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MULTIVARIATE VARIANCE RESIDUAL LIFE IN DISCRETE TIME

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1. INTRODUCTION

The concept of residual life is of special interest in reliability theory and survival analysis as it measures the life remaining to a device or an individual after it has attained a specific age. Various characteristics of residual life such as its mean, variance, coefficient of variation, higher moments and percentiles have been extensively studied in literature. Among these, the variance residual life has attracted many researchers including Dallas (1981), Karlin (1982), Chen *et al.* (1983), Gupta (1987), Gupta *et al.* (1987), Abouammoh *et al.* (1990), Adatia *et al.* (1991), Stein and Dattero (1999), Gupta and Kirmani (2000, 2004), Stoyanov and Al-Sadi (2004), Gupta (2006) and Nair and Sudheesh (2010) when lifetime is treated as a continuous random variable. These works emphasize the importance of variance residual life as (i) a reliability function useful in modelling lifetime data with special reference to inference procedures and characterizations (ii) a means to classify lifetime distributions through the monotonicity properties and (iii) through its relationship with the mean residual life in the same way as the mean to the variance; see Hall and Wellner (1981). In the discrete case also the topic in the univariate case has been well studied by several authors that includes Hitha and Nair (1989), Roy (2005), El-Arishy (2005), Sudheesh and Nair (2010), Khorashadizadeh *et al.* (2010) and Al-Zahrani *et al.* (2013). The only study that appears to be made in higher dimensional discrete case is that of Roy (2005) who characterized some bivariate discrete distributions by certain simple properties of the variance residual life. There are several multicomponent devices and systems in which the lifetimes of the components are measured as the number of time units completed, or the number of cycles in operation before failure. Also,

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in survival analysis, when continuous lifetimes are subjected to censoring, discrete data arise. In such cases, to ascertain the role of multivariate variance residual life and its properties, a systematic study of the topic does not appear to be available in reliability literature. This motivates the present study. Our objective is to make a theoretical exposition of the properties of the multivariate discrete variance residual life. It includes properties of the variance residual life, characterization of life distributions and classes of life distributions based on the monotonic properties of the variance residual life. As a by-product we also get some properties in the univariate case, that do not seem to have been discussed in the previous studies.

The paper is organized into five sections. In Section 2 we have included some preliminary definitions and results required in the sequel. This is followed by the definition and properties of the p-dimensional variance residual life function in Section 3. In Section 4 various classes of life distributions are discussed. The study ends with a brief conclusion in Section 5.

2. Preliminaries

Let $\mathbf{X}=(X_1,X_2,...,X_p)$ be a discrete random vector taking values in $\mathbf{N}^p$, $\mathbf{N}=(0,1,2,...)$ with survival function $S(\mathbf{x})=P[\mathbf{X}\geq\mathbf{x}]$ and probability mass function $f(\mathbf{x})=P[\mathbf{X}=\mathbf{x}]$ where $\mathbf{x}=(x_1,x_2,...,x_p)$ and the equalities and inequalities of vectors involved are taken component wise. The residual life of a multicomponent device whose lifetime $\mathbf{X}$ is defined as the vector

$$\mathbf{X}_{\mathbf{x}}=\Big(X_1-x_1|\mathbf{X}>\mathbf{x},X_2-x_2|\mathbf{X}>\mathbf{x},...,X_p-x_p|\mathbf{X}>\mathbf{x}\Big). \tag{1}$$

The mean residual life function of $\mathbf{X}$ is given by the vector

$$\Big(m_1(\mathbf{x}),m_2(\mathbf{x}),...,m_p(\mathbf{x})\Big),$$

where

$$\begin{aligned} m_i(\mathbf{x})&=E[X_i-x_i|\mathbf{X}>\mathbf{x}];\ i=1,2,...,p;\ x_i=-1,0,1,2,...\\ &=\frac{1}{S(\mathbf{x}+\mathbf{e}_p)}\sum_{t=x_i+1}^{\infty}S(t,\mathbf{x}_{(i)}+\mathbf{e}_{p-1}),\end{aligned}$$

where $\mathbf{x}_{(i)}=\mathbf{x}-\{x_i\}$ and $\mathbf{e}_p$ is the p- dimensional vector with unity as its elements. It can be seen that

$$\frac{S(x_1+2,x_2+1,...,x_p+1)}{S(x_1+1,x_2+1,...,x_p+1)}=\frac{m_1(\mathbf{x})-1}{m_1(x_1+1,\mathbf{x}_{(1)})}. \tag{2}$$

Further the mean residual life vector determines $S(\mathbf{x})$ uniquely through the formula

$$\begin{aligned}
S(\mathbf{x}) &= \prod_{r=0}^{x_1-1} \frac{m_1(r-1,\mathbf{x}_{(1)})-1}{m_1(r,\mathbf{x}_{(1)})} \prod_{r=0}^{x_2-1} \frac{m_2(0,r-1,x_3,...,x_p)-1}{m_2(0,r,...,x_p)} \cdots \\
&\qquad \prod_{r=0}^{x_p-1} \frac{m_p(0,...,0,r-1)-1}{m_p(0,...,0,r)} \\
&= \prod_{r=0}^{x_2-1} \frac{m_2(x_1,r-1,x_3,...,x_p)-1}{m_2(r,\mathbf{x}_{(2)})} \cdots \prod_{r=0}^{x_1-1} \frac{m_1(r-1,0,...,0)-1}{m_1(r,0,...,0)} \\
&\cdots \\
&= \prod_{r=0}^{x_p-1} \frac{m_p(x_1,...,x_{p-1},r-1)-1}{m_p(x_1,...,x_{p-1},r)} \cdots \prod_{r=0}^{x_{p-1}-1} \frac{m_{p-1}(0,...,0,r-1,0)-1}{m_{p-1}(0,...,0,r-1,0)}. \qquad (3)
\end{aligned}$$

These are discrete analogues of the results in Arnold and Zahedi (1988).

Corresponding to the vector $\mathbf{X}$, we can define a vector $\mathbf{Y} = (Y_1, Y_2, ..., Y_p)$ in $\mathbf{N}^p$ such that the distribution of $\mathbf{Y}$ is specified by the conditional probability mass functions

$$\begin{aligned}
g_1(x_1|\mathbf{Y}_{(1)} > \mathbf{x}_{(1)}) &= \frac{P[X_1 > x_1|\mathbf{X}_{(1)} > \mathbf{x}_{(1)}]}{E[X_1|\mathbf{X}_{(1)} > \mathbf{x}_{(1)}]} \\
g_2(x_2|\mathbf{Y}_{(2)} > \mathbf{x}_{(2)}) &= \frac{P[X_2 > x_2|\mathbf{X}_{(2)} > \mathbf{x}_{(2)}]}{E[X_2|\mathbf{X}_{(2)} > \mathbf{x}_{(2)}]} \\
&\cdots \\
g_p(x_p|\mathbf{Y}_{(p)} > \mathbf{x}_{(p)}) &= \frac{P[X_p > x_p|\mathbf{X}_{(p)} > \mathbf{x}_{(p)}]}{E[X_p|\mathbf{X}_{(p)} > \mathbf{x}_{(p)}]}.
\end{aligned}$$

The above definitions are extensions to the multivariate case of the concept of continuous bivariate equilibrium distributions discussed in Gupta and Sankaran (1998), Nair and Preeth (2008) and Navarro and Sarabia (2010). Notice that the above conditional probability mass functions lead to a multivariate distribution if and only if

$$\frac{P[Y_i > x_i|\mathbf{Y}_{(i)} > \mathbf{x}_{(i)}]}{P[Y_j > x_j|\mathbf{Y}_{(j)} > \mathbf{x}_{(j)}]} = \frac{A_j(\mathbf{x}_{(j)})}{A_i(\mathbf{x}_{(i)})},$$

where $A_i(.)$ and $A_j(.)$ are survival functions. The distribution of $\mathbf{Y}$ is called the multivariate equilibrium distribution of the random vector $\mathbf{X}$.

3. Multivariate variance residual life

Let $\mathbf{X}$ be the p-dimensional random vector defined in Section 2. Then the variance residual life of X_i is defined as

$$\sigma_i^2(\mathbf{x}) = E\left[(X_i - x_i)^2 | \mathbf{X} > \mathbf{x}\right] - m_i^2(\mathbf{x});\ x_i = -1, 0, 1, 2, \ldots;\ i = 1, 2, \ldots, p. \tag{4}$$

The evaluation of (4) can be accomplished by the formula

$$\sigma_i^2(\mathbf{x}) = \frac{2}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} \sum_{u_i=t_i+1}^{\infty} S(x_1+1, \ldots, x_{i-1}+1, u_i, x_{i+1}+1, \ldots, x_p+1) - m_i(\mathbf{x})(m_i(\mathbf{x})-1). \tag{5}$$

To prove this, we note that

$$\begin{aligned}
&E[(X_i - x_i)^2 | \mathbf{X} > \mathbf{x}] \\
&= \frac{1}{S(\mathbf{x}+\mathbf{e})} \sum_{t_1=x_1+1}^{\infty} \cdots \sum_{t_p=x_p+1}^{\infty} (t_i - x_i)^2 f(\mathbf{x}) \\
&= \frac{1}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} (t_i - x_i)^2 \Big[S(x_1+1, \ldots, x_{i-1}+1, t_i, x_{i+1}+1, \ldots, x_p+1) \\
&\qquad - S(x_1+1, \ldots, t_i+1, \ldots, x_p+1) \Big] \\
&= \frac{1}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} [2(t_i - x_i) - 1] S(x_1+1, \ldots, x_{i-1}+1, t_i, \ldots, x_p+1) \\
&= \frac{2}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} (t_i - x_i) S(x_1+1, \ldots, t_i, \ldots, x_p+1) + m_i(\mathbf{x}) \\
&= \frac{2}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i=x_i+1}^{\infty} \sum_{u_i=t_i+1}^{\infty} S(x_1+1, \ldots, x_{i-1}+1, u_i, x_{i+1}+1, \ldots, x_p+1) + m_i(\mathbf{x}).
\end{aligned} \tag{6}$$

Substituting (6) into (4), we have (5). In the bivariate case

$$E\left[(X_1 - x_1)^2 | X_1 > x_1, X_2 > x_2\right] = \frac{2}{S(x_1+1, x_2+1)} \sum_{t_1=x_1+1}^{\infty} \sum_{u=t_1+1}^{\infty} S(u, x_2+1) + m_1(x_1, x_2) \tag{7}$$

and similarly,

$$E\left[(X_2 - x_2)^2 | X_1 > x_1, X_2 > x_2\right] = \frac{2}{S(x_1+1, x_2+1)} \sum_{t_2=x_2+1}^{\infty} \sum_{u=t_2+1}^{\infty} S(x_1+1, u) + m_2(x_1, x_2), \tag{8}$$

from which $\sigma_1^2(x_1, x_2)$ and $\sigma_2^2(x_1, x_2)$ are computed. In the univariate case, for X_1,

$$\sigma_1^2(x_1) = \frac{2}{S(x_1+1)} \sum_{t=x_1+1}^{\infty} \sum_{u=t+1}^{\infty} S(u) - m_1(x_1)(m_1(x_1)-1). \tag{9}$$

EXAMPLE 1. *For the bivariate geometric distribution (Nair and Nair, 1988) with survival function*

$$S(x_1, x_2) = q_1^{x_1} q_2^{x_2} \theta^{x_1 x_2};\ x_i = 0, 1, 2, \ldots;\ 0 < q_i < 1;\ 0 \le \theta \le 1; 1-\theta \le (1-q_1\theta)(1-q_2\theta); \tag{10}$$

$i = 1, 2$, *we have*

$$m_i(x_1, x_2) = \left(1 - q_i \theta^{x_{3-i}+1}\right)^{-1};\ i = 1, 2$$

and

$$\frac{2}{S(x_1+1, x_2+1)} = \frac{2q_1\theta^{x_2+1}}{1-q_1\theta^{x_2+1}}.$$

Thus from (5)*, when* $p = 2, i = 1$*, we obtain*

$$\sigma_1^2(x_1, x_2) = \frac{q_1\theta^{x_2+1}}{(1-q_1\theta^{x_2+1})^2}$$

and similarly for $i = 2$

$$\sigma_2^2(x_1, x_2) = \frac{q_2\theta^{x_1+1}}{(1-q_2\theta^{x_1+1})^2}.$$

3.1. *Properties of variance residual life*

1. If $i_1, i_2, \ldots, i_r;\ r = 1, 2, \ldots, p$ are permutations of the integers $(1, 2, \ldots, r)$, the variance residual life of the marginal distributions of $\mathbf{X}$ are obtained from (5) by setting $x_{i_{r+1}} \ldots = -1$ whenever $r < p$. In particular

$$\sigma_i^2(-\mathbf{e}) = \sigma_i^2,$$

   the variance of the marginal distribution of X_i.

2. There exists a recurrence relation for $\sigma_i^2(\mathbf{x})$. Without loss of generality, we take $i = 1$ and state it as

$$\sigma_1^2(x_1+1, \mathbf{x}_{(1)}) = m_1(x_1+1, \mathbf{x}_{(1)}) \left[\frac{\sigma_1^2(\mathbf{x})}{m_1(\mathbf{x})-1} + m_1(\mathbf{x}) - m_1(x_1+1, \mathbf{x}_{(1)}) - 1 \right]. \tag{11}$$

PROOF. When $i = 1$, (5) can be written as

$$\sigma_1^2(\mathbf{x}) = \frac{2}{S(\mathbf{x}+\mathbf{e})} \sum_{t_1=x_1+1}^{\infty} m_1(t_1,\mathbf{x}_{(1)}) S(t_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}) - m_1(\mathbf{x})(m_1(\mathbf{x})-1).$$

Thus,

$$\begin{aligned}
&\left\{\sigma_1^2(\mathbf{x}) + m_1(\mathbf{x})[m_1(\mathbf{x})-1]\right\} S(x_1+1,\ldots,x_p+1) \\
&\quad - \left\{\sigma_1^2(x_1+1,\mathbf{x}_{(1)}) + m_1(x_1+1,\mathbf{x}_{(1)})\left[m_1(x_1+1,\mathbf{x}_{(1)})-1\right]\right\} \\
&\qquad S(x_1+2,x_2+1,\ldots,x_p+1) \\
&\qquad = 2m_1(x_1+1,\mathbf{x}_{(1)}) S(x_1+2,x_2+1,\ldots,x_p+1). \qquad (12)
\end{aligned}$$

Dividing (12) by $S(x_1+1,\ldots,x_p+1)$ and using (2), we obtain

$$\begin{aligned}
&\left\{\sigma_1^2(\mathbf{x}) + m_1(\mathbf{x})[m_1(\mathbf{x})-1]\right\} \\
&- \left\{\sigma_1^2(x_1+1,\mathbf{x}_{(1)}) + m_1(x_1+1,\mathbf{x}_{(1)})\left[m_1(x_1+1,\mathbf{x}_{(1)})-1\right]\right\} \frac{m_1(\mathbf{x})-1}{m_1(x_1+1,\mathbf{x}_{(1)})} \\
&= 2(m_1(\mathbf{x})-1).
\end{aligned}$$

Simplifying the above equation, we get (11). When $n = 1$, we have the univariate result as

$$\sigma_1^2(x_1) = (m_1(x_1)-1)\left[\frac{\sigma_1^2(x_1+1)}{m_1(x_1+1)} + m_1(x_1+1) - m_1(x_1) + 1\right]. \qquad (13)$$

□

3. The variance residual life function can be expressed in terms of mean residual life function as the following theorem shows.

THEOREM 1.

$$\sigma_1^2(\mathbf{x}) = E\left[m_1(X_1,\mathbf{x}_{(1)})(m_1(X_1-1,\mathbf{x}_{(1)})-1)|\mathbf{X} > \mathbf{x}\right]. \qquad (14)$$

PROOF. From (12), we can write

$$\begin{aligned}
&\left\{\sigma_1^2(\mathbf{x}) + m_1(\mathbf{x})(m_1(\mathbf{x})-1)\right\} S(\mathbf{x}+\mathbf{e}) \\
&= \sigma_1^2(x_1+1,\mathbf{x}_{(1)}) + m_1^2(x_1+1,\mathbf{x}_{(1)}) + m_1(x_1+1,\mathbf{x}_{(1)}) S(x_1+2,x_2+1,\ldots,x_p+1).
\end{aligned}$$

Dividing by $S(\mathbf{x}+\mathbf{e})$, we have

$$\sigma_1^2(\mathbf{x})=\left[1-\frac{f(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{S(\mathbf{x}+\mathbf{e})}\right]\Big[\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m^2(x_1+1,\mathbf{x}_{(1)})$$
$$+m_1(x_1+1,\mathbf{x}_{(1)})\Big]-m_1(\mathbf{x})(m_1(\mathbf{x})-1).$$

Now using (2)

$$\begin{aligned}\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-\sigma_1^2(\mathbf{x})&=m_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\\&\quad-\Big(\sigma_1^2(x_1+1,\mathbf{x}_{(1)})+m_1^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})\Big)\\&\quad\frac{1+m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})}{m_1(x_1+1,\mathbf{x}_{(1)})}-m_1(\mathbf{x})(m_1(\mathbf{x})-1)\\&=\frac{f(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{S(\mathbf{x}+\mathbf{e})}\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-\frac{f(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}{S(\mathbf{x}+\mathbf{e})}\\&\quad\Big(m_1(x_1+1,\mathbf{x}_{(1)})(1-m_1(\mathbf{x}))\Big).\end{aligned}$$

The last expression simplifies to

$$\sigma_1^2(\mathbf{x})S(\mathbf{x}+\mathbf{e})-\sigma_1^2(x_1+1,\mathbf{x}_{(1)})S(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})=\Big[m_1(x_1+1,\mathbf{x}_{(1)})$$
$$(m_1(\mathbf{x})-1)\Big]f(x_1+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}). \quad (15)$$

Adding the above identity for values of x_1

$$\sigma_1^2(\mathbf{x})S(\mathbf{x}+\mathbf{e}) = \sum_{t_1=x_1+1}^{\infty} m_1(t_1,\mathbf{x}_{(1)})(m_1(t_1-1,\mathbf{x}_{(1)})-1)f(t_1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1}),$$

which is same as (14). □

REMARK 2. *In the univariate case,* $(p=1)$

$$\sigma_1^2(x_1)=E\left[m_1(X_1)(m_1(X_1-1)-1)|X_1>x_1\right],$$

a formula that does not seem to have appeared in literature. It can be used for obtaining quick estimates of $\sigma_1^2(x_1)$ *based on the estimates of* $m_1(x_1)$.

4. A problem of traditional interest in modelling situations is to characterize life distributions by properties of reliability functions that enable easy identification of the appropriate model. We give some such properties in the following results.

THEOREM 3. *A random vector* $\mathbf{X}$ *in* $\mathbf{N}^p$ *has a variance residual life of the form*

$$\sigma_i^2(\mathbf{x}) = p_i(\mathbf{x}_{(i)});\ i = 1, 2, ..., p; \tag{16}$$

for all $\mathbf{x}$ *if and only if* $\mathbf{X}$ *follows the multivariate geometric distribution*

$$S(\mathbf{x}) = \prod_{i=1}^{p} q_i^{x_i} \prod_{i,j=1;i<j}^{p} q_{ij}^{x_i x_j} ... q_{12...p}^{x_1 x_2 ... x_p};\ x_i = 0, 1, 2, ...;\ 0 < q_i, q_{ij}, ..., q_{12...p} < 1 \tag{17}$$

and $1 - \sum_{i=1}^{p} q_i + \sum_{i,j=1;i<j}^{p} q_{ij} - ... + (-1)^p q_{12...p} \geq 0$.

PROOF. By direct calculation

$$\sigma_i^2(\mathbf{x}) = \frac{a(\mathbf{x}_{(i)})}{(1 - a(\mathbf{x}_{(i)}))^2}, \tag{18}$$

where

$$a(\mathbf{x}_{(i)}) = q_1 \prod_{j=2}^{p} q_{ij}^{x_j} \prod_{j,k=2;j<k}^{p} q_{ijk}^{x_j x_k} ... q_{12...p}^{x_2 ... x_p}.$$

This proves the "if" part. Now assume that (17) holds. Using (15) with suffix 1 replaced by i, we get

$$p_i(\mathbf{x}_{(i)}) = m_i(x_i + 1, \mathbf{x}_{(i)})(m_i(\mathbf{x}) - 1),$$

which cannot be true unless $m_i(\mathbf{x}) = a_i(\mathbf{x}_{(i)})$, a function independent of x_i. Taking $i = 1$ and $p = 1$, the mean residual life of X_1 is independent of x_1, say c_1. Then the survival function $S_1(.)$ of X_1 satisfies

$$S_1(x_1) = \prod_{t=0}^{x_1 - 1} \frac{m_1(x_1 - 1) - 1}{m_1(x_1)} = \left(\frac{c_1 - 1}{c_1}\right)^{x_1} = q_1^{x_1};\ 0 < q_1 < 1.$$

In general, $S_i(x_i) = q_i^{x_i}$. Similarly for $p = 2$ and $i = 1$, in the bivariate case

$$\begin{aligned} S_2(x_1, x_2) &= \prod_{t=0}^{x_1 - 1} \frac{m_1(t - 1, x_2) - 1}{m_1(t, x_2)} S(0, x_2) \\ &= [b_1(x_2)]^{x_1} q_2^{x_2};\ b_1(x_2) = \frac{a_1(x_2) - 1}{a_1(x_2)};\ 0 < b_1(x_2) < 1. \end{aligned} \tag{19}$$

Similarly working with $i = 2$ and $p = 2$, we obtain

$$S(x_1, x_2) = [b_2(x_1)]^{x_2} q_1^{x_1}. \tag{20}$$

From (19) and (20)

$$x_1 \log b_1(x_2) + x_2 \log q_2 = x_2 \log b_2(x_1) + x_1 \log q_1.$$

The left side of the above equation can be written as

$$x_1(\log b_1(x_2) - \log q_1) = x_2(\log b_2(x_1) - \log q_2),$$

which is linear in x_1 and hence right side must also be linear in x_1 and similarly for x_2. The only solution in this case is

$$b_1(x_2) = q_1 q_{12}^{x_2} \text{ for some } 0 < q_{12} < 1$$

and

$$b_2(x_1) = q_2 q_{12}^{x_1}.$$

This gives

$$S(x_1, x_2) = q_1^{x_1} q_2^{x_2} q_{12}^{x_1 x_2}.$$

Proceeding in this fashion, we arrive at (18) by mathematical induction and the proof is completed. In the bivariate case, the theorem reduces to the characterization result of the bivariate geometric distribution mentioned in (10). □

REMARK 4. *The property* $\left(\sigma_1^2(\mathbf{x}), ..., \sigma_p^2(\mathbf{x})\right) = (c_1, c_2, ..., c_p)$ *where the c's are independent of* $\mathbf{x}$ *is satisfied if and only if the distribution of* $\mathbf{X}$ *is specified by*

$$S(\mathbf{x}) = q_1^{x_1} q_2^{x_2} ... q_p^{x_p};\ 0 < q_i < 1;\ x_i = 0, 1, 2, ...;\ i = 1, 2, ..., p. \tag{21}$$

THEOREM 5. *A bivariate random vector* (X_1, X_2) *in* $\mathbf{N}^2$ *has variance residual life of the form*

$$\left(\sigma_1^2(x_1, x_2), \sigma_2^2(x_1, x_2)\right) = \begin{cases} (c_1\ c_2);\ x_1 > x_2 \\ (c_3\ c_4);\ x_2 > x_1 \\ (c_1\ c_4);\ x_1 = x_2, \end{cases} \tag{22}$$

where $c_i;\ i = 1, 2, 3, 4$ *are independent of* x_1 *and* x_2 *if and only if its survival function is*

$$S(x_1, x_2) = \begin{cases} q^{x_2} q_1^{x_1 - x_2};\ x_1 \geq x_2 \\ q^{x_1} q_2^{x_2 - x_1};\ x_2 \geq x_1;\ x_1, x_2 = 0, 1, 2, ... \\ \qquad 0 < q < q_1, q_2 < 1;\ 1 + q \geq q_1 + q_2. \end{cases} \tag{23}$$

PROOF. First we assume that the distribution of (X_1, X_2) is specified by (22). Then the mean residual life is calculated as

$$(m_1(x_1, x_2), m_2(x_1, x_2)) = \begin{cases} (k_1, k_2);\ \text{if } x_1 > x_2 \\ (k_3, k_4);\ \text{if } x_2 > x_1 \\ (k_1, k_4);\ \text{if } x_1 = x_2, \end{cases} \tag{24}$$

where $k_1 = (1-q_1)^{-1}$, $k_2 = \left(1-\frac{q}{q_1}\right)^{-1}$, $k_3 = \left(1-\frac{q}{q_2}\right)^{-1}$ and $k_4 = (1-q_2)^{-1}$. Also

$$\left(\sigma_1^2(x_1,x_2), \sigma_2^2(x_1,x_2)\right) = \begin{cases} \left(\dfrac{q_1}{(1-q_1)^2}, \dfrac{q}{q_1(1-\frac{q}{q_1})^2}\right); x_1 > x_2 \\ \left(\dfrac{q}{q_2(1-\frac{q}{q_2})^2}, \dfrac{q_2}{(1-q_2)^2}\right); x_2 > x_1 \\ \left(\dfrac{q_1}{(1-q_1)^2}, \dfrac{q_2}{(1-q_2)^2}\right); x_1 = x_2, \end{cases} \tag{25}$$

showing that it is of the form stated in (23). Conversely, assuming (23), we see from (16) with $p = 2$ that $\sigma_1^2(x_1, x_2) = (c_1, c_2)$ for $x_1 > x_2$ gives

$$m_1(x_1+1, x_2)(m_1(x_1,x_2)-1) = c_1$$

and similarly

$$m_2(x_1, x_2+1)(m_2(x_1,x_2)-1) = c_2.$$

The solutions of these equations must be of the form

$$(m_1(x_1,x_2), m_2(x_1,x_2)) = (k_1, k_2)$$

for some constants k_1 and k_2, both independent of x_1 and x_2. Similarly, we can work with the regions $x_2 > x_1$ and $x_1 = x_2$ to reach at (24). Substituting the values of $(m_1(.), m_2(.))$ in Formula (3), the bivariate geometric distribution of the form (23) is recovered. □

REMARK 6. *The p-variate version of* (23) *can be stated as*

$$S(\mathbf{x}) = q_{i_1}^{x_{i_1}} \left(\frac{q_{i_1 i_2}}{q_{i_1}}\right)^{x_{i_2}} \dots \left(\frac{q_{i_1 i_2 \dots i_p}}{q_{i_1 i_2 \dots i_{p-1}}}\right)^{x_{i_p}} ; x_{i_1} \geq x_{i_2} \geq \dots \geq x_{i_p}, \tag{26}$$

where $i_1, i_2, \dots, i_p$ *are the permutations of* $(1, 2, \dots, p)$,

$$0 < q_{i_1} < q_{i_1 i_2} < \dots < q_1 q_2 \dots q_p < 1,$$

and

$$1 - \sum_{i=1}^{p} q_i + \sum_{i<j} q_{ij} \dots + (-1)^p q_1 q_2 \dots q_p \geq 0.$$

The method of proof used in Theorem 5 is applicable in this case also, but with lengthy expressions for the mean and variance residual lives according to the various regions of the sample space required by $x_{i_1} \geq x_{i_2} \geq ... \geq x_{i_p}$. *Note that the variance residual life is piece-wise constant.*

The next theorem is on some special relationships between the variance and mean residual lives that characterizes some distributions.

THEOREM 7. *A random vector* $\mathbf{X}$ *taking values in* $\mathbf{N}^p$ *satisfies the property*

$$\sigma_i^2(\mathbf{x}) = k_i m_i(\mathbf{x})[m_i(\mathbf{x}) - 1];\ i = 1, 2, ..., p \tag{27}$$

for all $\mathbf{x}$ *if and only if the distribution of* $\mathbf{X}$ *is multivariate Waring with*

$$S(\mathbf{x}) = \frac{\left(A_0 + A_2 x_2 + ... + A_p x_p\right)_{x_1}}{\left(A_0 + A_1 + A_2 x_2 + ... + A_p x_p\right)_{x_1}} \cdots \frac{(B_0 + B_3 x_3)_{x_2} (C_0)_{x_3}}{(B_0 + B_2 + B_3 x_3)_{x_2} (C_0 + C_3)_{x_3}}, \tag{28}$$

$x_i = 0, 1, 2, ...;\ i = 1, 2, ..., p$ *and negative hyper geometric with*

$$S(\mathbf{x}) = \frac{\binom{\alpha_0 + \alpha_1 + \alpha_2 x_2 + ... + \alpha_p x_p - x_1}{\alpha_0 + \alpha_2 x_2 + ... + \alpha_p x_p - x_1}}{\binom{\alpha_0 + \alpha_1 + \alpha_2 x_2 + ... + \alpha_p x_p}{\alpha_0 + \alpha_2 x_2 + ... + \alpha_p x_p}} \cdots \frac{\binom{\beta_0 + \beta_1 + \beta_3 x_{p-3} - x_{p-1}}{\beta_0 + \beta_3 x_{p-3} - x_{p-1}}}{\binom{\beta_0 + \beta_1 + \beta_3 x_{p-3}}{\beta_0 + \beta_3 x_{p-3}}} \frac{\binom{\delta_0 + \delta_1 - x_p}{\delta_0 - x_p}}{\binom{\delta_0 + \delta_1}{\delta_0}}, \tag{29}$$

$$x_1 = 0, 1, 2, ..., \alpha_0; ...; x_p = 0, 1, 2, ..., \delta_0,$$

according as $k_i > 1$ *and* $0 < k_i < 1$.

PROOF. Since the proof of the theorem in the p- variate case is apparent from the tri-variate version, we consider the latter only, for brevity. Recall that

$$\sigma_i^2(\mathbf{x}) = \frac{2}{S(\mathbf{x}+\mathbf{e})} \sum_{t_i = x_i + 1}^{\infty} m_i(t_i, \mathbf{x}_{(i)}) S(t_i + 1, \mathbf{x}_{(i)} + \mathbf{e}_{p-1}) - m_i(\mathbf{x})(m_i(\mathbf{x}) - 1), \tag{30}$$

$i = 1, 2, ..., p$. Taking $p = 3$, $i = 1$ and $\mathbf{x} = (x_1, x_2, x_3)$, we can write the above identity when (27) holds as

$$(k+1) m_1(\mathbf{x})(m_1(\mathbf{x}) - 1)) = \frac{2}{S(\mathbf{x}+\mathbf{e}_3)} \sum_{t_1 = x_1 + 1}^{\infty} m_1(x_1, x_2, x_3) S(t_1 + 1, x_2 + 1, x_3 + 1).$$

$$(k+1)m_1(\mathbf{x})(m_1(\mathbf{x})-1)) = \frac{2}{S(\mathbf{x}+\mathbf{e}_3)} \sum_{t_1=x_1+1}^{\infty} m_1(x_1,x_2,x_3)S(t_1+1,x_2+1,x_3+1).$$

Hence

$$(k+1)m_1(\mathbf{x})(m_1(\mathbf{x})-1)S(\mathbf{x}+\mathbf{e}_3) - (k+1)m_1(x_1+1,x_2,x_3)$$
$$(m_1(x_1+1,x_2,x_3)-1)S(x_1+2,x_2+1,x_3+1) = 2S(x_1+2,x_2+1,x_3+1)$$
$$m_1(x_1+1,x_2,x_3).$$

Dividing by $S(x_1+1,x_2+1,x_3+1)$ and invoking (14) with $p=3$, we get, after some simplifications, that

$$(k+1)[m_1(\mathbf{x}) - m_1(x_1+1,x_2,x_3)] = 2$$

or

$$m_1(\mathbf{x}) - m_1(x_1+1,x_2,x_3) = \frac{k_1-1}{k_1+1}.$$

The solution of the above partial difference equation is

$$m_1(\mathbf{x}) = \alpha_1 x_1 + p_1(x_2,x_3),\ \alpha = \frac{k_1-1}{k_1+1}.$$

Likewise for $i=2$ and 3, we further have from (30)

$$m_2(\mathbf{x}) = \alpha_2 x_2 + p_2(x_1,x_3)$$

and

$$m_3(\mathbf{x}) = \alpha_3 x_3 + p_3(x_1,x_2).$$

From (3), the survival function is written as

$$S(\mathbf{x}) = \prod_{r=0}^{x_1-1} \frac{\alpha_1(r-1)+p_1(x_2,x_3)-1}{\alpha_1 r + p_1(x_2,x_3)} \prod_{r=0}^{x_2-1} \frac{\alpha_2(r-1)+p_2(0,x_3)-1}{\alpha_2 r + p_2(0,x_3)}$$
$$\prod_{r=0}^{x_3-1} \frac{\alpha_3(r-1)+p_3(0,0)-1}{\alpha_3 r + p_3(0,0)}$$

$$= \prod_{r=0}^{x_2-1} \frac{\alpha_2(r-1)+p_2(x_1,x_3)-1}{\alpha_2 r + p_2(x_1,x_3)} \prod_{r=0}^{x_3-1} \frac{\alpha_3(r-1)+p_3(x_1,0)-1}{\alpha_3 r + p_3(x_1,0)} \prod_{r=0}^{x_3-1} \frac{\alpha_1(r-1)+p_1(0,0)-1}{\alpha_1 r + p_1(0,0)}$$

$$= \prod_{r=0}^{x_3-1} \frac{\alpha_3(r-1)+p_3(x_1,x_2)-1}{\alpha_3 r + p_3(x_1,x_2)} \prod_{r=0}^{x_1-1} \frac{\alpha_1(r-1)+p_1(x_2,0)-1}{\alpha_1 r + p_1(x_2,0)} \prod_{r=0}^{x_2-1} \frac{\alpha_2(r-1)p_2(0,0)-1}{\alpha_2 r + p_2(0,0)}. \tag{31}$$

When $k>1, \alpha_i > 0$ the terms under the product symbol can be written in terms of the Pocchamer symbol

$$(t)_r = t(t+1)...(t+r-1).$$

Thus

$$S(\mathbf{x}) = \frac{\left(\frac{p_1(x_2,x_3)-1}{\alpha_1}-1\right)_{x_1}}{\left(\frac{p_1(x_2,x_3)}{\alpha_1}\right)_{x_1}} \frac{\left(\frac{p_2(0,x_3)-1}{\alpha_2}-1\right)_{x_2}}{\left(\frac{p_2(0,x_3)}{\alpha_2}\right)_{x_2}} \frac{\left(\frac{p_3(0,0)-1}{\alpha_3}-1\right)_{x_3}}{\left(\frac{p_3(0,0)}{\alpha_3}\right)_{x_3}}$$

and similarly the other two equivalent forms. A complete specification of $S(\mathbf{x})$ requires the solution of the functions $p_1(x_2,x_3)$, $p_2(x_1,x_3)$ and $p_3(x_1,x_2)$ for which we consider

$$\frac{S(x_1+1,x_2+1,x_3+1)}{S(x_1,x_2,x_3)} = \frac{S(x_1+1,x_2+1,x_3+1)}{S(x_1,x_2+1,x_3+1)} \frac{S(x_1,x_2+1,x_3+1)}{S(x_1,x_2,x_3+1)} \frac{S(x_1,x_2,x_3+1)}{S(x_1,x_2,x_3)}$$

$$= \frac{S(x_1+1,x_2+1,x_3+1)}{S(x_1+1,x_2,x_3+1)} \frac{S(x_1+1,x_2,x_3+1)}{S(x_1+1,x_2,x_3)} \frac{S(x_1+1,x_2,x_3)}{S(x_1,x_2,x_3)}$$

$$= \frac{S(x_1+1,x_2+1,x_3+1)}{S(x_1+1,x_2+1,x_3)} \frac{S(x_1+1,x_2+1,x_3)}{S(x_1,x_2+1,x_3)} \frac{S(x_1,x_2+1,x_3)}{S(x_1,x_2,x_3)}.$$

Converting the right hand expressions in terms of the mean residual life functions using (14) lead to three functional equations. One of these equations has the form

$$\frac{(m_1(x_1-1,x_2,x_3)-1)}{m_1(x_1,x_2,x_3)}\frac{(m_2(x_1-1,x_2-1,x_3)-1)}{m_2(x_1-1,x_2,x_3)}\frac{(m_3(x_1-1,x_2-1,x_3-1)-1)}{m_3(x_1-1,x_2-1,x_3)}=\frac{(m_1(x_1-1,x_2-1,x_3-1)-1)}{m_1(x_1,x_2-1,x_3-1)}$$

$$\frac{(m_2(x_1,x_2-1,x_3)-1)}{m_2(x_1,x_2,x_3)}\frac{(m_3(x_1,x_2-1,x_3-1)-1)}{m_3(x_1,x_2-1,x_3)}\frac{\frac{p_1(x_2,x_3)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2,x_3)}{\alpha_1}+x_1}$$

$$\frac{\frac{p_2(x_1-1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2}\frac{\frac{p_3(x_1,x_2)-1}{\alpha_3}+x_3-1}{\frac{p_3(x_1,x_2)}{\alpha_3}+x_3}$$

$$=\frac{\frac{p_1(x_2-1,x_3-1)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2-1,x_3-1)}{\alpha_1}+x_1}\frac{\frac{p_2(x_1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_2(x_1,x_3)}{\alpha_2}+x_2}$$

$$\frac{\frac{p_3(x_1,x_2-1)-1}{\alpha_3}+x_3-1}{\frac{p_3(x_1,x_2-1)}{\alpha_3}+x_3}$$

which can be rearranged into

$$\frac{\frac{p_1(x_2,x_3)-1}{\alpha_1}+x_1-1}{\frac{p_1(x_2,x_3)}{\alpha_1}+x_1}\frac{\frac{p_1(x_2-1,x_3-1)}{\alpha_1}+x_1}{\frac{p_1(x_2-1,x_3-1)-1}{\alpha_1}+x_1-1}$$

$$=\frac{\frac{p_1(x_1,x_3)-1}{\alpha_2}+x_2-1}{\frac{p_1(x_1,x_3)}{\alpha_2}+x_2}\frac{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2}{\frac{p_2(x_1-1,x_3)}{\alpha_2}+x_2-1}$$

$$\frac{\dfrac{p_3(x_1,x_2-1)-1}{\alpha_3}+x_3-1}{\dfrac{p_3(x_1,x_2-1)}{\alpha_3}+x_3}\;\frac{\dfrac{p_3(x_1,x_2)}{\alpha_3}+x_3}{\dfrac{p_3(x_1,x_2)-1}{\alpha_3}+x_3-1}.$$

The terms on the left side are linear in x_1 and therefore the functions $p_1(x_1,x_3)$ and $p_2(x_1,x_3)$ must be linear in x_1. Similar arguments using two other equations of the same kind reveals that $p_1(x_1,x_3)$, $p_2(x_1,x_3)$ and $p_3(x_1,x_2)$ should involve only linear terms in the respective variables. This enables to write the solution of the functional equations as

$$\begin{aligned}p_1(x_2,x_3)&=a_0+a_2x_2+a_3x_3\\ p_2(x_1,x_3)&=b_0+b_1x_1+b_3x_3\\ p_3(x_1,x_2)&=c_0+c_1x_1+c_2x_2.\end{aligned}$$

Substituting these in (31) and after renaming the constants, we get

$$S(x_1,x_2,x_3)=\frac{(A_0+A_2x_2+A_3x_3)_{x_1}}{(A_0+A_2x_2+A_3x_3+A_1)_{x_1}}\frac{(B_0+B_3x_3)_{x_2}}{(B_0+B_3x_3+B_2)_{x_2}}\frac{(C_0)_{x_3}}{(C_0+C_3)_{x_3}} \tag{32}$$

$$=\frac{(B_0+B_3x_3+B_1x_1)_{x_2}}{(B_0+B_3x_3+B_1x_1+B_2)_{x_2}}\frac{(C_0+C_1x_1)_{x_3}}{(C_0+C_1x_1+C_3)_{x_3}}\frac{(A_0)_{x_1}}{(A_0+A_1)_{x_1}} \tag{33}$$

$$=\frac{(C_0+C_1x_1+C_2x_2)_{x_3}}{(C_0+C_1x_1+C_2x_2+C_3)_{x_3}}\frac{(A_0+A_2x_2)_{x_1}}{(A_0+A_2x_2+A_1)_{x_1}}\frac{(B_0)_{x_2}}{(B_0+B_2)_{x_2}}, \tag{34}$$

as required. Now assuming the above distribution for $\mathbf{X}$, we have

$$\begin{aligned}m_1(\mathbf{x})=&\frac{(A_0+A_2(x_2+1)+A_3(x_3+1)+A_1)_{x_1+1}}{(A_0+A_2(x_2+1)+A_3(x_3+1))_{x_1+1}}\\ &\qquad\sum_{t=x_1+1}^{\infty}\frac{(A_0+A_2(x_2+1)+A_3(x_3+1))_t}{(A_0+A_2(x_2+1)+A_3(x_3+1)+A_1)_t}\\ =&\frac{A_0+A_2(x_2+1)+A_3(x_3+1)+A_1+x_1}{(A_1-1)},\end{aligned}$$

on using Waring expansion

$$\frac{1}{x-a}=\frac{1}{x}+\frac{a}{x(x+1)}+\frac{a(a+1)}{x(x+1)(x+2)}+\ldots$$

Likewise

$$\sigma_1^2(\mathbf{x}) = \frac{(A_0 + A_2(x_2+1) + A_3(x_3+1) + A_1 + x_1)}{(A_1-1)(A_1-2)} (A_0 + A_2(x_2+1) + A_3(x_3+1) + x_1 + 1).$$

Thus $\sigma_1^2(\mathbf{x}) = k_1 m_1(\mathbf{x})(m_1(\mathbf{x}) - 1)$, $k_1 = \dfrac{A_1}{A_1 - 2} > 1$.
Using (33) and (34) in the same way, $k_2 = \dfrac{B_1}{B_1 - 2} > 1$ and $k_3 = \dfrac{C_1}{C_1 - 2} > 1$.

When $k_i < 1$, α is negative. The proof runs along the same lines as in the Waring case, except that in (31) the terms form a descending factorial expression resulting in a hyper geometric function. The survival function takes the form

$$S(x_1, x_2, x_3) = \frac{\binom{\alpha_0+\alpha_1+\alpha_2 x_2+\alpha_3 x_3 - x_1}{\alpha_0+\alpha_2 x_2+\alpha_3 x_3 - x_1}}{\binom{\alpha_0+\alpha_2 x_2+\alpha_3 x_3+\alpha_1}{\alpha_0+\alpha_2 x_2+\alpha_3 x_3}} \frac{\binom{\beta_0+\beta_1+\beta_3 x_3 - x_2}{\beta_0+\beta_3 x_3 - x_2}}{\binom{\beta_0+\beta_1+\beta_3 x_3}{\beta_0+\beta_3 x_3}} \frac{\binom{\delta_0+\delta_1 - x_3}{\delta_0 - x_3}}{\binom{\delta_0+\delta_1}{\delta_0}}.$$

The mean and variance residual life functions are

$$m_1(\mathbf{x}) = \frac{\alpha_0 + \alpha_1 + \alpha_2(x_2+1) + \alpha_3(x_3+1) - x_1}{\alpha_1 + 1}$$

and

$$\sigma_1^2(\mathbf{x}) = \frac{(\alpha_0 + \alpha_1 + \alpha_2(x_2+1) + \alpha_3(x_3+1) - x_1)}{(\alpha_1+1)^2(\alpha_1+2)} (\alpha_0 + \alpha_2(x_2+1) + \alpha_3(x_3+1) - x_1 - 1)$$

and hence

$$\sigma_1^2(\mathbf{x}) = k_1 m_1(\mathbf{x})(m_1(\mathbf{x}) - 1);\ k_1 = \frac{\alpha_1}{\alpha_1 + 2} < 1.$$

This completes the proof. □

REMARK 8. *The value* $k_i = 1$ *omitted in the theorem corresponds to the multivariate geometric distribution* (17).

REMARK 9. *It is evident from the theorem that the multivariate Waring distribution and negative hyper geometric distributions are characterized by a linear mean residual life function and a quadratic variance residual life function in* $x_1, x_2, ..., x_p$.

REMARK 10. *The results in Theorem 7 are more general than that of Roy (2005) in which he has taken* $p = 2$ *and* $k_i = k$; $i = 1, 2$. *When* $p = 1$, *we have the characterization of univariate Waring and negative hyper-geometric models discussed in Hitha and Nair (1989).*

5. Let $\mathbf{X}$ and $\mathbf{Y}$ be two discrete random vectors defined on $\mathbf{N}^p$ with mean residual life of the ith components as $m_{\mathbf{X}_i}(\mathbf{x})$ and $m_{\mathbf{Y}_i}(\mathbf{x})$. The corresponding variance residual lives are denoted by $\sigma_i^2(\mathbf{x})$ and $\rho_i^2(\mathbf{x})$. Then we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in multivariate mean residual life if $m_{\mathbf{X}_i}(\mathbf{x}) \leq m_{\mathbf{Y}_i}(\mathbf{x})$, for $i = 1, 2, ..., p$ and all $\mathbf{x}$ in $\mathbf{N}^p$ and is denoted by $\mathbf{X} \leq_{MMRL} \mathbf{Y}$. Similarly, we say that $\mathbf{X}$ is less than $\mathbf{Y}$ in multivariate variance residual life if $\sigma_i^2(\mathbf{x}) \leq \rho_i^2(\mathbf{x})$, for $i = 1, 2, ..., p$ and all $\mathbf{x}$ in $\mathbf{N}^p$ and is denoted by $\mathbf{X} \leq_{MVRL} \mathbf{Y}$.
From Theorem 1, we see that

$$\mathbf{X} \leq_{MMRL} \mathbf{Y} \Rightarrow \mathbf{X} \leq_{MVRL} \mathbf{Y}.$$

6. Consider the equilibrium distribution of the vector $\mathbf{X}$ considered in Section 2. Denoting the mean residual life function of $\mathbf{Y}$ as

$$\mathbf{r}(\mathbf{x}) = \left(r_1(\mathbf{x}), r_2(\mathbf{x}), ..., r_p(\mathbf{x})\right)$$

where

$$r_i(\mathbf{x}) = E[Y_i - x_i | \mathbf{Y} > \mathbf{x}];\ i = 1, 2, ..., p.$$

We see that

$$\begin{aligned} r_i(\mathbf{x}) &= \frac{1}{S_{\mathbf{Y}}(\mathbf{x}+\mathbf{e})} \sum_{t=x_1+1}^{\infty} S_{\mathbf{Y}}(t, x_2+1, ..., x_p+1) \\ &= \frac{\sum_{t=x_1+1}^{\infty} \sum_{u=t+1}^{\infty} S(u, x_2+1, ..., x_p+1)}{\sum_{t=x_1+1}^{\infty} S(t, x_2+1, ..., x_p+1)}. \end{aligned}$$

With the aid of (6) and (4),

$$\sigma_1^2(\mathbf{x}) + m_i(\mathbf{x})(m_i(\mathbf{x}) - 1) = 2r_i(\mathbf{x})(m_i(\mathbf{x}) - 1). \tag{35}$$

Writing

$$C_i^2(\mathbf{x}) = \frac{\sigma_i^2(\mathbf{x})}{m_i(\mathbf{x})(m_i(\mathbf{x}) - 1)},$$

$$r_i(\mathbf{x}) = \frac{1}{2}\left(1 + C_i^2(\mathbf{x})\right) m_i(\mathbf{x}).$$

It may be noticed that in the discrete case, $C_i^2(\mathbf{x})$ enjoy properties analogous to the coefficient of variation of the residual life when $\mathbf{X}$ is continuous. For a discussion of the role of the coefficient of variation of residual life in reliability modelling, see Gupta and Kirmani (2000) and Gupta (2006).

4. Classes of life distributions based on variance residual life

Multivariate life distributions can be classified using the behaviour of their variance residual lives. In the multivariate case, there can be different ways of defining their monotonicity and as such we have an increasing(decreasing) multivariate variance residual life class MIVRL(MDVRL) corresponding to each mode of definition. Following Zahedi (1985) and Nair and Asha (1997), four different versions of classes are studied in this section.

A discrete random vector $\mathbf{X}$ defined on $\mathbf{N}^p$ is said to be

(i) MIVRL-1(MDVRL-1) if

$$\sigma_i^2(\mathbf{x}+\mathbf{t}) \geq (\leq) \sigma_i^2(\mathbf{x})$$

for all $\mathbf{x}$ and $\mathbf{t} = (t_1, t_2, ..., t_p)$ in $\mathbf{N}^p$ and $i = 1, 2, ..., p$.

(ii) MIVRL-2(MDVRL-2) if

$$\sigma_i^2(x_1, x_2, ..., x_{i-1}, x_i + t, x_{i+1}, ...x_p) \geq (\leq) \sigma_i^2(\mathbf{x}),$$

for all $\mathbf{x}$ and $t \in \mathbf{N}$ and $i = 1, 2, ..., p$.

(iii) MIVRL-3(MDVRL-3) if

$$\sigma_i^2(x_1 + t, x_2 + t, ..., x_n + t) \geq (\leq) \sigma_i^2(\mathbf{x}),$$

for all $n \leq p$; $i = 1, 2, ..., p$ and $t \in \mathbf{N}$.

(iv) MIVRL-4(MDVRL-4) if

$$\sigma_i^2(x+t, x+t, ..., x+t) \geq (\leq) \sigma_i^2(x, x, ..., x),$$

for all $x, t \in \mathbf{N}$ and $i = 1, 2, ..., p$.

The interpretation of (i) is that the variance residual life of a p- component device where the components are of different ages increase(decrease) with different intensities. In (ii) the variance residual life increases when a working component is replaced by a younger one, whereas in (iii), the components are initially of different ages and the variance residual life is reckoned after the same time for all of them. Lastly in (iv), the variances are compared after the same time when initially they are of the same age.

From the definitions, it is easy to see that

$$\begin{aligned}&\text{MIVRL-2(MDVRL-2)} \Leftarrow \text{MIVRL-1(MDVRL-1)}\\ &\Rightarrow \text{MIVRL-3(MDVRL-3)} \Rightarrow \text{MIVRL-4(MDVRL-4)}.\end{aligned}$$

Further, MDVRL-1 and MIVRL-1 are simultaneously satisfied when $\sigma_i^2(\mathbf{x}) = k_i$, a constant independent of $\mathbf{x}$. In this case, the distribution of $\mathbf{X}$ is multivariate geometric in Remark 4. Likewise, $\mathbf{X}$ is both MIVRL-2 and MDVRL-2 if and only if $\sigma_i^2(\mathbf{x}) = p_i(\mathbf{x}_{(i)})$; $i = 1, 2, ..., p$ so that the corresponding distribution is as in (17). The multivariate geometric distribution in Remark 4 satisfies the property of being both MDVRL-3 and MIVRL-3. Finally, the $\mathbf{X}$ is both MIVRL-4 and MDVRL-4 is satisfied if and only if

$$A_i(x+t) = A_i(x),$$

where $A_i(x) = \sigma^2(x, x, ..., x)$. The above is a univariate functional equation, from which a multivariate solution is difficult to emerge. The Waring and negative hyper geometric laws are respectively MIVRL-*k* and MDVRL-*k* for $k = 1, 2, 3, 4$ so that all the classes are well defined.

Some properties of the MIVRL and MDVRL classes are given below. It may be noted that various classes based on the multivariate mean residual life can also be defined in the same manner as with $\sigma^2(\mathbf{x})$. Accordingly, we say that $\mathbf{X}$ belongs to the

(i) MIMRL-1(MDMRL-1) class if

$$m_i(\mathbf{x}+\mathbf{t}) \geq (\leq) m_i(\mathbf{x});\ i = 1, 2, ..., p.$$

(ii) MIMRL-2(MDMRL-2) class if

$$m_i(x_1, ..., x_{i-1}, x_i + t, x_{i+1}, ..., x_p) \geq (\leq) m_i(\mathbf{x});\ i = 1, 2, ..., p.$$

(iii) MIMRL-3(MDMRL-3) class if

$$m_i(x_1 + t, x_2 + t, ..., x_n + t) \geq (\leq) m_i(\mathbf{x});\ n \leq p;\ i = 1, 2, ..., p.$$

(iv) MIMRL-4(MDMRL-4) class if

$$m_i(x + t, x + t, ..., x + t) \geq (\leq) m_i(\mathbf{x});\ x, t \in \mathbf{N};\ i = 1, 2, ..., p.$$

THEOREM 11. *The random vector* $\mathbf{X}$ *is MIVRL-2(MDVRL-2) if and only if*

$$\sigma_i^2(x_1, ..., x_{i-1}, x_i + 1, ..., x_p) \geq (\leq) m_i(x_1, x_2, ..., x_{i-1}, x_i + 1, x_{i+1}, ..., x_p)(m_i(\mathbf{x}) - 1)$$

for $i = 1, 2, ..., p$.

PROOF. We have from (11)

$$\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-\sigma_1^2(\mathbf{x})=\frac{m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})+1}{m_1(x_1+1,\mathbf{x}_{(1)})}$$
$$\left[\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x}-1))\right]. \quad (36)$$

Since the above is an identity, $\mathbf{X}$ is MIVRL if and only if

$$\sigma_1^2(x_1+1,\mathbf{x}_{(1)})\geq m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1).$$

The proof of $i=2,3,\ldots,p$ is similar. □

THEOREM 12. $\mathbf{X}$ *is MIMRL-2(MDMRL-2)*$\Rightarrow$ $\mathbf{X}$ *is MIVRL-2(MDVRL-2).*

PROOF. Using (5), we write

$$\sigma_1^2(x_1+1,\mathbf{x}_{(1)})-m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1)=\frac{2}{S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}$$
$$\sum_{t=x_1+2}^{\infty}\sum_{u=t+1}^{\infty}S(u,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})-m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)})-1)$$
$$-m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1)$$

$$=\frac{2}{S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty}m_1(t,\mathbf{x}_{(1)})S(t+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})-m_1(x_1+1,\mathbf{x}_{(1)})$$
$$\left[m_1(x_1+1,\mathbf{x}_{(1)})+m_1(\mathbf{x})-2\right]$$

$$=\frac{2}{S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty}\left[m_1(t,\mathbf{x}_{(1)})-m_1(x_1+1,\mathbf{x}_{(1)})\right]$$
$$+\frac{2m_1(x_1+1,\mathbf{x}_{(1)})}{S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\left[\sum_{t=x_1+2}^{\infty}S(t,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})-S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})\right]$$
$$-m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)})+m_1(\mathbf{x})-2)$$

$$=\frac{2}{S(x_1+2,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})}\sum_{t=x_1+2}^{\infty}\left[m_1(t,\mathbf{x}_{(1)})-m_1(x_1+1,\mathbf{x}_{(1)})\right]S(t+1,\mathbf{x}_{(1)}+\mathbf{e}_{p-1})$$
$$+m_1(x_1+1,\mathbf{x}_{(1)})(m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})).$$

When $\mathbf{X}$ is MDMRL-2, $m_1(t,\mathbf{x}_{(1)}) \leq m_1(x_1+1,\mathbf{x}_{(1)})$ for all $t \geq x_1+2$ and also $m_1(x_1+1,\mathbf{x}_{(1)}) \leq m_1(\mathbf{x})$. Hence the expression on the right is negative and hence by Theorem 1, $\mathbf{X}$ is MDVRL-2. The case of $i=2,3,\ldots$ is similar and so is the case of MIVRL-2. □

The above result gives only a sufficient condition for $\mathbf{X}$ to be MIVRL-2, besides being the implication among the MMRL and MVRL classes. A stronger result is presented in the next theorem.

THEOREM 13. *Suppose that $S(\mathbf{x})$ is strictly decreasing. Then $\mathbf{X}$ is MIVRL-2(MDVRL-2) if and only if the vector $\mathbf{Y}$ is MIMRL-2(MDMRL-2).*

PROOF. Using (35), we can write for $i=1$

$$2\Big(r_1(x_1+1,\mathbf{x}_{(1)})-r_1(\mathbf{x})\Big)=2\left[\frac{\sigma^2(x_1+1,\mathbf{x}_{(1)})+m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})}{2m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})}-1\right]$$

$$=\frac{\sigma_1^2(x_1+1,\mathbf{x}_{(1)})}{m_1(x_1+1,\mathbf{x}_{(1)})m_1(x_1+1,\mathbf{x}_{(1)})}-1. \tag{37}$$

Hence

$$\frac{\sigma_1^2(x_1+1,\mathbf{x}_{(1)})}{m_1(x_1+1,\mathbf{x}_{(1)})(m_1(\mathbf{x})-1)} = \frac{\Big[m_1(x_1+1,\mathbf{x}_{(1)})-1\Big]\Big[1+r_1(x_1+1,\mathbf{x}_{(1)})-r_1(\mathbf{x})\Big]}{m_1(\mathbf{x})-1}.$$

Also from (37)

$$\frac{\sigma_1^2(\mathbf{x})}{m_1(\mathbf{x})(m_1(\mathbf{x})-1)}=2\Big[r_1(\mathbf{x})-r_1(x_1-1,\mathbf{x}_{(1)})+1\Big].$$

The last two equations provide

$$\sigma_1^2(\mathbf{x})-\sigma_1^2(x_1+1,\mathbf{x}_{(1)})=\Big(m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})+1\Big)\Big[m_1(\mathbf{x})-m_1(x_1+1,\mathbf{x}_{(1)})\Big]\Big[1+r_1(x_1+1,\mathbf{x}_{(1)})-r_1(\mathbf{x})\Big]$$

$$=\frac{\Big[m_1(x_1+1,\mathbf{x}_{(1)})-m_1(\mathbf{x})+1\Big]}{1+r_1(\mathbf{x})-r_1(x_1-1,\mathbf{x}_{(1)})}\Big[r_1(x_1-1,\mathbf{x}_{(1)})-r_1(\mathbf{x})\Big].$$

On using the identity

$$m_1(\mathbf{x})=r_1(\mathbf{x})\Big[1+r_1(\mathbf{x})-r_1(x_1-1,\mathbf{x}_{(1)})\Big],$$

further simplification yields

$$\sigma_1^2(\mathbf{x}) - \sigma_1^2(x_1+1,\mathbf{x}_{(1)}) = \Big(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\Big)(m_1(\mathbf{x}) - r_1(\mathbf{x}))$$

$$= m_1(\mathbf{x})\Big(m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1\Big)\Big(r_1(x_1-1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})\Big). \tag{38}$$

By (2), $m_1(x_1+1,\mathbf{x}_{(1)}) - m_1(\mathbf{x}) + 1 > 0$ since $S(\mathbf{x})$ is strictly decreasing. Moreover the sign of the left side of (38) is the same as that of $r_1(x_1-1,\mathbf{x}_{(1)}) - r_1(\mathbf{x})$. This proves the assertion for $i = 1$. The same method applies to $i = 2, 3, \ldots, p$. □

REMARK 14. *Equation* (38) *reveals that* $\mathbf{X}$ *is MIVRL-2(MDVRL-2) if and only if*

$$m_i(\mathbf{x}) \geq (\leq) r_i(\mathbf{x}).$$

This is also equivalent to the statement

$$\mathbf{X} \text{ is MIVRL-2(MDVRL-2)} \iff \mathbf{X} \geq_{MMRL} (\leq_{MMRL}) \mathbf{Y}.$$

The above result helps us to provide bounds on the variance residual life as stated in the following theorem.

THEOREM 15. *(a) If* (X_1, X_2) *is MIVRL-2(MDVRL-2), then VRL components have the lower (upper) bounds as follows.*

$$\sigma_1^2(x_1,x_2) \geq (\leq) m_1(x_1,x_2)[m_1(x_1,x_2-1) - 1] \tag{39}$$

$$\sigma_2^2(x_1,x_2) \geq (\leq) m_2(x_1,x_2)[m_2(x_1-1,x_2) - 1]. \tag{40}$$

PROOF. The proof follows from Theorem 11 by noting that the left side of (36) is positive when (X_1, X_2) is MIVRL-2 and negative when MDVRL-2. Now change x_1+1 to x_1 for (a) and x_2+1 to x_2 for (b). □

Some other properties of the MIVRL-2 class are given below.

1. MDVRL-2 class is not closed under mixing. To see this, let $(\sigma_1^2(\mathbf{x}), \sigma_2^2(\mathbf{x}), \ldots, \sigma_p^2(\mathbf{x}))$ be the variance residual life of a mixture of two life distributions. Assume that the mixture is MDVRL-2. Then $\sigma_1^2(\mathbf{x})$ is decreasing in x_1 for all $x_2, \ldots, x_p$ and hence $\sigma_1^2(x_1)$ of X_1 is decreasing. But, in general univariate VRL is not closed under the formation of mixtures. Hence $\sigma_1^2(\mathbf{x})$ is not decreasing. Hence $\mathbf{X}$ is not DVRL.

2. Both MDVRL-2 and MIVRL-2 classes are not closed under convolutions. The proof is similar to previous one and hence omitted.

In the absence of multivariate discrete concepts corresponding to their univariate counterparts, extension of some of the univariate results to the multivariate case is still open.

5. Conclusion

In this paper, we have presented several properties of the different versions of multivariate variance residual life in discrete time. Also the classification of life distributions based on the monotonicity of the concept were discussed. It is hoped that the results will be useful in modelling and analysis of discrete multivariate data, which is not much seen in reliability literature. Inference procedure for estimating the variance residual life is a problem to be discussed in this context. Some attempts are being made in this direction and will be reported in a separate work.

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SUMMARY

Among various characteristics of residual life, the concept of variance residual life in the univariate case has been extensively discussed in reliability literature. In the present work we extend this notion to the discrete multivariate case and study its properties. Different versions of classes of multivariate distributions based on the monotonicity of variance residual life are also presented along with some characterizations.

Keywords: Multivariate variance residual life; Geometric, Waring and negative hyper geometric distributions; Increasing (decreasing) variance residual life classes; Multivariate equilibrium models.