\frac{n!}{(n-r)!}\left(\frac{kc}{\alpha}\right)^r\prod_{i=1}^{r}\frac{x_{(i)}^{c-1}}{\left(1+\frac{x_{(i)}^c}{\alpha}\right)}\exp(-kS_r), \tag{10}$$

where

$$S_r=\sum_{i=1}^{r}\log\left(1+\frac{x_{(i)}^c}{\alpha}\right)+(n-r)\log\left(1+\frac{x_{(r)}^c}{\alpha}\right).$$

LEMMA 1. *S_r is complete and sufficient for the distribution given at (3). Moreover, the pdf of S_r is given by*

$$f_1(s_r;k)=\frac{k^r}{\Gamma r}s_r^{r-1}\exp(-ks_r). \tag{11}$$

PROOF. It follows from (10) and factorization theorem, that S_r is sufficient for k. It is complete also as its distribution belongs to exponential family. Since $F(x_i;k,c,\alpha)$ is uniform over $(0,1)$, $U_i=F(x_i;k,c,\alpha)$ is also uniform over $(0,1)$ and so $y_i=\log\left(1+\frac{x_{(i)}^c}{\alpha}\right)$ follows exponential distribution with mean life $(1/k)$. Let us consider the transformation $Z_i=(n-i+1)\left\{Y_{(i)}-Y_{(i-1)}\right\}, i=1,2,...,r$. Obviously, $\sum_{i=1}^{r}Z_i=S_r$. Since $Z_i's$ are exponential random variables with mean life $(1/k)$, using the additive property of gamma variates $S_r\sim\gamma(1/k,r)$ and the pdf of S_r is given by (11). □

The following theorem provides the UMVUE of powers of k.

THEOREM 2. *For $p\in(-\infty,\infty)(p\neq 0)$, the UMVUE of k^p is given by*

$$\widetilde{k}_{II}^p=\begin{cases}\left\{\frac{\Gamma(r)}{\Gamma(r-p)}\right\}S_r^{-p} & (p<r)\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF. Theorem follows from Lemma 1 and using the fact that

$$E(S_r^{-p}) = \frac{\Gamma(r-p)}{\Gamma(p)} k^p.$$

□

Next theorem provides the UMVUE of $R(t)$.

THEOREM 3. *The UMVUE of $R(t)$ is given by*

$$\widetilde{R}_{II}(t) = \begin{cases} \left(1 - \frac{\log\left(1+\frac{t^c}{\alpha}\right)}{S_r}\right)^{r-1}, & \log\left(1+\frac{t^c}{\alpha}\right) < S_r \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. From (5) we have

$$\begin{aligned} R(t) &= \left(1+\frac{t^c}{\alpha}\right)^{-k} \\ &= \exp\left\{-klog\left(1+\frac{t^c}{\alpha}\right)\right\} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left\{klog\left(1+\frac{t^c}{\alpha}\right)\right\}^i. \end{aligned}$$

Using Lemma 1 of Chaturvedi and Tomer (2002), we have

$$R_{II}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left\{\log\left(1+\frac{t^c}{\alpha}\right)\right\}^i \tilde{k}_{II}^i.$$

Theorem follows now on using Theorem 1. □

The UMVUE of the pdf $f(x;k,c,\alpha)$ given at (3) is provided in the next corollary.

COROLLARY 4. *The UMVUE of sampled pdf $f(x;k,c,\alpha)$ at a specified point x is given by*

$$\widetilde{f}_{II}(x;k,c,\alpha) = \begin{cases} \frac{(r-1)cx^{c-1}}{\alpha\left(1+\frac{x^c}{\alpha}\right)S_r}\left(1 - \frac{\log\left(1+\frac{x^c}{\alpha}\right)}{S_r}\right)^{r-2}, & \log\left(1+\frac{x^c}{\alpha}\right) < S_r \\ 0, & \textit{otherwise.} \end{cases}$$

PROOF. The expectation of $\int_t^\infty \widetilde{f}_{II}(x;k,c,\alpha)dx$ with respect to S_r is R(t). Thus

$$\widetilde{R}_{II}(t) = \int_t^\infty \widetilde{f}_{II}(x;k,c,\alpha)dx$$

or

$$\widetilde{f}_{II}(t;k,c,\alpha) = -\frac{d\widetilde{R}_{II}(t)}{dt}. \tag{12}$$

The result now follows from Theorem 3 and (12). □

Let X and Y be two independent random variables following the classes of distributions $f_1(x;k_1,c_1,\alpha_1)$ and $f_2(y;k_2,c_2,\alpha_2)$, respectively, where

$$f_1(x;k_1,c_1\alpha_1) = \frac{k_1 c_1 x^{c_1-1}}{\alpha_1\left(1+\frac{x_1^c}{\alpha_1}\right)}\exp\left(-k_1\log\left(1+\frac{x_1^c}{\alpha_1}\right)\right); x>0, k_1, c_1, \alpha_1>0$$

and

$$f_2(y;k_2,c_2,\alpha_2) = \frac{k_2 c_2 y^{c_2-1}}{\alpha_2\left(1+\frac{y_2^c}{\alpha_2}\right)}\exp\left(-k_2\log\left(1+\frac{y_2^c}{\alpha_2}\right)\right); y>0, k_2, c_2, \alpha_2>0.$$

Let n items on X and m items on Y are put on a life test and failure times of first r_1 and r_2 units are observed on X and Y respectively. From Corollary (4), it follows that the UMVUE's of $f_1(x;k_1,c_1\alpha_1)$ and $f_2(y;k_2,c_2,\alpha_2)$ at specified points x and y are given by

$$\widetilde{f}_{1II}(x;k_1,c_1,\alpha_1) = \begin{cases} \frac{(r_1-1)c_1 x^{c_1-1}}{\alpha_1\left(1+\frac{x^{c_1}}{\alpha_1}\right)S_{r_1}}\left(1-\frac{\log\left(1+\frac{x^{c_1}}{\alpha_1}\right)}{S_{r_1}}\right)^{r_1-2}, & \log\left(1+\frac{x^{c_1}}{\alpha_1}\right)<S_{r_1} \\ 0, & \textit{otherwise.} \end{cases}$$

and

$$\widetilde{f}_{2II}(y;k_2,c_2,\alpha_2) = \begin{cases} \frac{(r_2-1)c_2 y^{c_2-1}}{\alpha_2\left(1+\frac{y^{c_2}}{\alpha_2}\right)T_{r_2}}\left(1-\frac{\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{T_{r_2}}\right)^{r_2-2}, & \log\left(1+\frac{y^{c_2}}{\alpha_2}\right)<T_{r_2} \\ 0, & \textit{otherwise.} \end{cases}$$

In the next theorem, we provide the UMVUE of stress strength reliability P.

THEOREM 5. *The UMVUE of P is given by*

$$\tilde{P}_{II} = \begin{cases} (r_2-1)\int_o^U (1-z)^{r_2-2}\left\{1-S_{r_1}^{-1}\log\left\{1+\alpha_1^{-1}\left(\alpha_2\left(\exp(zT_{r_2}-1)\right)\right)^{\frac{c_1}{c_2}}\right\}\right\}^{r_1-1} dz, \\ \quad if \quad \alpha_1\left(\exp(S_{r_1}-1)^{\frac{1}{c_1}}\right) < \alpha_2\left(\exp(T_{r_2}-1)^{\frac{1}{c_2}}\right) \\ (r_2-1)\int_o^1 (1-z)^{r_2-2}\left\{1-S_{r_1}^{-1}\log\left\{1+\alpha_1^{-1}\left(\alpha_2\left(\exp(zT_{r_2}-1)\right)\right)^{\frac{c_1}{c_2}}\right\}\right\}^{r_1-1} dz, \\ \quad if \quad \alpha_1\left(\exp(S_{r_1}-1)^{\frac{1}{c_1}}\right) > \alpha_2\left(\exp(T_{r_2}-1)^{\frac{1}{c_2}}\right) \end{cases}$$

where $U = -T_{r_2}^{-1}\log\left\{1+\alpha_2^{-1}\left(\alpha_1\left(\exp\left(zS_{r_1}-1\right)\right)\right)^{\frac{c_2}{c_1}}\right\}$.

PROOF. From the arguments similar to those adopted in Corollary 4

$$\begin{aligned} \widetilde{P}_{II} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{f}_{1II}(x;k_1,c_1,\alpha_1)\widetilde{f}_{2II}(y;k_2,c_2,\alpha_2)\,dx\,dy \\ &= \int_{y=0}^{\infty} \widetilde{R}_{1II}(y)\left\{\frac{-d}{dy}\widetilde{R}_{2II}(y)\right\} dy \\ &= \int_{y=0}^{M}\left(1-\frac{\log\left(1+\frac{y^{c_1}}{\alpha_1}\right)}{S_{r_1}}\right)^{r_1-1}\frac{(r_2-1)c_2y^{c_2-1}}{\alpha_2\left(1+\frac{y^{c_2}}{\alpha_2}\right)T_{r_2}}\left(1-\frac{\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{T_{r_2}}\right)^{r_2-2} dy, \quad (13) \end{aligned}$$

where $M = min\left\{\left(\alpha_1\left(\exp(S_{r_1}-1)\right)\right)^{\frac{1}{c_1}}, \left(\alpha_2\left(\exp(T_{r_2}-1)\right)\right)^{\frac{1}{c_2}}\right\}$.

When $M = \left(\alpha_1\left(\exp(S_{r_1}-1)\right)\right)^{\frac{1}{c_1}}$, putting $\frac{\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{T_{r_2}} = z$ in (13), we get

$$\tilde{P}_{II} = (r_2-1)\int_o^U (1-z)^{r_2-2}\left\{1-S_{r_1}^{-1}\log\left\{1+\alpha_1^{-1}\left(\alpha_2\left(\exp(zT_{r_2}-1)\right)\right)^{\frac{c_1}{c_2}}\right\}\right\}^{r_1-1} dz \quad (14)$$

When $M = \left(\alpha_2\left(\exp(T_{r_2}-1)\right)\right)^{\frac{1}{c_2}}$ for the same transformation of variables, we get

$$\tilde{P}_{II} = (r_2-1)\int_o^1 (1-z)^{r_2-2}\left\{1-S_{r_1}^{-1}\log\left\{1+\alpha_1^{-1}\left(\alpha_2\left(\exp(zT_{r_2}-1)\right)\right)^{\frac{c_1}{c_2}}\right\}\right\}^{r_1-1} dz \quad (15)$$

Theorem follows now on combining (14) and (15). □

COROLLARY 6. *When X and Y belong to the same families of distributions (X and Y are independent random variables)*

$$\widetilde{P}_{II} = \begin{cases} (r_2-1)\int_o^{\frac{S_{r_1}}{T_{r_2}}} (1-z)^{r_2-2}\left\{1-\frac{T_{r_2}}{S_{r_1}}z\right\}^{r_1-1} dz, & \text{if } S_{r_1} < T_{r_2} \\ (r_2-1)\int_o^{1} (1-z)^{r_2-2}\left\{1-\frac{T_{r_2}}{S_{r_1}}z\right\}^{r_1-1} dz, & \text{if } S_{r_1} > T_{r_2}. \end{cases}$$

3.1. *Maximum likelihood estimation*

3.1.1. *When c and α are known*

In the following theorem, we derive the MLE's of powers of k.

THEOREM 7. *For $p \in (-\infty, \infty)(p \neq 0)$, the MLE of k^p is given by*

$$\hat{k}_{II}^p = \left(\frac{r}{S_r}\right)^p.$$

PROOF. Taking natural logarithm of the both sides of (10), differentiating it with respect to k, equating the differential coefficient equal to zero and solving for k, we get

$$\hat{k}_{II} = \frac{r}{S_r}.$$

The result now follows from the invariance property of MLE's. □

THEOREM 8. *The MLE of reliability function $R(t)$ is given by*

$$\widehat{R}_{II}(t) = \exp\left\{-\frac{r}{S_r}\log\left(1+\frac{t^c}{\alpha}\right)\right\}.$$

PROOF. The result follows from the expression of $R(t)$, Theorem 7 and the invariance property of the MLE. □

COROLLARY 9. *The MLE of the pdf $f(x;k,c,\alpha)$ at a specified point x is*

$$\widehat{f}_{II}(x;k,c,\alpha) = \frac{rcx^{c-1}}{\alpha\left(1+\frac{x^c}{\alpha}\right)S_r}\exp\left\{-\frac{r}{S_r}\log\left(1+\frac{x^c}{\alpha}\right)\right\}.$$

PROOF. The proof is similar to that of Corollary 4. □

THEOREM 10. *The MLE of P is given by*

$$\widehat{P}_{II} = \int_{o}^{\infty} \exp\left\{-\frac{r_1}{S_{r_1}} \log\left(1+\alpha_1^{-1}\left(\alpha_2\left(\exp\left(\frac{zT_{r_2}}{r_2}\right)-1\right)\right)^{\frac{c_1}{c_2}}\right)\right\} e^{-z} dz.$$

PROOF.

$$\begin{aligned}\widehat{P}_{II} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \widetilde{f}_{1II}(x;k_1,c_1,\alpha_1)\widetilde{f}_{2II}(y;k_2,c_2,\alpha_2)dx\,dy \\ &= \int_{y=0}^{\infty} \widetilde{R}_{1II}(y)\left\{\frac{-d}{dy}\widetilde{R}_{2II}(y)\right\}dy \\ &= \int_{y=0}^{\infty} \exp\left\{-\frac{r_1}{S_{r_1}}\log\left(1+\frac{y_1^c}{\alpha_1}\right)\right\} \frac{r_2 c_2 y^{c_2-1}}{\alpha\left(1+\frac{y^{c_2}}{\alpha_2}\right)T_{r_2}} \exp\left\{-\frac{r_2}{T_{r_2}}\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)\right\}dy.\end{aligned} \quad (16)$$

Theorem now follows on substituting $\frac{r_2}{T_{r_2}}\log\left(1+\frac{y^{c_2}}{\alpha_2}\right) = z.$ □

COROLLARY 11. *When X and Y belong to same families of distributions (X and Y are independent random variables)*

$$\widehat{P}_{II} = \frac{r_2 S_{r_1}}{r_2 S_{r_1} + r_1 T_{r_2}}.$$

3.1.2. *When c and α are unknown*

From (10), the log-likelihood is given by

$$\begin{aligned}LogL = \log\left\{\frac{n!}{(n-r)!}\right\} + r\,log k + r\,log c - r\log\alpha + (c-1)\sum_{i=1}^{r}\log(x_{(i)}) \\ -\sum_{i=1}^{r}\log(1+\frac{x^{c_i}}{\alpha}) - kS_r.\end{aligned} \quad (17)$$

Differentiating w.r.t. k, c, and α and equating to 0 we get following three equations

$$\frac{r}{k} - S_r = 0 \quad (18)$$

$$\frac{r}{c} + \sum_{i=1}^{r}\log(x_{(i)}) - (k+1)\sum_{i=1}^{r}\frac{x_{(i)}^c \log(x_{(i)})}{\alpha\left(1+\frac{x_{(i)}^c}{\alpha}\right)} - k(n-r)\frac{x_{(r)}^c \log(x_r)}{\alpha\left(1+\frac{x_{(r)}^c}{\alpha}\right)} = 0 \quad (19)$$

$$-\frac{r}{\alpha}+\frac{(k+1)}{\alpha^2}\sum_{i=1}^{r}\frac{x_{(i)}^{c}}{\left(1+\frac{x_{(i)}^{c}}{\alpha}\right)}+k\frac{x_{(r)}^{c}}{\alpha^2\left(1+\frac{x_{(r)}^{c}}{\alpha}\right)}=0. \tag{20}$$

From (18), we have the MLE of k as follows

$$\hat{k}_{II}=\frac{r}{S_r}. \tag{21}$$

MLE's of c and α (say $\hat{c}_{II}$ and $\hat{\alpha}_{II}$) can be obtained by solving (19) and (20) simultaneously and using (21).

THEOREM 12. *The MLE of $R(t)$ is given by*

$$\widehat{R}_{II}(t)=\exp\left\{-\frac{r}{S_r}\log\left(1+\frac{t^{\hat{c}}}{\hat{\alpha}}\right)\right\}.$$

PROOF. Theorem follows from expression of $R(t)$ on using MLE of k,c and α and invariance property of MLE's. □

COROLLARY 13. *The MLE of the pdf $f(x;k,c,\alpha)$ at a specified point x is*

$$\widehat{f}_{II}(x;k,c,\alpha)=\frac{r\hat{c}x^{\hat{c}-1}}{\hat{\alpha}\left(1+\frac{x^{\hat{c}}}{\hat{\alpha}}\right)S_r}\exp\left\{-\frac{r}{S_r}\log\left(1+\frac{x^{\hat{c}}}{\hat{\alpha}}\right)\right\}.$$

THEOREM 14. *The MLE of P is given by*

$$\widehat{P}_{II}=\int_{o}^{\infty}\exp\left\{-\frac{r_1}{S_{r_1}}\log\left(1+\hat{\alpha}_1^{-1}\left(\hat{\alpha}_2\left(\exp\left(\frac{zT_{r_2}}{r_2}\right)-1\right)\right)^{\frac{\hat{c}_1}{\hat{c}_2}}\right)\right\}e^{-z}dz.$$

PROOF. Theorem follows from Theorem (10) and invariance property of MLE's.□

3.2. *Interval estimation*

The Fisher's information matrix is given by

$$I(\theta)=-E\begin{bmatrix}\frac{\delta^2\log L}{\delta\alpha^2} & \frac{\delta^2\log L}{\delta\alpha\delta c} & \frac{\delta^2\log L}{\delta\alpha\delta k}\\ \frac{\delta^2\log L}{\delta c\delta\alpha} & \frac{\delta^2\log L}{\delta c^2} & \frac{\delta^2\log L}{\delta c\delta k}\\ \frac{\delta^2\log L}{\delta k\delta\alpha} & \frac{\delta^2\log L}{\delta k\delta c} & \frac{\delta^2\log L}{\delta k^2}\end{bmatrix}.$$

where $\theta=[k,c,\alpha]^T$. Since it is difficult to obtain the above expectations, we use the observed information matrix by dropping the expectation sign. The asymptotic variance-covariance matrix is the inverse of $I(\hat{\theta})$ where $\hat{\theta}=(\hat{k},\hat{c},\hat{\alpha})^T$. Let us denote by $\hat{\sigma}^2(\hat{k}),\hat{\sigma}^2(\hat{c})$

and $\hat{\sigma}^2(\hat{\alpha})$ be the estimated variances of $,\hat{c}$ and $\hat{\alpha}$ respectively. Using the asymptotic normality of MLE's the $100(1-\epsilon)\%$ confidence intervals for k, c and α are given by $(\hat{k}-Z_{\epsilon/2}\hat{\sigma}(\hat{k}), \hat{k}+Z_{\epsilon/2}\hat{\sigma}(\hat{k})),(\hat{c}-Z_{\epsilon/2}\hat{\sigma}(\hat{c}), \hat{c}+Z_{\epsilon/2}\hat{\sigma}(\hat{c}))$ and $(\hat{\alpha}-Z_{\epsilon/2}\hat{\sigma}(\hat{\alpha}), \hat{\alpha}+Z_{\epsilon/2}\hat{\sigma}(\hat{\alpha}))$ respectively where $Z_{\epsilon/2}$ is the upper $100(1-\epsilon)$ percentile point of standard normal distribution.

4. Estimation under type-I censoring

Let $0 \le X_{(1)} \le X_{(2)} \le .... \le X_{(n)}$ be the failure times of n items under test and lifetime of each item has the pdf given by (2). Test begins at time 0 and operates till $X_{(1)} = x_{(1)}$, the first failure time. Failed item is replaced by a new one and system operates further till $X_{(2)} = x_{(2)}$, and so on. The experiment is terminated at time t_o.

THEOREM 15. *If $N(t_o)$ be the number of items that failed before time t_o, then*

$$P[N(t_o) = r/t_0] = \frac{\exp\left\{-nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}\left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}^r}{r!}. \tag{22}$$

PROOF. For $Y'_{(i)}s, i = 1,2,...,n$ defined in Section 2, lest us make the transformation $W_1 = Y_{(1)}, W_2 = Y_{(2)} - Y_{(1)}, ..., W_n = Y_{(n)} - Y_{(n-1)}$.
We have shown that $W'_i s$ are i.i.d. rv's having exponential distribution with mean $(1/k)$. By the definition of $N(t_o)$

$$\begin{aligned}
&P[N(t_o) = r/t_o] \\
&\quad = P[X_{(r)} \le t_o] - P[X_{(r+1)} \le t_o] \\
&\quad = P\left[Y_{(r)} \le \log\left(1+\frac{t_0^c}{\alpha}\right)\right] - P\left[Y_{(r+1)} \le \log\left(1+\frac{t_0^c}{\alpha}\right)\right] \\
&\quad = P\left[nk\sum_{i=1}^{r+1} W_i \ge nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right] - P\left[nk\sum_{i=1}^{r} W_i \ge nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right].
\end{aligned}$$

Using a result of Patel, Kapadia and Owen (1976, pp. 244), we have

$$\begin{aligned}
&P[N(t_o) = r/t_o] \\
&= \frac{1}{\Gamma(r+1)}\int_{nk\log\left(1+\frac{t_0^c}{\alpha}\right)}^{\infty} u^r e^{-u}\,du - \frac{1}{\Gamma(r)}\int_{nk\log\left(1+\frac{t_0^c}{\alpha}\right)}^{\infty} u^{r-1}e^{-u}\,du \\
&= \exp\left\{-nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}\left[\sum_{j=0}^{r}\frac{\left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}^j}{j!} - \sum_{j=0}^{r-1}\frac{\left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}^j}{j!}\right].
\end{aligned}$$

and the theorem follows. □

THEOREM 16. *For $p \in (0, \infty)$, the UMVUE of k^p is given by*

$$\widetilde{k}_I^p = \begin{cases} \frac{r!}{(r-p)!}\left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}^{-p}, & \text{if } p \le r \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. It follows from (22) and factorization theorem that r is complete and sufficient for α. The theorem now follows from the result that

$$E\left[r(r-1)...(r-p+1)\right] = \left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}^p.$$

□

THEOREM 17. *The UMVUE of R(t) is given by*

$$\widetilde{R}_I(t) = \begin{cases} \left[1 - \frac{\log\left(1+\frac{t^c}{\alpha}\right)}{n\log\left(1+\frac{t_0^c}{\alpha}\right)}\right]^r, & \text{if } \log\left(1+\frac{t^c}{\alpha}\right) \le n\log\left(1+\frac{t_o^c}{\alpha}\right) \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. Using Theorem (16)

$$\begin{aligned} \widetilde{R}_I(t) &= \exp\left\{-k\log\left(1+\frac{t^c}{\alpha}\right)\right\} \\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left\{\log\left(1+\frac{t^c}{\alpha}\right)\right\}^i \tilde{k}_I^i \\ &= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\left\{\frac{\log\left(1+\frac{t^c}{\alpha}\right)}{n\log\left(1+\frac{t_o^c}{\alpha}\right)}\right\}^i \end{aligned}$$

and the theorem follows. □

COROLLARY 18. *The UMVUE of the sampled pdf at a specified point x is*

$$\widetilde{f}_I(x;k,c,\alpha) = \begin{cases} \frac{rcx^{c-1}}{n\alpha\log\left(1+\frac{t_0^c}{\alpha}\right)\left(1+\frac{x^c}{\alpha}\right)}\left[1 - \frac{\log\left(1+\frac{x^c}{\alpha}\right)}{n\log\left(1+\frac{t_0^c}{\alpha}\right)}\right]^{r-1} \\ \qquad\qquad \text{if } \log\left(1+\frac{x^c}{\alpha}\right) \le n\log\left(1+\frac{t_o^c}{\alpha}\right) \\ 0, \qquad \text{otherwise.} \end{cases}$$

PROOF. Theorem follows from Theorem (17) and the techniques adopted in the proof of Corollary (4). □

As in Section 3, let X and Y be two independent rv's following the classes of distributions $f_1(x;k_1,c_1,\alpha_1)$ and $f_2(y;k_2,c_2,\alpha_2)$, respectively. Let n items on X and m items on Y are put on life tests and r_1 and r_2 be the number of failures before time t_o and t_{oo}, respectively. It follows from Corollary (18) that the UMVUE's of $f_1(x;k_1,c_1,\alpha_1)$ and $f_2(y;k_2,c_2,\alpha_2)$ at a specified point x and y respectively, are given by

$$\widetilde{f}_I(x;k_1,c_1,\alpha_1) = \frac{r_1 c_1 x^{c_1-1}}{n\alpha_1 \log\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)\left(1+\frac{x^{c_1}}{\alpha}\right)} \left[1-\frac{\log\left(1+\frac{x^{c_1}}{\alpha_1}\right)}{n\log\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)}\right]^{r_1-1}$$

and

$$\widetilde{f}_I(y;k_2,c_2,\alpha_2) = \frac{r_2 c_2 y^{c_2-1}}{m\alpha_2 \log\left(1+\frac{t_o^{c_2}}{\alpha_2}\right)\left(1+\frac{y^{c_2}}{\alpha}\right)} \left[1-\frac{\log\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{m\log\left(1+\frac{t_o^{c_2}}{\alpha_2}\right)}\right]^{r_2-1}.$$

In the next theorem, we provide the UMVUE of P.

THEOREM 19. *The UMVUE of P is given by*

$$\tilde{P}_I = \begin{cases} r_2\int_o^U (1-z)^{r_2-1}\left\{1-\frac{\log\left\{1+\alpha_1^{-1}\left\{\alpha_2\left(\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)^{mz}-1\right)\right\}^{\frac{c_1}{c_2}}\right\}}{n\log\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)}\right\}^{r_1} dz, \\ \quad \text{if } \left\{\alpha_1\left(\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)^n-1\right)\right\}^{\frac{1}{c_1}} < \left\{\alpha_2\left(\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)^m-1\right)\right\}^{\frac{1}{c_2}} \\ r_2\int_o^1 (1-z)^{r_2-1}\left\{1-\frac{\log\left\{1+\alpha_1^{-1}\left\{\alpha_2\left(\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)^{mz}-1\right)\right\}^{\frac{c_1}{c_2}}\right\}}{n\log\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)}\right\}^{r_1} dz, \\ \quad \text{if } \left\{\alpha_1\left(\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)^n-1\right)\right\}^{\frac{1}{c_1}} > \left\{\alpha_2\left(\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)^m-1\right)\right\}^{\frac{1}{c_2}} \end{cases}$$

where

$$U = \frac{\log\left\{1+\alpha_2^{-1}\left\{\alpha_1\left(\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)^n-1\right)\right\}^{\frac{c_2}{c_1}}\right\}}{m\log\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)}.$$

COROLLARY 20. *When X and Y belong to the same families of distributions (X and Y are independent random variables)*

$$\widetilde{P}_I = \begin{cases} r_2 \int_o^{\frac{n}{m}} (1-z)^{r_2-1} \left\{1-\frac{m}{n}z\right\}^{r_1} dz, & \text{if } m > n \\ r_2 \int_o^1 (1-z)^{r_2-1} \left\{1-\frac{m}{n}z\right\}^{r_1} dz, & \text{if } m < n. \end{cases}$$

4.1. *Maximum likelihood estimation*

4.1.1. *When c and α are known*

In the following theorem, we derive the MLE's of powers of k.

THEOREM 21. *For $p \in (-\infty, \infty)(p \neq 0)$, the MLE of k^p is given by*

$$\hat{k}_I^p = \left(\frac{r}{n \log\left(1+\frac{t_o^c}{\alpha}\right)} \right)^p.$$

PROOF. Taking natural logarithm of the both sides of (22), differentiating it with respect to k, equating the differential coefficient equal to zero and solving for k, we get

$$\hat{k}_I = \left(\frac{r}{n \log\left(1+\frac{t_o^c}{\alpha}\right)} \right).$$

The result now follows from the invariance property of MLE's. □

THEOREM 22. *The MLE of reliability function $R(t)$ is given by*

$$\widehat{R}_I(t) = \exp\left\{ -\frac{r \log\left(1+\frac{t^c}{\alpha}\right)}{n \log\left(1+\frac{t_o^c}{\alpha}\right)} \right\}.$$

PROOF. The result follows from the expression of $R(t)$, Theorem 21 and the invariance property of the MLE. □

COROLLARY 23. *The MLE of the pdf $f(x;k,c,\alpha)$ at a specified point x is*

$$\widehat{f}_I(x;k,c,\alpha) = \frac{r c x^{c-1}}{\alpha\left(1+\frac{x^c}{\alpha}\right) n \log\left(1+\frac{t_o^c}{\alpha}\right)} \exp\left\{ -\frac{r \log\left(1+\frac{x^c}{\alpha}\right)}{n \log\left(1+\frac{t_o^c}{\alpha}\right)} \right\}.$$

PROOF. The proof is similar to that of Corollary 4. □

THEOREM 24. *The MLE of P is given by*

$$\widehat{P}_I = \int_o^\infty \exp\left[-\frac{r_1 \ln\left\{1+\alpha_1^{-1}\left\{\alpha_2\left(\exp\left(\frac{m\log\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)z}{r_2}\right)-1\right)^{\frac{c_1}{c_2}}\right\}\right\}}{n\log\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)} \right] e^{-z} dz.$$

PROOF.

$$\begin{aligned}
\widehat{P}_I &= \int_{y=0}^\infty \int_{x=y}^\infty \widehat{f}_{1I}(x;k_1,c_1,\alpha_1)\widehat{f}_{2I}(y;k_2,c_2,\alpha_2)dx\,dy \\
&= \int_{y=0}^\infty \widehat{R}_{1I}(y)\left\{\frac{-d}{dy}\widehat{R}_{2I}(y)\right\} dy \\
&= \int_0^\infty \exp\left\{-\frac{r_1\ln\left(1+\frac{y_1^c}{\alpha_1}\right)}{n\ln\left(1+\frac{t_o^{c_1}}{\alpha_1}\right)}\right\} \frac{r_2 c_2 y^{c_2-1}}{\alpha_2\left(1+\frac{y^{c_2}}{\alpha_2}\right) m\ln\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)} \\
&\qquad \exp\left\{-\frac{r_2\ln\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{m\ln\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)}\right\} dy. \qquad (23)
\end{aligned}$$

Theorem now follows on substituting $\frac{r_2\ln\left(1+\frac{y^{c_2}}{\alpha_2}\right)}{m\ln\left(1+\frac{t_{oo}^{c_2}}{\alpha_2}\right)} = z.$ □

COROLLARY 25. *When X and Y belong to same families of distributions (X and Y are independent random variables)*

$$\widehat{P}_I = \frac{n r_2}{m r_1 + n r_2}.$$

4.1.2. When c and α are unknown

From (22), the log-likelihood is given by

$$\log P = \left\{-nk\log\left(1+\frac{t_o^c}{\alpha}\right)\right\} + r\log\left\{nk\log\left(1+\frac{t_o^c}{\alpha}\right)\right\} - \log r!. \qquad (24)$$

Differentiating (24) with respect to k, c and α and equating to 0 we get the following equations

$$-n\log\left(1+\frac{t_o^c}{\alpha}\right)+\frac{r}{k}=0 \tag{25}$$

$$\frac{nkt_o^c}{\alpha^2\left(1+\frac{t_o^c}{\alpha}\right)}-\frac{rnkt_o^c}{\alpha^2\left(1+\frac{t_o^c}{\alpha}\right)nk\log\left(1+\frac{t_o^c}{\alpha}\right)}=0 \tag{26}$$

$$\frac{-nkt_o^c\log t_o}{\alpha\left(1+\frac{t_o^c}{\alpha}\right)}+\frac{rt_o^c log t_o}{\alpha\left(1+\frac{t_o^c}{\alpha}\right)\log\left(1+\frac{t_o^c}{\alpha}\right)}=0. \tag{27}$$

Solving the above equations simultaneously using the numerical techniques we get the MLE's of k, c and α say $\hat{k}$, $\hat{c}$ and $\hat{\alpha}$.

THEOREM 26. *The MLE of reliability function $R(t)$ is given by*

$$\widehat{R}_I(t)=\exp\left\{-\frac{r\log\left(1+\frac{t^{\hat{c}}}{\hat{\alpha}}\right)}{n\log\left(1+\frac{t_o^{\hat{c}}}{\hat{\alpha}}\right)}\right\}.$$

COROLLARY 27. *The MLE of the pdf $f(x;k,c,\alpha)$ at a specified point x is*

$$\widehat{f}_I(x;\hat{k},\hat{c},\hat{\alpha})=\frac{r\hat{c}x^{\hat{c}-1}}{\hat{\alpha}\left(1+\frac{x^{\hat{c}}}{\hat{\alpha}}\right)n\log\left(1+\frac{t_o^{\hat{c}}}{\hat{\alpha}}\right)}\exp\left\{-\frac{r\log\left(1+\frac{x^{\hat{c}}}{\hat{\alpha}}\right)}{n\log\left(1+\frac{t_o^{\hat{c}}}{\hat{\alpha}}\right)}\right\}.$$

THEOREM 28. *The MLE of P is given by*

$$\widehat{P}_I=\int_o^\infty \exp\left[-\frac{r_1\ln\left\{1+\hat{\alpha}_1^{-1}\left\{\hat{\alpha}_2\left(\exp\left(\frac{m\log\left(1+\frac{t_{oo}^{\hat{c}_2}}{\hat{\alpha}_2}\right)z}{r_2}\right)-1\right)^{\frac{\hat{c}_1}{\hat{c}_2}}\right\}\right\}}{n\log\left(1+\frac{t_o^{\hat{c}_1}}{\hat{\alpha}_1}\right)}\right]e^{-z}dz.$$

4.2. Interval estimation

The Fisher's information matrix is given by

$$I(\theta)=-E\begin{bmatrix}\frac{\delta^2\log L}{\delta\alpha^2} & \frac{\delta^2\log L}{\delta\alpha\delta c} & \frac{\delta^2\log L}{\delta\alpha\delta k}\\ \frac{\delta^2\log L}{\delta c\delta\alpha} & \frac{\delta^2\log L}{\delta c^2} & \frac{\delta^2\log L}{\delta c\delta k}\\ \frac{\delta^2\log L}{\delta k\delta\alpha} & \frac{\delta^2\log L}{\delta k\delta c} & \frac{\delta^2\log L}{\delta k^2}\end{bmatrix},$$

where $\theta = [k, c, \alpha]^T$. Since it is difficult to obtain the above expectations, we use the observed information matrix by dropping the expectation sign. The asymptotic variance-covariance matrix is the inverse of $I(\hat{\theta})$ where $\hat{\theta} = (\hat{k}, \hat{c}, \hat{\alpha})^T$. Let us denote by $\hat{\sigma^2}(\hat{k}), \hat{\sigma^2}(\hat{c})$ and $\hat{\sigma^2}(\hat{\alpha})$ be the estimated variances of $, \hat{c}$ and $\hat{\alpha}$ respectively. Using the asymptotic normality of MLE's the $100(1-\epsilon)\%$ confidence intervals for k, c and α are given by $(\hat{k} - Z_{\epsilon/2}\hat{\sigma}(\hat{k}), \hat{k} + Z_{\epsilon/2}\hat{\sigma}(\hat{k}))$,$(\hat{c} - Z_{\epsilon/2}\hat{\sigma}(\hat{c}), \hat{c} + Z_{\epsilon/2}\hat{\sigma}(\hat{c}))$ and $(\hat{\alpha} - Z_{\epsilon/2}\hat{\sigma}(\hat{\alpha}), \hat{\alpha} + Z_{\epsilon/2}\hat{\sigma}(\hat{\alpha}))$ respectively where $Z_{\epsilon/2}$ is the upper $100(1-\epsilon)$ percentile point of standard normal distribution.

5. Testing procedures for different statistical hypotheses

In this section, we develop the test procedure for testing statistical hypotheses for the parameter k and P. Suppose we want to test the hypothesis $H_0 : k = k_0$ against the alternative $H_1 : k \neq k_0$ under type II censoring. From (10)

$$\mathrm{Sup}_{\theta_0} L(k|\underline{x}) = n(n-1)....(n-r+1)\left\{\frac{ck_0}{\alpha}\right\}^r \prod_{i=1}^{r} \frac{x_i^{c-1}}{1+\frac{x_i^c}{\alpha}} \exp(-k_0 S_r)$$

and

$$\mathrm{Sup}_{\theta} L(k|\underline{x}) = n(n-1)....(n-r+1)\left(\frac{r}{S_r}\right)^r \exp(-r).$$

The likelihood ratio is given by

$$\lambda(\underline{x}) = \left(\frac{S_r \alpha_0}{r}\right)^r \exp(-\alpha_0 S_r + r) \tag{28}$$

The first term on the right side of (28) is monotonically increasing in S_r, whereas, the second one is monotonically decreasing. Using the fact that $\frac{2S_r}{k_0}$ follows χ^2_{2r} and denoting by β the probability of type I error, the critical region is given by

$$\{0 < S_r < \lambda_0\} \bigcup \left\{\lambda_0' < S_r < \infty\right\}$$

where

$$\lambda_0 = \frac{k_0}{2}\chi^2_{2r}\left(1 - \frac{\beta}{2}\right)$$

and

$$\lambda_0' = \frac{k_0}{2}\chi^2_{2r}\left(\frac{\beta}{2}\right).$$

For type I censoring, a similar procedure can be used to find the critical region. Denoting by r, a poisson rv with parameter $nk\log\left(1+\frac{t_0^c}{\alpha}\right)$. The critical region is given by $\left\{r < \lambda_1 \text{ or } r > \lambda_1'\right\}$, r follows a Poisson distribution with parameter $\left\{nk\log\left(1+\frac{t_0^c}{\alpha}\right)\right\}$.

Now suppose we want to test the null hypothesis $H_0 : k \leq k_0$ against $H_1 : k > k_0$ under type II censoring. It is easy to see that the family of sampled pdf has monotonic likelihood in S_r. Thus, the uniformly most powerful critical region is given by

$$S_r \leq \acute{\lambda}_0,$$

where

$$\acute{\lambda}_0 = \frac{k_0}{2}\chi^2_{2r}(1-\beta).$$

Under type I censoring, the critical region is $r \geq \acute{\lambda}_1$, where $P(r \geq \acute{\lambda}_1) = \beta$.

Now suppose we want to test the null hypothesis $H_0 : P = P_0$ against $H_1 : P \neq P_0$ under type II censoring. It is easy to see that

$$P = \frac{k_2}{k_1 + k_2},$$

for $c_1 = c_2$ and $\alpha_1 = \alpha_2$. For $\delta = \frac{P_0}{1-P_0}, H_0$ is equivalent to $H_0 : k_1 = \delta k_2$, so that $H_1 : k_1 \neq \delta k_2$. For a generic[2] constant η

$$L(k_1, k_2 \big| \underline{x}, \underline{y}) = \eta k_1^{r_1} k_2^{r_2} \exp\left\{-(k_1 S_{r_1} + k_2 T_{r_2})\right\}.$$

Under H_0

$$\hat{k}_{1II} = \frac{\delta(r_1 + r_2)}{\delta S_{r_1} + T_{r_2}}, \qquad \hat{k}_{2II} = \frac{(r_1 + r_2)}{\delta S_{r_1} + T_{r_2}}.$$

Thus

$$\text{Sup}_{H_0} L(k_1, k_2 \big| \underline{x}, \underline{y}) = \frac{\eta}{\left(S_{r_1} + T_{r_2}/\delta\right)^{r_1+r_2}} \exp(-(r_1 + r_2)).$$

Over the entire parametric space $\Theta = \{(k_1, k_2)/k_1, k_2 > 0\}$

$$\text{Sup}_{\Theta} L(k_1, k_2 \big| \underline{x}, \underline{y}) = \frac{\eta}{S_{r_1}^{r_1} T_{r_2}^{r_2}} \exp(-(r_1 + r_2)).$$

On using the fact that $\frac{S_{r_1}}{T_{r_2}}$ follows $\frac{r_1 k_2}{r_2 k_1} F_{2r_1, 2r_2}(.)$ where $F_{a,b}(.)$, is the F-statistic with (a, b) degrees of freedom, the critical region is given by

---

[2] By generic constant we represent here a group of normalizing constants which includes all constants arising at each step. This help us to get rid of writing different normalizing constants at each step.

$$\left\{\left(\frac{S_{r_1}}{T_{r_2}}<\lambda_2\right)\cup\left(\frac{S_{r_1}}{T_{r_2}}>\lambda_2'\right)\right\},$$

where

$$\lambda_2=\frac{r_1}{\delta r_2}F_{2r_1,2r_2}\left(1-\frac{\beta}{2}\right)$$

and

$$\lambda_2'=\frac{r_1}{\delta r_2}F_{2r_1,2r_2}\left(\frac{\beta}{2}\right).$$

6. NUMERICAL FINDINGS

In this section we obtained estimates of parameters and reliability functions for simulated as well as real data. Result obtained for two different censoring schemes has been verified here. Estimates of reliability functions has been obtained and compared for different sample sizes and has been represented in tabulated form.

Firstly we verified results obtained under type-II censoring. We generated 10,000 samples of size 50 each from the distribution given in (3) with $\alpha=3; c=2; k=1$ using inverse transform sampling and obtained estimates of k and $R(t)$ by setting $r=35$, $p=1$, $t=0.75$(in hours). These estimates are provided in Table 1 (see Appendix). The sampled pdf $f(x)$, MLE and UMVUE of $f(x)$ are plotted in Figure 2 between the interval $[0,1]$. Next we generated 1,000 samples of sizes 30 and 40 respectively each from the distribution given at (3) with $\alpha_1=\alpha_2=3; n=30; m=40; c_1=c_2=2.5; k_1=k_2=1$. Estimates of stress-strength reliability $P=P(X>Y)$ are obtained by setting $r=19$ and $s=30$ respectively. Estimates of P are: $[\hat{P},\tilde{P}]=[0.494745, 0.497046]$. Value of P based on the samples is found to be 0.5. Similar calculations have been carried out under type I censoring scheme. Estimates of k and $R(t)$ have been obtained by setting $t_0=1.5$, $p=1$, $t=0.75$ (in hours). These estimates are provided in Table 2. The sampled pdf $f(x)$, MLE and UMVUE of $f(x)$ are plotted in Figure 3 between the interval $[0,1]$.

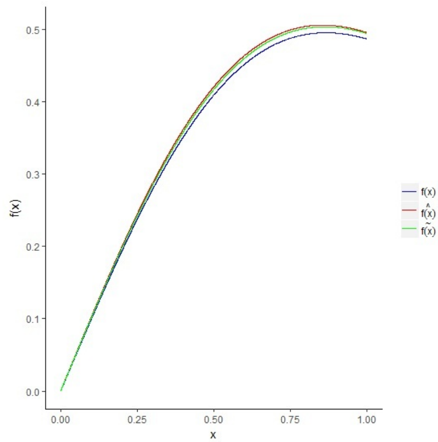

Figure 2 – MLE and UMVUE of sampled pdf.

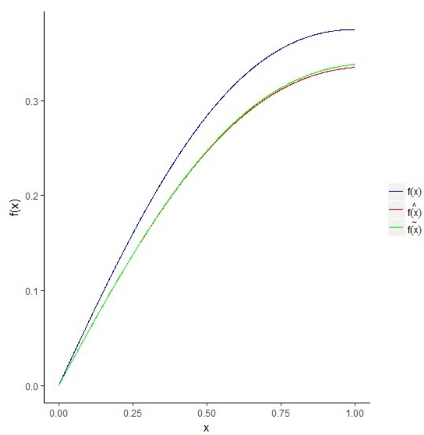

Figure 3 – MLE and UMVUE of sampled pdf.

Estimates of stress-strength reliability $P = P(X > Y)$ are obtained by setting $t_0 = 3.5, t_{00} = 4$ respectively. Estimates of P are: $[\hat{P}, \tilde{P}] = [0.50112, 0.499922]$. Value of P based on the samples is found to be 0.5.

For checking the authenticity of tests derived in Section 5, we generated a sample of size 50 with $\alpha = 3; c = 2; k = 1$ using inverse transform sampling. The null hypothesis to be tested is $H_0 : k = 1$ against $H_1 : k \neq 1$. Setting $r = 35$, the value of test statistic i.e. s_r comes out to be 26.9556. Using the chi-square table, at 5 percent level of significance, the values of λ_0 and $\lambda_0^{'}$ come out to be 24.3788 and 47.5116 respectively. Hence we may accept the null hypothesis.

Let us suppose the null hypothesis to be tested is $H_0 : k \leq 1$ against $H_1 : k > 1$. Setting $r = 35$, the value of test statistic i.e. s_r comes out to be 26.9556. Using the chi-square table, at 5 percent level of significance, the value of $\lambda_0^{'}$ comes out to be 25.8696. Hence we may accept the null hypothesis.

Now let us suppose we want to test the null hypothesis $H_0 : P = P_0$ against $H_1 : P \neq P_0$. We generated two samples of size 30 and 40 respectively with $\alpha_1 = \alpha_2 = 3; n = 30; m = 40; c_1 = c_2 = 2.5; k_1 = k_2 = 1$. Setting $r = 19$; $s = 30$, the value of test statistic S_r/T_s is found to be 0.8591. Using the F-table, the values of λ_2 and $\lambda_2^{'}$ are calculated as 0.3472 and 1.1119 respectively at 5 percent level of significance. Hence we may accept the null hypothesis. The similar calculations may be done for testing the above hypotheses under type-I censoring scheme.

We constructed Table 5 to compare the performance of two estimators of reliability function $R(t)$ for different (n, r) and t under type-II censoring. The similar comparison has been done under type-I censoring scheme in Table 6. We constructed Table 7 and Table 8, which contain estimates of k and $R(t)$ obtained under type II and type I censoring respectively for decreasing number of observed failures and sample size 100.

Next we considered an example of real data. This data consists the maximum flood levels (in millions of cubic feet per second) for the Susquehanna River of Harrisburg over 20 four-year periods (Dumonceaux and Antle, 1973). This data can also be found in (Cheng and Amin, 1981, 1983): (0.654 0.613 0.315 0.449 0.297 0.402 0.379 0.423 0.379 0.3235 0.269 0.740 0.418 0.412 0.494 0.416 0.338 0.392 0.484 0.265.)

(Shao, 2004) has showed in his paper that the three-parameter Burr distribution fits this data better in comparison to Weibull and Pareto distributions. He provided the MLE of k, c and α as follows $[\hat{k}, \hat{c}, \hat{\alpha}] = [0.142, 6.434, 1350.844]$.

Under type-II censoring scheme, we obtained estimates of k and $R(t)$, MSE and variance by setting $r = 15$; $t = 0.5$ (in hours). These estimates are given in Table 3. Under type-I censoring scheme,we obtained estimates of k and $R(t)$, MSE and variance by setting $t_o = 5$; $t = 0.5$ (in hours). These estimates are given in Table 4.

7. Remarks and conclusion

Seeing Figure 1 we observed that hazard rate is a decreasing function of t for every value of k. From Figure 2 and Figure 3 it is evident that the estimators of sampled pdf obtained under type II censoring fit better to the actual model than the estimators of sampled pdf obtained under type I censoring. From Table 5 and Table 6 it is clear that estimators obtained under type-II censoring perform better than the estimators obtained under type-I censoring. Seeing Table 7 and Table 8, we observed that MSE and variance increase rapidly as we decrease the number of observed failures in the case of type I censoring as compared to type II censoring. Hence we conclude that estimated accuracy affects much in the case of type I censoring when the number of observed failures decreases as compared to type II censoring.

ACKNOWLEDGEMENTS

We are thankful to the referee for valuable comments and suggestions, which led to considerable improvement in the original manuscript. We are also thankful to Department of Science and Technology, Government of India for financial support.

Appendix

Tables

TABLE 1
Estimates of k and R(t), variance and MSE under type II censoring.

$\hat{k}$	1.031713
$\tilde{k}$	1.002236
$R(t)$	0.841788
$\hat{R(t)}$	0.842105
$\tilde{R(t)}$	0.83793
$MSE[\hat{R(t)}]$	0.000682
$V[\tilde{R(t)}]$	0.00064

TABLE 2
Estimates of k and R(t), variance and MSE under type I censoring.

$\hat{k}$	1.306162
$\tilde{k}$	0.494053
$R(t)$	0.842105
$\hat{R(t)}$	0.799042
$\tilde{R(t)}$	0.79854
$MSE[\hat{R(t)}]$	0.002011
$V[\tilde{R(t)}]$	0.002056

TABLE 3
Estimates of k and R(t), variance and MSE under type II censoring.

$\hat{k}$	0.100231
$\tilde{k}$	0.093549
$R(t)$	0.999466
$\hat{R}(t)$	0.999623
$\tilde{R}(t)$	0.999648
$MSE[\hat{R}(t)]$	2.471063e-08
$V[\tilde{R}(t)]$	3.324842e-08

TABLE 4
Estimates of k and R(t), variance and MSE under type I censoring.

$\hat{k}$	0.092043
$\tilde{k}$	0.092043
$R(t)$	0.999465
$\hat{R}(t)$	0.970877
$\tilde{R}(t)$	0.970852
$MSE[\hat{R}(t)]$	0.0008172771
$V[\tilde{R}(t)]$	0.0008187055

TABLE 5

Estimates of R(t) and P, corresponding variances and MSE for different (n, r).

(n,r)	(30,19)				(40,30)				(50,35)			
t	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$
1	0.9140	0.9097	8.21E-05	0.0001796	0.9066	0.9037	0.000270	0.0003738	0.9197	0.9175	1.16E-05	3.10E-05
2	0.8240	0.8161	3.28E-04	0.0006777	0.8099	0.8047	0.0010361	0.0014001	0.8352	0.8312	4.70E-05	1.18E-04
3	0.7223	0.7116	7.65E-04	0.001475	0.7020	0.6951	0.0023021	0.0030193	0.7394	0.7339	1.12E-04	2.60E-04
4	0.6213	0.6090	1.32E-03	0.0023514	0.5963	0.5885	0.0037529	0.0047631	0.6435	0.6370	1.98E-04	4.20E-04
5	0.5282	0.5159	1.87E-03	0.003085	0.5005	0.4928	0.0050369	0.0061803	0.5545	0.5478	2.88E-04	5.59E-04
6	0.4466	0.4352	2.33E-03	0.0035631	0.4177	0.4108	0.0059475	0.0070587	0.4756	0.4692	3.69E-04	6.56E-04
7	0.3771	0.3671	2.65E-03	0.0037786	0.3483	0.3425	0.0064399	0.0074030	0.4078	0.4020	4.32E-04	7.06E-04
8	0.3188	0.3106	2.84E-03	0.0037815	0.2910	0.2865	0.0065704	0.0073294	0.3503	0.3453	4.74E-04	7.17E-04
9	0.2703	0.2640	2.91E-03	0.0036375	0.2441	0.2408	0.0064327	0.0069777	0.3020	0.2979	4.98E-04	6.99E-04
10	0.2302	0.2257	2.89E-03	0.0034053	0.2058	0.2036	0.0061194	0.0064681	0.2615	0.2583	5.06E-04	6.64E-04

TABLE 6

Estimates of R(t) and P, corresponding variances and MSE for different (n, t_0).

(n,t_0)	(50,4.5)				(40,3.5)				(30,2.5)			
t	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$	$\tilde{R}(t)$	$\hat{R}(t)$	$V(\tilde{R}(t))$	$MSE(\hat{R}(t))$
1	0.8836	0.8837	1.78E-02	1.78E-02	0.8563	0.8566	1.12E-02	1.13E-02	0.8500	0.8506	1.0E-02	1.01E-02
2	0.6938	0.6948	7.03E-02	7.08E-02	0.6319	0.6338	4.13E-02	4.21E-02	0.6171	0.6209	3.55E-02	3.70E-02
3	0.5489	0.5511	8.93E-02	9.06E-02	0.4704	0.4742	4.85E-02	5.02E-02	0.4510	0.4585	4.04E-02	4.34E-02
4	0.4491	0.4524	8.48E-02	8.67E-02	0.3651	0.3703	4.29E-02	4.51E-02	0.3438	0.3541	3.45E-02	3.84E-02
5	0.3789	0.3829	7.38E-02	7.60E-02	0.2943	0.3006	3.50E-02	3.74E-02	0.2726	0.2847	2.73E-02	3.15E-02
6	0.3275	0.3321	5.32E-02	6.51E-02	0.2446	0.2515	2.81E-02	3.04E-02	0.2230	0.2363	2.13E-02	2.54E-02
7	0.2884	0.2935	6.27E-02	5.55E-02	0.2081	0.2154	2.26E-02	2.48E-02	0.1871	0.2010	1.67E-02	2.05E-02
8	0.2578	0.2632	4.53E-02	4.77E-02	0.1804	0.1880	1.83E-02	2.05E-02	0.1600	0.1743	1.32E-02	1.67E-02
9	0.2332	0.2388	3.9E-02	4.12E-02	0.1587	0.1664	1.51E-02	1.7E-02	0.1390	0.1535	1.06E-02	1.38E-02
10	0.2130	0.2188	3.38E-02	3.59E-02	0.1413	0.1491	1.25E-02	1.44E-02	0.1223	0.1368	8.68E-03	1.16E-02

TABLE 7
Estimates of k and R(t) for different number of observed failures, their corresponding variances and MSE.

r	$\hat{k}$	$\tilde{k}$	R(t)	$\tilde{R(t)}$	$\hat{R(t)}$	$MSE\hat{R(t)}$	$V\tilde{R(t)}$
75	1.013251	0.999741	0.842105	0.840362	0.842141	0.000291	0.000283
60	1.016345	0.999405	0.842105	0.839963	0.842191	0.000371	0.000358
50	1.021046	1.000625	0.842105	0.839339	0.842020	0.000462	0.000443
40	1.025707	1.000064	0.842105	0.838731	0.842094	0.000572	0.000542
30	1.036973	1.002407	0.842105	0.837248	0.841762	0.000807	0.000750

TABLE 8
Estimates of k and R(t) for different censoring times, their corresponding variances and MSE.

t_0	$\hat{k}$	$\tilde{k}$	R(t)	$\tilde{R(t)}$	$\hat{R(t)}$	$MSE\hat{R(t)}$	$V\tilde{R(t)}$
3.5	1.307094	0.494405	0.842105	0.798865	0.798614	0.001948	0.001970
3	1.040453	0.541392	0.842105	0.836315	0.836137	0.000107	0.000109
2.5	0.760967	0.600179	0.842105	0.877454	0.877343	0.001314	0.001306
2	0.484255	0.674531	0.842105	0.920173	0.920117	0.006138	0.006130
1.5	0.239859	0.765908	0.842105	0.959629	0.959610	0.013832	0.013828

REFERENCES

A. M. AWAD, M. K. GHARRAF (1986). *Estimation of $P(Y < X)$ in the Burr case: A comparative study*. Communications in Statistics - Simulation and Computation, 15, no. 2, pp. 389–403.

D. J. BARTHOLOMEW (1957). *A problem in life testing*. Journal of the American Statistical Association, 52, no. 2, pp. 350–355.

D. J. BARTHOLOMEW (1963). *The sampling distribution of an estimate arising in life testing*. Technometrics, 5, pp. 361–374.

A. P. BASU (1964). *Estimates of reliability for some distributions useful in life testing*. Technometrics, 6, pp. 215–219.

I. BURR (1942). *Cumulative frequency functions*. Annals of Mathematical Statistics, 13, pp. 215–232.

A. CHAO (1982). *On comparing estimators of $Pr(X > Y)$ in the exponential case*. IEEE Transaction on Reliability, R, no. 26, pp. 389–392.

A. CHATURVEDI, A. MALHOTRA (2017). *Inference on the parameters and reliability characteristics of three parameter Burr distribution based on records*. Applied Mathematics and Information Sciences, 11, no. 3, pp. 1–13.

A. CHATURVEDI, U. RANI (1997). *Estimation procedures for a family of density functions representing various life-testing models*. Metrika, 46, pp. 213–219.

A. CHATURVEDI, U. RANI (1998). *Classical and Bayesian reliability estimation of the generalized Maxwell failure distribution*. Journal of Statistical Research, 32, pp. 113–120.

A. CHATURVEDI, K. G. SINGH (2006). *Bayesian estimation procedures for a family of lifetime distributions under squared-error and entropy losses*. Metron, 64, no. 2, pp. 179–198.

A. CHATURVEDI, K. G. SINGH (2008). *A family of lifetime distributions and related estimation and testing procedures for the reliability function*. Journal of Applied Statistical Science, 16, no. 2, pp. 35–40.

A. CHATURVEDI, K. SURINDER (1999). *Further remarks on estimating the reliability function of exponential distribution under type I and type II censorings*. Brazilian Journal of Probability and Statistics, 13, no. 2, pp. 29–39.

A. CHATURVEDI, N. TIWARI, S. K. TOMER (2002). *Robustness of the sequential testing procedures for the generalized life distributions*. Brazilian Journal of Probability and Statistics, 16, pp. 7–24.

A. CHATURVEDI, S. K. TOMER (2002). *Classical and Bayesian reliability estimation of the negative binomial distribution*. Journal of Applied Statistical Science, 11, pp. 33–43.

A. CHATURVEDI, S. K. TOMER (2003). *UMVU estimation of the reliability function of the generalized life distributions*. Statistical Papers, 44, no. 3, pp. 301–313.

A. CHATURVEDI, S. VYAS (2017). *Estimation and testing procedures of reliability functions of exponentiated distributions under censorings*. Statistica, 77, no. 1, pp. 13–31.

R. CHENG, N. AMIN (1981). *Maximum likelihood estimation of parameters in the inverse gaussian distribution with unknown origin*. Technometrics, 23, pp. 257–263.

R. CHENG, N. AMIN (1983). *Estimating parameters in continuous univariate distributions with a shifted origin*. Journal of the Royal Statistical Society: Series B, 45, pp. 394–403.

K. CONSTANTINE, M. KARSON, S. K. TSE (1986). *Estimators of $P(Y < X)$ in the gamma case*. Communications in Statistics-Simulation and Computation, 15, pp. 365–388.

R. DUMONCEAUX, C. ANTLE (1973). *Discrimination between the lognormal and Weibull distributions*. Technometrics, 15, pp. 923–926.

N. JOHNSON (1975). *Letter to the editor*. Technometrics, 17, p. 393.

G. D. KELLEY, J. A. KELLEY, W. SCHUCANY (1976). *Efficient estimation of* $P(Y < X)$ *in the exponential case*. Technometrics, 18, pp. 359–360.

E. PUGH (1963). *The best estimate of reliability in the exponential case*. Operations Research, 11, pp. 57–61.

Y. S. SATHE, S. P. SHAH (1981). *On estimation* $P(Y < X)$ *for the exponential distribution*. Communications in Statistics-Theory and Methods, 10, pp. 39–47.

Q. SHAO (2004). *Notes on maximum likelihood estimation for the three-parameter Burr XII distribution*. Computational Statistics & Data Analysis, 45, pp. 675–687.

H. TONG (1974). *A note on the estimation of* $P(Y < X)$ *in the exponential case*. Technometrics, 16, p. 625.

H. TONG (1975). *Letter to the editor*. Technometrics, 17, p. 393.

R. K. TYAGI, S. K. BHATTACHARYA (1989a). *A note on the MVU estimation of reliability for the Maxwell failure distribution*. Estadistica, 41, pp. 73–79.

R. K. TYAGI, S. K. BHATTACHARYA (1989b). *Bayes estimator of the Maxwell's velocity distribution function*. Statistica, 49, pp. 563–567.

F. WANG, J. KEATS, W. ZIMMER (1996). *Maximum likelihood estimation of the Burr XII distribution with censored and uncensored data*. Microelectronics Reliability, 36, pp. 359–362.

L. WANG, Y. SHI (2010). *Empirical Bayes inference for the Burr model based on records*. Applied Mathematical Sciences, 4, no. 34, pp. 1663 – 1670.

D. WINGO (1983). *Maximum likelihood methods for fitting the Burr Type XII distribution parameters to life test data*. Biometrical Journal, 25, pp. 77–84.

D. WINGO (1993). *Maximum likelihood estimation of Burr XII distribution parameters under type II censoring*. Microelectronics Reliability, 33, pp. 1251–1257.

W. ZIMMER, J. KEATS, F. WANG (1998). *The Burr XII distribution in reliability analysis*. Journal of Quality Technology, 30, pp. 386–394.

SUMMARY

A three parameter Burr distribution is considered. Two measures of reliability are discussed. Point and interval estimation procedures are developed for the parameters, and reliability functions under type II and type I censoring. Two types of point estimators namely- uniformly minimum variance unbiased estimators (UMVUES) and maximum likelihood estimators (MLES) are derived. Asymptotic variance-covariance matrix and confidence intervals for MLE's are obtained. Testing procedures are also developed for various hypotheses.

Keywords: Three parameter Burr distribution; Point estimation; Interval estimation; Type-II and type-I censoring, Testing of hypotheses.