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ON A GENERALIZED LINDLEY DISTRIBUTION

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1. INTRODUCTION

The Lindley distribution due to Lindley (1958) has been studied and generalized by many authors in recent years. The developments include Varghese and Vaidyanathan (2016) providing an analysis of simple step-stress accelerated life test data from Lindley distribution under type-I censoring; Shibu and Irshad (2016) and Maya and Irshad (2017) proposing a generalization referred to as the new extended generalized Lindley distribution. Nedjar and Zeghdoudi (2016) studying various mathematical properties and simulations of a distribution proposed in Zeghdoudi and Nedjar (2015).

The distribution studied by Nedjar and Zeghdoudi (2016) is referred to as the gamma Lindley (GaL) distribution. It has the probability density function specified by

$$f_1(x) = \frac{\theta^2 \left[(\beta + \beta\theta - \theta)x + 1 \right] e^{-\theta x}}{\beta(1+\theta)} \tag{1}$$

for $x > 0$, $\theta > 0$ and $\beta > 0$. The properties of this distribution studied by Nedjar and Zeghdoudi (2016) include quantile function, Lorenz curve, entropies, limiting distributions of extreme order statistics, estimation by method of moments and estimation by maximum likelihood method. Nedjar and Zeghdoudi (2016) also presented simulation studies to assess the performance of the estimators and a real data application.

We would like to point out that the GaL distribution is a particular case of or the same as at least two known distributions. Firstly, consider the extended generalized Lindley distribution due to Torabi *et al.* (2014) specified by the probability density function

$$f_2(x) = \frac{e^{-\theta x^{\beta}} \beta \left[\alpha x^{\alpha\beta} + \gamma x^{(\alpha+1)\beta} \right] \theta^{\alpha+1}}{x(\gamma+\theta)\Gamma(\alpha+1)} \tag{2}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$, $\theta > 0$ and $\gamma > 0$. Clearly, (1) is the particular case of (2) for $\alpha = 1$, $\beta = 1$ and $\gamma = \beta + \beta\theta - \theta$.

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Secondly, consider the two-parameter Lindley distribution due to Shanker *et al.* (2013) specified by the probability density function

$$f_3(x) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x} \tag{3}$$

for $x > 0$, $\theta > 0$ and $\alpha > -\theta$. Clearly, (1) is the same as (3) after reparametrization.

Most of the mathematical properties studied by (including properties not studied by) Nedjar and Zeghdoudi (2016) have already been studied by Shanker *et al.* (2013) and Torabi *et al.* (2014). Shanker *et al.* (2013) studied moments, related measures, failure rate function, mean residual life function, stochastic orderings, estimation by method of moments and estimation by maximum likelihood method. Shanker *et al.* (2013) also presented four real data applications. Torabi *et al.* (2014) studied shapes of the probability density and hazard rate functions, stochastic orderings, moments, moment generating function, characteristic function, mean residual life function, scaled total time function, Lorenz curve, Bonferroni curve, bivariate generalizations, estimation by maximum likelihood method and estimation by minimum spacing distance method. Torabi *et al.* (2014) also presented simulation studies to assess the performance of the estimators and a real data application.

In the rest of this note, we point out that most of the results presented in Nedjar and Zeghdoudi (2016) are either incorrect or not useful. In Section 2 below, we point out problems with results on quantiles given in Section 2 of Nedjar and Zeghdoudi (2016). In Section 3 below, we point out problems with results on the Lorenz curve given in Section 3 of Nedjar and Zeghdoudi (2016). In Section 4 below, we point out problems with results on the Rényi entropy given in Section 5 of Nedjar and Zeghdoudi (2016). In Section 5 below, we point out problems with maximum likelihood estimation presented in Section 6 of Nedjar and Zeghdoudi (2016). In Section 6 below, we point out problems with simulation results presented in Section 8 of Nedjar and Zeghdoudi (2016). In Section 7 below, we point out problems with real data applications presented in Section 8 of Nedjar and Zeghdoudi (2016).

2. Quantiles

The quantiles of the GaL distribution are the solutions of the equation

$$1 - \frac{[(\beta+\beta\theta-\theta)(\theta x+1)+\theta]\,e^{-\theta x}}{\beta(1+\theta)} = p, \tag{4}$$

see equation (3) in Nedjar and Zeghdoudi (2016).

Tables 1 and 2 in Nedjar and Zeghdoudi (2016) provided tabulations of the solutions of (4) for various values of θ, β and p. However, some of the values given in these tables appear inaccurate or completely incorrect.

We recomputed the solutions of (4) for the combinations considered in Nedjar and Zeghdoudi (2016). The solutions were computed using Maple, which allows for arbitrary precision. The solutions are given in Tables 1 and 2 below.

The quantile values given in Table 1 of Nedjar and Zeghdoudi (2016) for $\theta = 0.05$, $\beta = 1$ appear completely incorrect. Also many of the values reported in Table 2 of Nedjar and Zeghdoudi (2016) are inaccurate, for example, when $(p, \theta, \beta) =$

$(0.99, 0.1, 0.1)$, $(p, \theta, \beta) = (0.01, 0.1, 0.5)$, $(p, \theta, \beta) = (0.01, 3, 1)$, $(p, \theta, \beta) = (0.15, 3, 1)$, $(p, \theta, \beta) = (0.3, 3, 1)$, $(p, \theta, \beta) = (0.35, 3, 1)$, $(p, \theta, \beta) = (0.6, 3, 1)$, $(p, \theta, \beta) = (0.65, 3, 1)$, $(p, \theta, \beta) = (0.75, 3, 1)$, $(p, \theta, \beta) = (0.8, 3, 1)$, $(p, \theta, \beta) = (0.05, 5, 1)$, $(p, \theta, \beta) = (0.2, 5, 1)$, $(p, \theta, \beta) = (0.25, 5, 1)$, $(p, \theta, \beta) = (0.3, 5, 1)$, $(p, \theta, \beta) = (0.45, 5, 1)$, $(p, \theta, \beta) = (0.8, 5, 1)$, $(p, \theta, \beta) = (0.85, 5, 1)$ and $(p, \theta, \beta) = (0.9, 5, 1)$.

TABLE 1
Q_1, Q_2 *and* Q_3 *of the GaL distribution.*

Parameters	$p = 0.25$	$p = 0.5$	$p = 0.75$
$\theta = 0.01$, $\beta = 0.1$	86.284	157.743	259.046
$\theta = 0.1$, $\beta = 0.5$	7.825	14.904	24.999
$\theta = 0.05$, $\beta = 1$	18.275	32.606	52.886

TABLE 2
Quantiles of the GaL distribution.

p	$\theta = 0.1$, $\beta = 0.1$	$\theta = 0.1$, $\beta = 0.5$	$\theta = 3$, $\beta = 1$	$\theta = 5$, $\beta = 1$
0.01	0.11055	0.50675	0.00447	0.00241
0.05	0.56408	2.09502	0.02273	0.01230
0.10	1.15838	3.69516	0.04656	0.02523
0.15	1.78629	5.12962	0.07160	0.03888
0.20	2.45189	6.49139	0.09802	0.05335
0.25	3.16005	7.82533	0.12600	0.06869
0.30	3.91662	9.15961	0.15572	0.08504
0.35	4.72873	10.51554	0.18745	0.10256
0.40	5.60526	11.91199	0.22151	0.12145
0.45	6.55739	13.36786	0.25831	0.14190
0.50	7.59952	14.90389	0.29836	0.16429
0.55	8.75052	16.54484	0.34235	0.18895
0.60	10.03602	18.32216	0.39118	0.21642
0.65	11.49189	20.27821	0.44613	0.24746
0.70	13.17063	22.47326	0.50913	0.28317
0.75	15.15358	24.99924	0.58300	0.32522
0.80	17.57676	28.00798	0.67264	0.37647
0.85	20.69504	31.78135	0.78699	0.44215
0.90	25.07970	36.94570	0.94631	0.53418
0.95	32.54941	45.47726	1.21442	0.68999
0.99	49.78520	64.40922	1.82223	1.04644

Furthermore, Theorem 1 in Nedjar and Zeghdoudi (2016) gave an explicit expression for the solution of (4) in terms of the Lambert W distribution. This theorem appears incorrect because the Lambert W distribution is defined as the solution of $W(p)e^{W(p)} = p$ and (4) cannot be expressed in this form.

3. Lorenz curve

In Section 3 of Nedjar and Zeghdoudi (2016), they derived an expression for the Lorenz curve Lorenz (1905) of a GaL random variable. But the expression appears incorrect.

For a random variable X having the probability density function given by (1),

$$F(x)E[X \mid X \leq x] = \frac{2\beta(1+\theta)-\theta}{\theta\beta(1+\theta)} - \frac{\theta^2 e^{-\theta x}}{\beta(1+\theta)} \left[\frac{(\beta+\beta\theta-\theta)\left(2+2\theta x+\theta^2 x^2\right)}{\theta^3} + \frac{1+\theta x}{\theta^2}\right].$$

Hence, it follows from the definition of the Lorenz curve,

$$L(p) = \frac{F(x)E\left[X \mid X \leq F^{-1}(p)\right]}{E(X)},$$

that the correct expression for the Lorenz curve of a GaL random variable is

$$L(p) = 1 - \frac{\theta^3 e^{-\theta x}}{2\beta(1+\theta)-\theta} \left[\frac{(\beta+\beta\theta-\theta)\left(2+2\theta x+\theta^2 x^2\right)}{\theta^3} + \frac{1+\theta x}{\theta^2}\right],$$

where $x = F^{-1}(p)$.

4. RÉNYI ENTROPY

In Section 5 of Nedjar and Zeghdoudi (2016), they derived an expression for the Rényi entropy Renyi (1961) of a GaL random variable. But the expression appears incorrect.

For the probability density function given by (1),

$$\int_0^\infty f^\gamma(x)dx = \frac{\theta^{2\gamma-1} e^a \Gamma(\gamma+1, a)}{\gamma\beta^\gamma(1+\theta)^\gamma a^\gamma},$$

where $a = \gamma\theta/(\beta+\beta\theta-\theta)$ and

$$\Gamma(a, x) = \int_x^\infty t^{a-1} e^{-t} dt$$

denotes the incomplete gamma function. Hence, it follows from the definition of the Rényi entropy,

$$J(\gamma) = \frac{1}{1-\gamma} \log\left[\int_0^\infty f^\gamma(x)dx\right],$$

that the correct expression for the Rényi entropy of a GaL random variable is

$$J(\gamma) = \frac{1}{1-\gamma} \log\left[\frac{\theta^{2\gamma-1} e^a \Gamma(\gamma+1, a)}{\gamma\beta^\gamma(1+\theta)^\gamma a^\gamma}\right].$$

5. MAXIMUM LIKELIHOOD ESTIMATION

In Section 6 of Nedjar and Zeghdoudi (2016), they considered maximum likelihood estimation of the parameters of the GaL distribution. They derived maximum likelihood estimates based on having a single observation from the distribution. These results have no practical use because data sets are usually of size greater than one.

Furthermore, some of the results in Section 6 of Nedjar and Zeghdoudi (2016) do not appear correct. For example, according to their equation (23),

$$E\left(\frac{1}{X}\right)=\frac{2\beta(1+\theta)-\theta}{\theta\beta(1+\theta)}$$

and

$$E\left(\frac{1}{1+X}\right)=\frac{e^{\theta}}{\beta(\theta+1)}\left(2\beta+\beta\theta-\theta-\beta\theta^2+\theta^2\right),$$

where X is a GaL random variable. Actually,

$$E\left(\frac{1}{X}\right)=\infty$$

and

$$E\left(\frac{1}{1+X}\right)=\frac{\theta(\beta+\beta\theta-\theta)}{\beta(\theta+1)}+\frac{\theta^2 e^{\theta}(1-\beta-\beta\theta+\theta)}{\beta(\theta+1)}\Gamma(0,\theta).$$

In the rest of this section, we derive the maximum likelihood estimates of β and θ when $x_1, x_2, \ldots, x_n$ is a random sample from the GaL distribution. The log-likelihood function is

$$\begin{aligned}\log L(\theta,\beta)=2n\log\theta+&\sum_{i=1}^{n}\log[(\beta+\beta\theta-\theta)x_i+1]-\theta\sum_{i=1}^{n}x_i\\&-n\log\beta-n\log(1+\theta).\end{aligned}$$

The normal equations for the two parameters are:

$$\begin{aligned}\frac{\partial\log L(\theta,\beta)}{\partial\theta}&=\frac{2n}{\theta}+(\beta-1)\sum_{i=1}^{n}\frac{x_i}{(\beta+\beta\theta-\theta)x_i+1}-\sum_{i=1}^{n}x_i-\frac{n}{1+\theta}=0,\\\frac{\partial\log L(\theta,\beta)}{\partial\beta}&=(1+\theta)\sum_{i=1}^{n}\frac{x_i}{(\beta+\beta\theta-\theta)x_i+1}-\frac{n}{\beta}=0.\end{aligned}$$

Rearranging these equations, we can see that the maximum likelihood estimators for θ and β are

$$\widehat{\theta}=\frac{1}{\overline{x}},\quad\widehat{\beta}=\frac{1}{1+\overline{x}},$$

where $\overline{x}$ denotes the sample mean. The variances and the covariance of these estimators can be based on the observed information matrix. Simple calculations show that

$$\begin{aligned}
\mathrm{Var}\left(\widehat{\theta}\right) &\approx \frac{I_{22}}{I_{11}I_{22}-I_{12}^2}, \\
\mathrm{Var}\left(\widehat{\beta}\right) &\approx \frac{I_{11}}{I_{11}I_{22}-I_{12}^2}, \\
\mathrm{Cov}\left(\widehat{\theta},\widehat{\beta}\right) &\approx -\frac{I_{12}}{I_{11}I_{22}-I_{12}^2},
\end{aligned}$$

where

$$\begin{aligned}
I_{11} &= \frac{2n}{\widehat{\theta}^2} - \frac{n}{\left(1+\widehat{\theta}\right)^2} + \left(\widehat{\beta}-1\right)^2 \sum_{i=1}^{n} x_i^2, \\
I_{12} &= -\sum_{i=1}^{n} x_i + \left(\widehat{\beta}-1\right)\left(1+\widehat{\theta}\right) \sum_{i=1}^{n} x_i^2, \\
I_{22} &= -\frac{n}{\widehat{\beta}^2} + \left(1+\widehat{\theta}\right)^2 \sum_{i=1}^{n} x_i^2.
\end{aligned}$$

6. Simulation study

Since the maximum likelihood estimates given in Section 6 of Nedjar and Zeghdoudi (2016) are for single observations the simulations presented in their Section 8 may not be correct. Here, we recalculate their simulation results using the results in Section 5. Tables 3 to 6 give the biases and mean squared errors of $\widehat{\theta}$ and $\widehat{\beta}$ for the combinations of parameter values and sample sizes considered in Nedjar and Zeghdoudi (2016). The simulation scheme described in Nedjar and Zeghdoudi (2016) was used for computing the biases and mean squared errors.

TABLE 3
Biases of $\widehat{\theta}$.

Parameters	$n=10$	$n=30$	$n=50$
$\theta=1, \beta=6$	-0.444	-0.465	-0.473
$\theta=1, \beta=0.1$	8.043	7.776	7.776
$\theta=1, \beta=0.75$	-0.183	-0.223	-0.233
$\theta=0.1, \beta=1$	-0.044	-0.047	-0.047
$\theta=0.5, \beta=1$	-0.180	-0.196	-0.196
$\theta=3, \beta=1$	-0.388	-0.518	-0.540
$\theta=3, \beta=0.5$	2.772	2.419	2.372
$\theta=0.5, \beta=0.5$	-0.087	-0.113	-0.119
$\theta=0.1, \beta=0.5$	-0.042	-0.044	-0.044

The biases and mean squared errors reported in Tables 3 to 6 appear substantially different from those in Tables 3 and 4 of Nedjar and Zeghdoudi (2016).

TABLE 4
Biases of $\widehat{\beta}$.

Parameters	$n=10$	$n=30$	$n=50$
$\theta=1, \beta=6$	-5.651	-5.655	-5.655
$\theta=1, \beta=0.1$	0.797	0.797	0.797
$\theta=1, \beta=0.75$	-0.309	-0.318	-0.318
$\theta=0.1, \beta=1$	-0.947	-0.949	-0.950
$\theta=0.5, \beta=1$	-0.761	-0.765	-0.768
$\theta=3, \beta=1$	-0.290	-0.290	-0.292
$\theta=3, \beta=0.5$	0.342	0.341	0.341
$\theta=0.5, \beta=0.5$	-0.213	-0.224	-0.224
$\theta=0.1, \beta=0.5$	-0.445	-0.447	-0.447

TABLE 5
Mean squared errors of $\widehat{\theta}$.

Parameters	$n=10$	$n=30$	$n=50$
$\theta=1, \beta=6$	0.217	0.226	0.229
$\theta=1, \beta=0.1$	70.804	63.123	60.593
$\theta=1, \beta=0.75$	0.105	0.069	0.067
$\theta=0.1, \beta=1$	0.002	0.002	0.002
$\theta=0.5, \beta=1$	0.039	0.040	0.039
$\theta=3, \beta=1$	0.915	0.485	0.431
$\theta=3, \beta=0.5$	10.880	6.728	5.990
$\theta=0.5, \beta=0.5$	0.026	0.018	0.017
$\theta=0.1, \beta=0.5$	0.002	0.002	0.002

TABLE 6
Mean squared errors of $\widehat{\beta}$.

Parameters	$n=10$	$n=30$	$n=50$
$\theta=1, \beta=6$	31.904	31.973	31.996
$\theta=1, \beta=0.1$	0.636	0.635	0.635
$\theta=1, \beta=0.75$	0.103	0.102	0.103
$\theta=0.1, \beta=1$	0.899	0.901	0.902
$\theta=0.5, \beta=1$	0.579	0.587	0.591
$\theta=3, \beta=1$	0.088	0.086	0.086
$\theta=3, \beta=0.5$	0.119	0.117	0.117
$\theta=0.5, \beta=0.5$	0.049	0.050	0.051
$\theta=0.1, \beta=0.5$	0.198	0.200	0.200

7. Real data applications

Since the maximum likelihood estimates given in Section 6 of Nedjar and Zeghdoudi (2016) are for single observations the real data applications presented in their Section 8 may also not be correct.

Nedjar and Zeghdoudi (2016) considered two real data sets. For the first data 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2, the maximum likelihood estimates obtained using Section 5 are $\widehat{\theta} = 0.036$, $\widehat{\beta} = 0.035$ with the log-likelihood equal to -64.738. For the second data 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653, the maximum likelihood estimates obtained using Section 5 are $\widehat{\theta} = 0.006$, $\widehat{\beta} = 0.006$ with the log-likelihood equal to -154.590. These numbers are very different from those given in Section 8 of Nedjar and Zeghdoudi (2016). Hence, the GaL distribution may not give the best fit to the two data sets, as Nedjar and Zeghdoudi (2016) claimed.

These two real data sets are also considered in Shanker *et al.* (2017). Hence, the GaL distribution, a reparameterization of that due to Shanker *et al.* (2013), cannot be supposed to give best fits for these data sets.

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Summary

Nedjar and Zeghdoudi (2016) studied various properties of a distribution which they claim to have introduced. Here, we point out that the distribution is a particular case of at least two known distributions. Most of the properties derived by Nedjar and Zeghdoudi are already known. We also point out that most of the results presented by Nedjar and Zeghdoudi are either incorrect or not useful.

Keywords: Gamma Lindley distribution; Lindley distribution; Mathematical properties.