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# NEW EXTENDED GENERALIZED LINDLEY DISTRIBUTION: PROPERTIES AND APPLICATIONS

R. Maya [1]
*Department of Statistics, University College, Trivandrum -695 034, India.*
M. R. Irshad
*Department of Statistics, University College, Trivandrum -695 034, India.*

## 1. Introduction

Lindley (1958) suggested an one parameter distribution to illustrate the diffference between fiducial distribution and posterior distribution and has the following probability density function (pdf),

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x > 0,\ \theta > 0. \tag{1}$$

Ghitany *et al.* (2008) developed different properties of Lindley distribution and showed that the Lindley distribution fits better than the exponential distribution based on the waiting times before service of the bank customers. Sankaran (1970) used Lindley distribution as the mixing distribution of a Poisson parameter and the resulting distribution is known as the Poisson-Lindley distributions. Zakerzadeh and Dolati (2009) have obtained a generalized Lindley distribution and discussed its various properties and applications. Ghitany *et al.* (2013) and Nadarajah *et al.* (2011) have recently proposed two parameter extensions of the Lindley distribution named as the generalized Lindley and power Lindley distributions respectively. A discrete form of Lindley distribution was introduced by Gomez, Ojeda (2011) by discretizing the continuous Lindley distribution. Ali *et al.* (2013) have considered Bayesian analysis of the Lindley model via informative and non-informative priors under different loss functions. Elbatal, Elgarhy (2013) have investigated most of the statistical properties of the transmuted quasi-Lindley distribution, and have obtained the maximum likelihood estimates of the parameters. Location parameter extension of Lindley distribution is extensively discussed by Monsef (2015). Kadilar, Cakmakyapan (2016) introduced in the literature, the Lindley family of distributions. Nedjar and Zehdoudi (2016) introduced gamma Lindley distribution and studied some important properties of their proposed generalization.
The aim of this paper is to introduce a new extended generalized Lindley distribution ($NEGLD$) by mixing two gamma distributions and is given in section 2.

[1] Corresponding Author. E-mail: mayasekharan@gmail.com

In section 3, we briefly discuss some of the statistical and reliability properties of $NEGLD$. In section 4, we estimate the unknown parameters by using maximum likelihood estimation (MLE) and method of moment estimation (MME) and find out the asymptotic confidence interval of the parameters. Application of the introduced model is discussed in section 5, and it is shown that the $NEGLD$ model fits better than the other models.

## 2. Proposed model

DEFINITION 1. *Let X be a non negative random variable obtained from the mixture of two gamma distributions, namely gamma* $(\alpha, \theta)$ *and gamma* $(\alpha$-1, $\theta)$ *with mixing probabilities* $p_1 = \frac{\theta k}{\eta^\delta + \theta k}$ *and* $p_2 = \frac{\eta^\delta}{\eta^\delta + \theta k}$ *respectively, the corresponding pdf has the form*

$$f(x;\theta,\alpha,k,\eta,\delta) = \frac{\theta^2}{\eta^\delta + \theta\ k}\left\{\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\delta(\theta x)^{\alpha-2}}{\theta\Gamma(\alpha-1)}\right\}e^{-\theta x}; x > 0, \quad (2)$$

$\theta > 0$, $\alpha > 1$, $\delta > 0$, $k \geq 0$, $\eta \geq 0$, *subject to k and* $\eta$ *are not allowed to be simultaneously zeros. We call this model as new extended generalized Lindley distribution (NEGLD).*

The cumulative distribution function (cdf) of $NEGLD$ is given by

$$F(x;\theta,\alpha,k,\eta,\delta) = \frac{1}{(\eta^\delta + \theta k)}\Big\{\theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_{\alpha-1}(\theta x)\Big\}, \quad (3)$$

where

$$\gamma(a,b) = \int_0^b t^{a-1}e^{-t}dt$$

is the lower incomplete gamma function and

$$\gamma_a(b) = \frac{\gamma(a,b)}{\Gamma(a)}.$$

The survival function associated with (3) is obtained as

$$\overline{F}(x;\theta,\alpha,k,\eta,\delta) = 1 - \frac{1}{(\eta^\delta + \theta k)}\Big\{\theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_{\alpha-1}(\theta x)\Big\}. \quad (4)$$

Figures 1-5 illustrate some of the possible shapes of the density function of $NEGLD$ for selected values of the parameters $\theta, \alpha, k, \eta$ and $\delta$.

*Figure 1* – $\alpha = 2, k = 8, \eta = 3, \delta = 1.5$

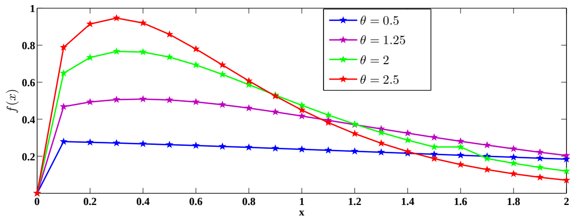



*Figure 2* – $\theta = 2, k = 2, \eta = 5, \delta = 1.5$

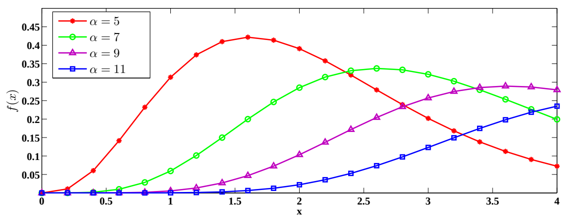



*Figure 3* – $\theta = 3, \alpha = 5, \eta = 6, \delta = 0.5$

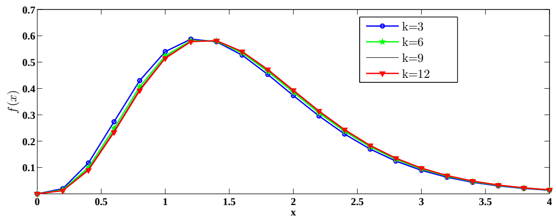



*Figure 4* – $\theta = 3, \alpha = 5, k = 9, \delta = 3.5$

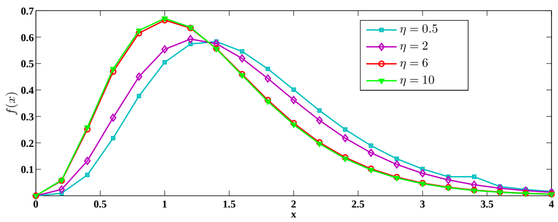
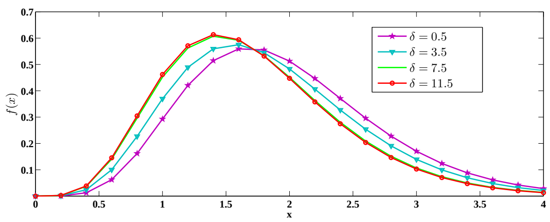



*Figure 5* – $\theta = 3.5, \alpha = 7, k = 3, \eta = 2$



## 3. Properties of the new extended generalized Lindley distribution

### 3.1. Moments and recurrence relation

If $X$ has the $NEGLD$ with density function given in (2), then the $r^{th}$ raw moment $\mu_r^{'}$ about origin is given by

$$\begin{aligned}\mu_r^{'} &= E(X^r)\\ &= \int_0^\infty x^r f(x;\theta,\alpha,k,\eta,\delta)dx \\ &= \frac{\Gamma(\alpha+r-1)}{(\eta^\delta+\theta k)\Gamma(\alpha)}\left\{\frac{(\alpha-1)(\eta^\delta+\theta k)+\theta k r}{\theta^r}\right\}.\end{aligned} \tag{5}$$

Thus the mean, variance and coefficient of variation of $NEGL$ random variable are respectively given by

$$\mu = \frac{\alpha}{\theta} - \frac{\eta^\delta}{\theta(\eta^\delta+\theta k)}, \tag{6}$$

$$\sigma = \frac{1}{\theta^2(\eta^\delta+\theta k)^2}\left\{(\alpha-1)\eta^{2\delta}+\alpha\theta k(\theta k+2\eta^\delta)\right\} \tag{7}$$

and

$$\zeta = \frac{\sqrt{(\alpha-1)\eta^{2\delta}+\alpha\theta k(\theta k+2\eta^\delta)}}{\alpha(\eta^\delta+\theta k)-\eta^\delta}\times 100. \tag{8}$$

It can be easily seen that $\mu_r^{'}$ satisfies the recursive property through the relationship

$$\mu_{r+1}^{'} = \frac{(\alpha+r-1)}{\theta}\mu_r^{'} + \frac{k\Gamma(\alpha+r)}{\theta^r(\eta^\delta+\theta k)\Gamma(\alpha)}, \tag{9}$$

which is immediate from (5).

### 3.2. Characteristic function

If $X$ has the $NEGLD$, then the characteristic function of $X$ is given by

$$\begin{aligned}\phi_X(t) = E(e^{itX}) &= \int_0^\infty e^{itx} f(x;\theta,\alpha,k,\eta,\delta)dx \\ &= \frac{1}{\eta^\delta+\theta k}\left\{\sum_{j=0}^{\infty}(-1)^j\binom{-\alpha}{j}\left(\frac{it}{\theta}\right)^j\left\{\theta k+\eta^\delta\left(1-\frac{it}{\theta}\right)\right\}\right\}.\end{aligned} \tag{10}$$

*3.3. Conditional moments*

The $r^{th}$ conditional moment of the $NEGL$ distribution is given by

$$M_r^{'} = E(X^r|X > x) = \frac{\int_x^\infty t^r f(t;\theta,\alpha,k,\eta,\delta)dt}{\overline{F}(x;\theta,\alpha,k,\eta,\delta)}.$$

Now

$$\int_x^\infty t^r f(t;\theta,\alpha,k,\eta,\delta)dt = \frac{1}{(\eta^\delta+\theta k)}\left\{\frac{k}{\Gamma(\alpha)\theta^{r-1}}\Gamma(\alpha+r,\theta x)+\frac{\eta^\delta}{\Gamma(\alpha-1)\theta^r}\Gamma(\alpha+r-1,\theta x)\right\}.$$

Therefore

$$M_r^{'} = \frac{\frac{k}{\Gamma(\alpha)\theta^{r-1}}\Gamma(\alpha+r,\theta x)+\frac{\eta^\delta}{\Gamma(\alpha-1)\theta^r}\Gamma(\alpha+r-1,\theta x)}{\eta^\delta+\theta k-\left\{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_{\alpha-1}(\theta x)\right\}}. \tag{11}$$

*3.4. Hazard rate function*

Let $X$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$, then the hazard rate function is given by $h(x) = \frac{f(x)}{\overline{F}(x)}$.
The hazard rate function for $NEGL$ random variable is given by

$$h(x) = \frac{\theta^2\left(\frac{k\left(\theta x\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta^\delta\left(\theta x\right)^{\alpha-2}}{\theta\Gamma(\alpha-1)}\right)e^{-\theta x}}{\eta^\delta+\theta k-\left\{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_{\alpha-1}(\theta x)\right\}}. \tag{12}$$

*3.5. Vitality function and Mean residual life function*

The vitality function is a very useful tool in modeling life-time data. This function, together with mean residual life function plays an important role in engineering and biomedical science. If $X$ is a non-negative random variable having cdf $F(x)$ with pdf $f(x)$, the vitality function associated with the random variable $X$ is defined as

$$V(x) = E\left(X|X > x\right) = \frac{1}{\overline{F}(x)}\int_x^\infty tf(t)\ dt. \tag{13}$$

In the reliability context clearly, (13) can be interpreted as the average life span of components whose age exceeds $x$. It may be noted that the hazard rate reflects the risk of sudden death within a life span, whereas the vitality function

provides a more direct measure to describe the failure pattern in the sense that it is expressed in terms of increased average life span.
The vitality function of the $NEGL$ random variable is given by

$$\begin{aligned} V(x) &= E(X|X>x) \\ &= \frac{\int_x^\infty t f(t;\theta,\alpha,k,\eta,\delta)dt}{\overline{F}(x;\theta,\alpha,k,\eta,\delta)} \\ &= \frac{\alpha k\theta\Gamma_{\alpha+1}(\theta x)+\eta^\delta(\alpha-1)\Gamma_\alpha(\theta x)}{\theta\Big\{\eta^\delta+\theta k-\Big(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_{\alpha-1}(\theta x)\Big)\Big\}}, \end{aligned} \tag{14}$$

which is immediate from (11)
Mean residual life time is the expected value of the remaining life time of a unit if it has survived an age $x$. If $X$ is a non-negative random variable having cdf $F(x)$ with pdf $f(x)$, the mean residual life time is given by

$$\begin{aligned} m(x) &= E(X-x|X>x) \\ &= \frac{\int_x^\infty \overline{F}(t)dt}{\overline{F}(x)}. \end{aligned}$$

The mean residual function of the $NEGL$ random variable is given by

$$\begin{aligned} m(x) &= E(X-x|X>x) = E(X|X>x)-x \\ &= V(x)-x \\ &= \frac{\alpha k\theta\Gamma_{\alpha+1}(\theta x)+\eta^\delta(\alpha-1)\Gamma_\alpha(\theta x)}{\theta\Big\{\eta^\delta+\theta k-\Big(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_{\alpha-1}(\theta x)\Big)\Big\}}-x. \end{aligned} \tag{15}$$

### *3.6. Geometric vitality function*

It is well known that geometric mean is a very useful average in situations such as analyzing data related to the sizes of bacterial populations, because concentrations of bacteria may vary over a wide range. Besides being used by biologists, geometric mean is also used in many other fields such as analyzing income data in the context of Economics, modeling financial data etc. Geometric vitality function is used for finding the geometric mean of the residual life-time.
For a non-negative random variable $X$ admitting an absolutely continuous distribution function, with $E(\log X)<\infty$, the geometric vitality function is defined as

$$\begin{aligned} \log G(x) &= E\left(\log X|X>x\right) \\ &= \frac{1}{\overline{F}(x)}\int_x^\infty \log t f(t)\; dt. \end{aligned} \tag{16}$$

The geometric vitality function of $NEGL$ random variable is given by

$$\log G(x) = \frac{\int_x^\infty \log t f(t;\theta,\alpha,k,\eta,\delta)\ dt}{\overline{F}(x;\theta,\alpha,k,\eta,\delta)}.$$

Now

$$\begin{aligned}
&\int_x^\infty \log t f(t;\theta,\alpha,k,\eta,\delta)\ dt\\
&= \frac{\theta}{\eta^\delta+\theta k}\Bigg\{\frac{k}{\Gamma(\alpha)}\Big(\Gamma^{'}(\alpha,\theta x) - \log\theta\Gamma(\alpha,\theta x)\Big)\\
&\quad + \frac{\eta^\delta}{\theta\Gamma(\alpha-1)}\Big(\Gamma^{'}(\alpha-1,\theta x) - \log\theta\Gamma(\alpha-1,\theta x)\Big)\Bigg\}.
\end{aligned}$$

Therefore

$$\log G(x) = \frac{\theta\Bigg\{\frac{k}{\Gamma(\alpha)}\Big(\Gamma^{'}(\alpha,\theta x) - \log\theta\Gamma(\alpha,\theta x)\Big) + \frac{\eta^\delta}{\theta\Gamma(\alpha-1)}\Big(\Gamma^{'}(\alpha-1,\theta x) - \log\theta\Gamma(\alpha-1,\theta x)\Big)\Bigg\}}{\eta^\delta+\theta k - \Big(\theta k\gamma_\alpha(\theta x) + \eta^\delta\gamma_{\alpha-1}(\theta x)\Big)}, \tag{17}$$

where

$$\Gamma^{'}(a,b) = \int_b^\infty y^{a-1}\log y\ e^{-y}dy.$$

### 3.7. *Mean inactivity time and reversed hazard rate function*

The dual concept of the mean residual life is called mean inactivity time or reversed mean residual life. It is the average failure of a device or a lifetime of the system under the condition that the failure has occurred before the time $x$. If $X$ is a non-negative random variable having cdf $F(x)$ with pdf $f(x)$, the mean inactivity time is given as

$$m_F(x) = \frac{\int_0^x F(t)dt}{F(x)}.$$

The mean inactivity time of $NEGL$ random variable is obtained as follows

$$\begin{aligned}
m_F(x) &= \frac{\int_0^x F(t;\theta,\alpha,k,\eta,\delta)\ dt}{F(x;\theta,\alpha,k,\eta,\delta)}\\
&= x - \frac{\int_0^x tf(t;\theta,\alpha,k,\eta,\delta)dt}{F(x;\theta,\alpha,k,\eta,\delta)}.
\end{aligned} \tag{18}$$

Now

$$\int_0^x tf(t;\theta,\alpha,k,\eta,\delta)dt = \frac{1}{\theta(\eta^\delta+\theta k)}\Big\{\theta k\alpha\gamma_{\alpha+1}(\theta x)+\eta^\delta(\alpha-1)\gamma_\alpha(\theta x)\Big\}.$$

Therefore

$$m_F(x) = x - \frac{\theta k\alpha\gamma_{\alpha+1}(\theta x)+\eta^\delta(\alpha-1)\gamma_\alpha(\theta x)}{\theta\Big(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_{\alpha-1}(\theta x)\Big)}. \tag{19}$$

Another reliability measure in connection with the mean inactivity time is the reversed hazard rate. The reversed hazard rate specifies the instantaneous rate of death or failure at time $x$, given that it failed before time $x$. Let $X$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$, then the reversed hazard rate function is given by $r(x) = \frac{f(x)}{F(x)}$. $r(x)dx$ is the probability of falling in the interval ($x$-$dx$, $x$), when a unit is found failed at time $x$. This concept is useful in casualty, insurance,reliability and forensic science to predict times of occurrences of events.
The reversed hazard rate function for $NEGL$ random variable is given by

$$r(x) = \frac{\theta\Big\{k\theta\big(\theta x\big)^{\alpha-1}\Gamma(\alpha-1)+\eta^\delta\big(\theta x\big)^{\alpha-2}\Gamma(\alpha)\Big\}e^{-\theta x}}{\theta k\gamma(\alpha,\theta x)\Gamma(\alpha-1)+\eta^\delta\gamma(\alpha-1,\theta x)\Gamma(\alpha)}. \tag{20}$$

### *3.8. Entropy Measures*

Shannon (1948) defined the basic measure of uncertainty associated with the random variable $X$ and is given by

$$H(X) = -\int_0^\infty f(x)\ \log f(x)\ dx.$$

The entropy is interpreted as the expected uncertainty contained in $f(x)$ about the predictability of an outcome of the random variable$X$.

#### *3.8.1. Sharma-Mittal entropy*

Several generalizations of Shannon's entropy have been put forward by researchers. A two-parametric generalization of Shannon's entropy is called Sharma-Mittal entropy (see, Sharma and Mittal (1977)) and is given as

$$H_{\nu,\beta}(X) = \frac{1}{1-\beta}\Big\{\Big(\int_0^\infty f^\nu(x)\ dx\Big)^{\frac{1-\beta}{1-\nu}}-1\Big\}.$$

It generalizes Tsallis, Rényi and Boltzmann-Gibbs entropies.

Sharma-Mittal entropy for $NEGL$ random variable is given by

$$H_{\nu,\beta}(X) = \frac{1}{1-\beta}\left\{\Big(\int_0^\infty f^{\nu}(x;\theta,\alpha,k,\eta,\delta)\ dx\Big)^{\frac{1-\beta}{1-\nu}} - 1\right\}.$$

Now

$$\int_0^\infty f^{\nu}(x;\theta,\alpha,k,\eta,\delta)dx = \left(\frac{\theta^2}{\eta^\delta+\theta k}\right)^{\nu}\left(\frac{k}{\Gamma(\alpha)}\right)^{\nu}\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\alpha-1)}\right)^{j} \times \int_0^\infty \Big(\theta x\Big)^{\nu(\alpha-1)-j} e^{-\theta\nu x}dx.$$

Therefore,

$$H_{\nu,\beta}(X) = \frac{1}{1-\beta}\left\{\left(\frac{k\theta^{\frac{2\nu-1}{\nu}}}{\nu^{\frac{(\alpha-1)\nu+1}{\nu}}(\eta^\delta+\theta k)\Gamma(\alpha)}\right)^{\frac{\nu(1-\beta)}{1-\nu}} \times \left(\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\alpha-1)}\right)^{j}\frac{\Gamma\Big(\nu(\alpha-1)-j+1\Big)}{\nu^{-j}}\right)^{\frac{1-\beta}{1-\nu}} - 1\right\}. \tag{21}$$

### *3.8.2. Mathai-Haubold entropy*

Mathai and Haubold (Mathai and Haubold (2006)) introduced the generalized information measure

$$M_\rho(X) = \frac{1}{\rho-1}\left\{\int_0^\infty f^{2-\rho}(x)\ dx - 1\right\},\ \text{for } 0<\rho<2, \rho\neq 1.$$

As $\rho \to 1$, Mathai-Haubold entropy reduces to Shannon's entropy.

Mathai-Haubold entropy for $NEGL$ random variable is given by

$$M_\rho(X) = \frac{1}{\rho-1}\left\{\left(\frac{k\theta^{\frac{3-2\rho}{2-\rho}}}{(2-\rho)^{\alpha+\frac{\rho-1}{2-\rho}}(\eta^\delta+\theta k)\Gamma(\alpha)}\right)^{2-\rho} \times \sum_{j=0}^{2-\rho}\binom{2-\rho}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\alpha-1)}\right)^{j}\frac{\Gamma\Big(\alpha(2-\rho)+\rho-1-j\Big)}{(2-\rho)^{-j}} - 1\right\}. \tag{22}$$

### *3.8.3. Residual entropy*

If we think of $X$ as the life-time of a new unit, then $H(X)$ can be viewed as a useful tool for measuring the associated uncertainty. However, if a unit is known to have survived upto age $t$, then $H(X)$ is no longer useful for measuring the uncertainty about remaining life-time of the unit. In this scenario, Ebrahimi, Pellerey (1995) followed by Ebrahimi (1996) have proposed the concept of residual entropy and is defined as

$$H(f;x) = \log \overline{F}(x) - \frac{1}{\overline{F}(x)} \int_x^\infty f(t)\ \log f(t)\ dt.$$

Residual entropy for $NEGL$ random variable is given by

$$H(f;x) = \log \overline{F}(x;\theta,\alpha,k,\eta,\delta) - \frac{\int_x^\infty f(t;\theta,\alpha,k,\eta,\delta)\ \log f(t;\theta,\alpha,k,\eta,\delta)\ dt}{\overline{F}(x;\theta,\alpha,k,\eta,\delta)}.$$

Now

$$\begin{aligned}
&\int_x^\infty f(t;\theta,\alpha,k,\eta,\delta)\ \log f(t;\theta,\alpha,k,\eta,\delta) dt \\
&= \int_x^\infty f(t;\theta,\alpha,k,\eta,\delta) \left\{ \log\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) + \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)(\theta t)^{-1}}{k\theta\Gamma(\alpha-1)}\right)^j \right. \qquad (23) \\
&\qquad \left. + \log\left(\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}\right) - \theta t \right\} dt \\
&= a_1 + a_2 + a_3 - a_4,
\end{aligned}$$

where

$$a_1 = \int_x^\infty \log\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) f(t;\theta,\alpha,k,\eta,\delta) dt, \qquad (24)$$

$$a_2 = \int_x^\infty f(t;\theta,\alpha,k,\eta,\delta) \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)(\theta t)^{-1}}{k\theta\Gamma(\alpha-1)}\right)^j dt, \qquad (25)$$

$$a_3 = \int_x^\infty f(t;\theta,\alpha,k,\eta,\delta) \log\left(\frac{k(\theta t)^{\alpha-1}}{\Gamma(\alpha)}\right) dt \qquad (26)$$

and

$$a_4 = \int_x^\infty f(t;\theta,\alpha,k,\eta,\delta)\ \theta\ t\ dt. \qquad (27)$$

From (24),

$$a_1 = \log\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) \overline{F}(x). \qquad (28)$$

From (25),

$$a_2 = \left(\frac{\theta^2}{\eta^\delta + \theta k}\right) \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)}{k\theta\Gamma(\alpha-1)}\right)^j \left\{ \frac{k\Gamma\Big(\alpha - j, \theta x\Big)}{\theta\Gamma(\alpha)} + \frac{\eta^\delta \Gamma\Big(\alpha - 1 - j, \theta x\Big)}{\theta^2 \Gamma(\alpha-1)} \right\}. \tag{29}$$

From (26),

$$a_3 = \log\left(\frac{k}{\Gamma(\alpha)}\right) \overline{F}(x;\theta,\alpha,k,\eta,\delta) + \frac{\theta^2}{(\eta^\delta + \theta k)} \left\{ \frac{k(\alpha-1)}{\theta} \frac{\Gamma'(\alpha,\theta x)}{\Gamma(\alpha)} + \frac{\eta^\delta(\alpha-1)}{\theta^2} \frac{\Gamma'(\alpha-1,\theta x)}{\Gamma(\alpha-1)} \right\}. \tag{30}$$

From (27),

$$a_4 = \frac{\theta^2}{(\eta^\delta + \theta k)} \left\{ \frac{k\Gamma(\alpha+1,\theta x)}{\theta\Gamma(\alpha)} + \frac{\eta^\delta \Gamma(\alpha,\theta x)}{\theta^2\Gamma(\alpha-1)} \right\}. \tag{31}$$

Substituting (28), (29), (30) and (31) in (23), we get

$$\begin{aligned} H(f;x) = {} & \log(\overline{F}(x;\theta,\alpha,k,\eta,\delta)) - \log\left(\frac{k\theta^2}{(\eta^\delta + \theta k)\Gamma(\alpha)}\right) \\ & - \frac{\theta^2}{(\eta^\delta + \theta k)\overline{F}(x;\theta,\alpha,k,\eta,\delta)} \left\{ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta\Gamma(\alpha)}{k\theta\Gamma(\alpha-1)}\right)^j \right. \\ & \left(\frac{k}{\theta} \frac{\Gamma\Big(\alpha - j,\theta x\Big)}{\Gamma(\alpha)} + \frac{\eta^\delta}{\theta^2} \frac{\Gamma\Big(\alpha-1-j,\theta x\Big)}{\Gamma(\alpha-1)}\right) + \frac{k(\alpha-1)}{\theta} \frac{\Gamma'(\alpha,\theta x)}{\Gamma(\alpha)} \\ & \left. + \frac{\eta^\delta(\alpha-1)}{\theta^2} \frac{\Gamma'(\alpha-1,\theta x)}{\Gamma(\alpha-1)} - \frac{k}{\theta} \frac{\Gamma(\alpha+1,\theta x)}{\Gamma(\alpha)} - \frac{\eta^\delta}{\theta^2} \frac{\Gamma(\alpha,\theta x)}{\Gamma(\alpha-1)} \right\}, \end{aligned} \tag{32}$$

where $\overline{F}(x;\theta,\alpha,k,\eta,\delta)$ is given in (4) and $\Gamma'(a,b) = \int_b^\infty t^{a-1} \log(t)\ e^{-t} dt$.

### *3.8.4. Past entropy*

In many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. For instance, if at time $x$, a system which is observed only at certain pre-assigned inspection times, is found to be down, then the uncertainty of the life of the system relies on the past, that is, on which instant in $(0,\ x)$ it has failed. Based on this idea, Di Crescenzo, Longobardi (2002) defined the past entropy over $(0,\ x)$.

If $X$ denotes the life-time of a component/system or of living organism, then the past entropy of $X$ at time $x$ is defined as

$$\overline{H}(f;x) = \log F(x) - \frac{1}{F(x)} \int_0^x f(t)\ \log f(t)\ dt.$$

Given that at time $x$ a component has failed, the past entropy measures the uncertainty about its past life. In forensic science, a life-time distribution truncated above $x$ is of utmost importance.
Past entropy for $NEGL$ random variable is given by

$$\begin{aligned}\overline{H}(f;x) &= \log(F(x;\theta,\alpha,k,\eta,\delta)) - \log\left(\frac{k\theta^2}{(\eta^\delta+\theta k)\Gamma(\alpha)}\right) \\ &- \frac{\theta^2}{(\eta^\delta+\theta k)F(x;\theta,\alpha,k,\eta,\delta)}\left\{\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{k\theta\Gamma(\alpha-1)}\right)^j\right. \\ &\left(\frac{k}{\theta}\frac{\gamma\Big(\alpha-j,\theta x\Big)}{\Gamma(\alpha)}+\frac{\eta^\delta}{\theta^2}\frac{\gamma\Big(\alpha-1-j,\theta x\Big)}{\Gamma(\alpha-1)}\right)+\frac{k(\alpha-1)}{\theta}\frac{\gamma^{'}(\alpha,\theta x)}{\Gamma(\alpha)} \\ &\left.+\frac{\eta^\delta(\alpha-1)}{\theta^2}\frac{\gamma^{'}(\alpha-1,\theta x)}{\Gamma(\alpha-1)}-\frac{k}{\theta}\frac{\gamma(\alpha+1,\theta x)}{\Gamma(\alpha)}-\frac{\eta^\delta}{\theta^2}\frac{\gamma(\alpha,\theta x)}{\Gamma(\alpha-1)}\right\},\end{aligned} \tag{33}$$

where $F(x;\theta,\alpha,k,\eta,\delta)$ is given in (3) and $\gamma^{'}(a,b)=\int\limits_0^b t^{a-1}\ \log(t)\ e^{-t}dt.$

## 4. Estimation

### *4.1. Method of Moment Estimation*

From (5), the $r^{th}$ raw moment about origin for the $NEGL$ random variable is given by

$$\mu_r^{'}=\frac{\Gamma(\alpha+r-1)}{(\eta^\delta+\theta k)\Gamma(\alpha)}\left\{\frac{(\alpha-1)(\eta^\delta+\theta k)+\theta kr}{\theta^r}\right\}.$$

Put $r=1,2,3,4$ and 5, the first five raw moments are obtained. Equating this raw moments to the corresponding sample moments; say $m_1^{'}$, $m_2^{'}$, $m_3^{'}$, $m_4^{'}$ and $m_5^{'}$, where $m_r^{'}=\frac{1}{n}\sum\limits_{i=1}^{n}X_i^r$, we get

$$\frac{1}{\theta(\eta^\delta+\theta k)}\Big\{\alpha(\eta^\delta+\theta k)-\eta^\delta\Big\}=m_1^{'}, \tag{34}$$

$$\frac{1}{\theta^2(\eta^\delta+\theta k)}\Big\{\alpha\Big(\alpha(\eta^\delta+\theta k)-\eta^\delta+\theta k\Big)\Big\}=m_2^{'}, \tag{35}$$

$$\frac{\alpha(\alpha+1)}{\theta^3(\eta^\delta+\theta k)}\Big\{\alpha(\eta^\delta+\theta k)-\eta^\delta+2\theta k\Big\}=m_3^{'}, \tag{36}$$

$$\frac{\alpha(\alpha+1)(\alpha+2)}{\theta^4(\eta^\delta+\theta k)}\Big\{\alpha(\eta^\delta+\theta k)-\eta^\delta+3\theta k\Big\}=m_4^{'} \tag{37}$$

and

$$\frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{\theta^5(\eta^\delta+\theta k)}\Big\{\alpha(\eta^\delta+\theta k)-\eta^\delta+4\theta k\Big\}=m_5^{'}. \tag{38}$$

Moment estimates of $NEGLD$ are obtained by solving this nonlinear system of equations from (34) to (38) by using Newton Raphson method.

### *4.2. Method of Maximum Likelihood Estimation*

Let $X_1, X_2, ..., X_n$ be a random sample of size $n$ from $NEGLD$ with unknown parameter vector $\Theta = \Big(\theta, \alpha, k, \eta, \delta\Big)$. The likelihood function for $\Theta$ is

$$\begin{aligned} l(\Theta) =& \prod_{i=1}^{n} f_i\Big(x; \theta, \alpha, k, \eta, \delta\Big) \\ =& \left(\frac{\theta^2}{\eta^\delta + \theta k}\right)^n e^{-\theta \sum\limits_{i=1}^{n} x_i} \Big(\Gamma(\alpha)\Gamma(\alpha-1)\Big)^{-n} \prod_{i=1}^{n} \Big(\Gamma(\alpha-1)k\theta^{\alpha-1}x_i^{\alpha-1} \\ &+ \Gamma(\alpha)\eta^\delta\theta^{\alpha-3}x_i^{\alpha-2}\Big). \end{aligned}$$

The partial derivatives of $\log l\big(\Theta\big)$ with respect to the parameters are

$$\begin{aligned} \frac{\partial \log l}{\partial \theta} =& \frac{2n}{\theta} - \frac{nk}{(\eta^\delta + \theta k)} - \sum_{i=1}^{n} x_i + \\ & \sum_{i=1}^{n} \left( \frac{(\alpha-1)k\Gamma(\alpha-1)\theta^{\alpha-2}x_i^{\alpha-1} + (\alpha-3)\eta^\delta\Gamma(\alpha)\theta^{\alpha-4}x_i^{\alpha-2}}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} \right), \end{aligned} \tag{39}$$

$$\begin{aligned} \frac{\partial \log l}{\partial \alpha} =& \sum_{i=1}^{n} \frac{k\theta^{\alpha-1}x_i^{\alpha-1}\Big(\Gamma'(\alpha-1) + \Gamma(\alpha-1)\log(\theta x_i)\Big)}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} + \\ & \sum_{i=1}^{n} \frac{\eta^\delta\theta^{\alpha-3}x_i^{\alpha-2}\Big(\Gamma'(\alpha) + \Gamma(\alpha)\log(\theta x_i)\Big)}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} \\ & - n\Big(\psi(\alpha) + \psi(\alpha-1)\Big), \end{aligned} \tag{40}$$

$$\frac{\partial \log l}{\partial k} = \frac{-n\theta}{(\eta^\delta + \theta k)} + \sum_{i=1}^{n} \left( \frac{\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1}}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} \right), \tag{41}$$

$$\frac{\partial \log l}{\partial \eta} = \frac{-n\delta\eta^{\delta-1}}{(\eta^\delta + \theta k)} + \sum_{i=1}^{n} \left( \frac{\delta\eta^{\delta-1}\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} \right) \tag{42}$$

and

$$\frac{\partial \log l}{\partial \delta} = \frac{-n\eta^\delta \ \log(\eta)}{(\eta^\delta + \theta k)} + \sum_{i=1}^{n} \left( \frac{\eta^\delta \ \log(\eta)\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}}{k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2}} \right). \tag{43}$$

The MLE of the parameters $\Theta = \Big(\theta, \alpha, k, \eta, \delta\Big)$ say $\widehat{\Theta} = \Big(\widehat{\theta}, \widehat{\alpha}, \widehat{k}, \widehat{\eta}, \widehat{\delta}\Big)$ are obtained by solving the equations $\frac{\partial \log l}{\partial \theta} = 0$, $\frac{\partial \log l}{\partial \alpha} = 0$, $\frac{\partial \log l}{\partial k} = 0$, $\frac{\partial \log l}{\partial \eta} = 0$ and $\frac{\partial \log l}{\partial \delta} = 0$. The above equations are not in a closed form. So the maximum likelihood estimates of the parameters must be obtained by using Newton Raphson method.

*4.3. Asymptotic Confidence Interval*

In this section, we present the asymptotic confidence intervals for the parameters of the *NEGLD*. Let $\widehat{\Theta} = \left(\widehat{\theta}, \widehat{\alpha}, \widehat{k}, \widehat{\eta}, \widehat{\delta}\right)$ be the maximum likelihood estimator of $\Theta = \left(\theta, \alpha, k, \eta, \delta\right)$. To obtain the asymptotic confidence intervals for these parameters, we can use the large sample property of MLE. Therefore $\left(\widehat{\theta}-\theta,\ \widehat{\alpha}-\alpha,\ \widehat{k}-k,\ \widehat{\eta}-\eta,\ \widehat{\delta}-\delta\right)^{\tau}$ is asymptotically normally distributed with mean vector $\underline{0} = \left(0,0,0,0,0\right)^{\tau}$ and estimated variance covariance matrix $\Delta^{-1}$, where $\Delta$ is the observed Fisher's information matrix given by

$$\Delta = \begin{pmatrix} -\frac{\partial^2 \log l}{\partial\theta^2} & -\frac{\partial^2 \log l}{\partial\theta\partial\alpha} & -\frac{\partial^2 \log l}{\partial\theta\partial k} & -\frac{\partial^2 \log l}{\partial\theta\partial\eta} & -\frac{\partial^2 \log l}{\partial\theta\partial\delta} \\ -\frac{\partial^2 \log l}{\partial\alpha\partial\theta} & -\frac{\partial^2 \log l}{\partial\alpha^2} & -\frac{\partial^2 \log l}{\partial\alpha\partial k} & -\frac{\partial^2 \log l}{\partial\alpha\partial\eta} & -\frac{\partial^2 \log l}{\partial\alpha\partial\delta} \\ -\frac{\partial^2 \log l}{\partial k\partial\theta} & -\frac{\partial^2 \log l}{\partial k\partial\alpha} & -\frac{\partial^2 \log l}{\partial k^2} & -\frac{\partial^2 \log l}{\partial k\partial\eta} & -\frac{\partial^2 \log l}{\partial k\partial\delta} \\ -\frac{\partial^2 \log l}{\partial\eta\partial\theta} & -\frac{\partial^2 \log l}{\partial\eta\partial\alpha} & -\frac{\partial^2 \log l}{\partial\eta\partial k} & -\frac{\partial^2 \log l}{\partial\eta^2} & -\frac{\partial^2 \log l}{\partial\eta\partial\delta} \\ -\frac{\partial^2 \log l}{\partial\delta\partial\theta} & -\frac{\partial^2 \log l}{\partial\delta\partial\alpha} & -\frac{\partial^2 \log l}{\partial\delta\partial k} & -\frac{\partial^2 \log l}{\partial\delta\partial\eta} & -\frac{\partial^2 \log l}{\partial\delta^2} \end{pmatrix}$$

The elements of $\Delta$ are given by

$$\Delta_{11} = \frac{2n}{\theta^2} - \frac{nk^2}{(\eta^\delta+\theta k)^2} - \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)\Big((\alpha-1)(\alpha-2)A_i + (\alpha-3)(\alpha-4)B_i\Big) - \Big((\alpha-1)A_i+(\alpha-3)B_i\Big)^2\Big\}, \tag{44}$$

$$\Delta_{12} = \Delta_{21} = -\frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)^2 + 2A_iB_i\Big(\psi(\alpha-1)-\psi(\alpha)\Big)\Big\}, \tag{45}$$

$$\Delta_{13} = \Delta_{31} = \frac{n\eta^\delta}{(\eta^\delta+\theta k)^2} - \frac{2}{k\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}, \tag{46}$$

$$\Delta_{14} = \Delta_{41} = -\frac{nk\delta\eta^{\delta-1}}{(\eta^\delta+\theta k)^2} + \frac{2\delta}{\eta\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}, \tag{47}$$

$$\Delta_{15} = \Delta_{51} = -\frac{nk\eta^\delta\ \log(\eta)}{(\eta^\delta+\theta k)^2} + \frac{2\log(\eta)}{\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}, \tag{48}$$

$$\Delta_{22} = n\Big(\psi^{'}(\alpha)+\psi^{'}(\alpha-1)\Big) - \sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\left\{A_i^2\psi^{'}(\alpha-1) + B_i^2\psi^{'}(\alpha) + A_iB_i\Big\{\psi^{'}(\alpha-1)+\psi^{'}(\alpha)+\Big(\psi(\alpha)-\psi(\alpha-1)\Big)^2\Big\}\right\}, \tag{49}$$

$$\Delta_{23} = \Delta_{32} = -\frac{1}{k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\alpha-1)-\psi(\alpha)\Big\}, \tag{50}$$

$$\Delta_{24} = \Delta_{42} = -\frac{\delta}{\eta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\alpha)-\psi(\alpha-1)\Big\}, \tag{51}$$

$$\Delta_{25} = \Delta_{52} = -\log(\eta)\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\alpha)-\psi(\alpha-1)\Big\}, \tag{52}$$

$$\Delta_{33} = -\frac{n\theta^2}{(\eta^\delta+\theta k)^2}+\frac{1}{k^2}\sum_{i=1}^{n}\frac{A_i^2}{(A_i+B_i)^2}, \tag{53}$$

$$\Delta_{34} = \Delta_{43} = -\frac{n\theta\delta\eta^{\delta-1}}{(\eta^\delta+\theta k)^2}+\frac{\delta}{\eta k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}, \tag{54}$$

$$\Delta_{35} = \Delta_{53} = -\frac{n\theta\eta^\delta\ \log(\eta)}{(\eta^\delta+\theta k)^2}+\frac{\log(\eta)}{k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}, \tag{55}$$

$$\Delta_{44} = \frac{n\delta\eta^{\delta-2}}{(\eta^\delta+\theta k)^2}\Big((\delta-1)\theta k-\eta^\delta\Big)-\frac{1}{\eta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2} \Big\{\Big(A_i+B_i\Big)B_i\delta(\delta-1)-(\delta B_i)^2\Big\}, \tag{56}$$

$$\Delta_{45} = \Delta_{54} = \frac{n}{(\eta^\delta+\theta k)^2}\Big\{\theta k\eta^{\delta-1}\Big(\delta\ \log(\eta)+1\Big)+\eta^{2\delta-1}\Big\}- \frac{1}{\eta}\sum_{i=1}^{n}\frac{A_iB_i\Big(1+\delta\ \log(\eta)\Big)+B_i^2}{(A_i+B_i)^2} \tag{57}$$

and

$$\Delta_{55} = \frac{n\theta k\eta^\delta\Big(\log(\eta)\Big)^2}{(\eta^\delta+\theta k)^2}-\sum_{i=1}^{n}\frac{A_iB_i\Big(\log(\eta)\Big)^2}{(A_i+B_i)^2}, \tag{58}$$

where

$$A_i = k\Gamma(\alpha-1)\theta^{\alpha-1}x_i^{\alpha-1},$$

$$B_i = \eta^\delta\Gamma(\alpha)\theta^{\alpha-3}x_i^{\alpha-2},$$

$$\psi^{'}(a) = \frac{\Gamma(a)\Gamma^{''}(a) - \Big(\Gamma^{'}(a)\Big)^2}{\Gamma^2(a)}$$

is the polygamma function and

$$\Gamma^{(n)}(a) = \int_0^\infty t^{a-1}\Big(\log(t)\Big)^n e^{-t}dt$$

is the $n^{th}$ order derivative of gamma function.
The estimated variance-covariance matrix of the parameters $\widehat{\theta}, \widehat{\alpha}, \widehat{k}, \widehat{\eta}$ and $\widehat{\delta}$ can be calculated by

$$\begin{bmatrix} \widehat{Var(\theta)} & \widehat{Cov(\theta,\alpha)} & \widehat{Cov(\theta,k)} & \widehat{Cov(\theta,\eta)} & \widehat{Cov(\theta,\delta)} \\ \widehat{Cov(\alpha,\theta)} & \widehat{Var(\alpha)} & \widehat{Cov(\alpha,k)} & \widehat{Cov(\alpha,\eta)} & \widehat{Cov(\alpha,\delta)} \\ \widehat{Cov(k,\theta)} & \widehat{Cov(k,\alpha)} & \widehat{Var(k)} & \widehat{Cov(k,\eta)} & \widehat{Cov(k,\delta)} \\ \widehat{Cov(\eta,\theta)} & \widehat{Cov(\eta,\alpha)} & \widehat{Cov(\eta,k)} & \widehat{Var(\eta)} & \widehat{Cov(\eta,\delta)} \\ \widehat{Cov(\delta,\theta)} & \widehat{Cov(\delta,\alpha)} & \widehat{Cov(\delta,k)} & \widehat{Cov(\delta,\eta)} & \widehat{Var(\delta)} \end{bmatrix} = \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \Delta_{15} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} & \Delta_{25} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} & \Delta_{35} \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & \Delta_{44} & \Delta_{45} \\ \Delta_{51} & \Delta_{52} & \Delta_{53} & \Delta_{54} & \Delta_{55} \end{bmatrix}^{-1}$$

The diagonal elements $\widehat{Var(\theta)}$, $\widehat{Var(\alpha)}$, $\widehat{Var(k)}$, $\widehat{Var(\eta)}$ and $\widehat{Var(\delta)}$ are the asymptotic variances of the estimators of $\theta, \alpha, k, \eta,$ and $\delta$ respectively. The approximate $100(1-\varphi)\%$ two-sided confidence intervals for $\theta, \alpha, k, \eta,$ and $\delta$ are $\widehat{\theta} \pm Z_{\frac{\varphi}{2}}\sqrt{\widehat{Var(\theta)}}$, $\widehat{\alpha} \pm Z_{\frac{\varphi}{2}}\sqrt{\widehat{Var(\alpha)}}$, $\widehat{k} \pm Z_{\frac{\varphi}{2}}\sqrt{\widehat{Var(k)}}$, $\widehat{\eta} \pm Z_{\frac{\varphi}{2}}\sqrt{\widehat{Var(\eta)}}$ and $\widehat{\delta} \pm Z_{\frac{\varphi}{2}}\sqrt{\widehat{Var(\delta)}}$ respectively, where $Z_{\frac{\varphi}{2}}$ is the upper $\frac{\varphi}{2}^{th}$ percentile of a standard normal distribution.
Here we use the likelihood ratio (LR) test to compare our model with its sub models for a given data. For testing $\eta = \delta = 1$, the LR statistic is given by

$$\omega = 2\Big(\log l(\widehat{\theta}, \widehat{\alpha}, \widehat{k}, \widehat{\eta}, \widehat{\delta}) - \log l(\tilde{\theta}, \tilde{\alpha}, \tilde{k}, 1, 1)\Big),$$

where $\widehat{\theta}$, $\widehat{\alpha}$, $\widehat{k}$, $\widehat{\eta}$ and $\widehat{\delta}$ are the unrestricted estimates and $\tilde{\theta}$, $\tilde{\alpha}$ and $\tilde{k}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_d$, where $\chi^2_d$ denote the upper $100d\%$ point of the $\chi^2$ distribution with 2 degrees of freedom.

## 5. Data Analysis

In this section, we demonstrate the usefulness of the *NEGLD* by fitting a real data set. The data set was given by Bjerkedal (1960). The data are the survival times of guinea pigs injected with different doses of tubercle bacilli. We used the data set obtained under the regimen 6.6. Guinea pigs are known to have high susceptibility to human tuberculosis. Even an infection initiated with a few virulent tubercle bacilli will lead to progressive disease and death. The data set consists of 72 observations and are listed below:

**Survival times of 72 guinea pigs under regimen 6.6**
0.12, 0.15, 0.22, 0.24, 0.24, 0.32, 0.32, 0.33, 0.34, 0.38, 0.38,0.43, 0.44, 0.48, 0.52, 0.53, 0.54, 0.54, 0.55, 0.56, 0.57, 0.58, 0.58, 0.59, 0.60, 0.60, 0.60, 0.60, 0.61, 0.62, 0.63, 0.65, 0.65, 0.67, 0.68, 0.70, 0.70, 0.72, 0.73, 0.75, 76, 0.76, 0.81, 0.83, 0.84, 0.85, 0.87, 0.91, 0.95, 0.96, 0.98, 0.99, 1.09, 1.10, 1.21, 1.27, 1.29, 1.31, 1.43, 1.46, 1.46, 1.75, 1.75, 2.11, 2.33, 2.58, 2.58, 2.63, 2.97, 3.41, 3.41, 3.76.

We fit the density functions of the new extended generalized Lindley distribution ($NEGLD$), Lindley distribution ($LD_1$) (Lindley (1958)), two-parameter Lindley distribution ($LD_2$) (Shanker *et al.* (2013)), generalized Lindley distribution ($GLD_1$)(Zakerzadeh and Dolati (2009)) and a new generalized Lindley distribution ($GLD_2$) (Abouammoh *et al.* (2015)). The pdfs of the $LD_1$, $LD_2$, $GLD_1$ and $GLD_2$ distributions are respectively given as

$$f_1(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\ x>0,\ \theta>0,$$

$$f_2(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\ x>0,\ \theta>0,\ \alpha>-\theta,$$

$$f_3(x;\alpha,\theta,\gamma) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)}{(\gamma+\theta)\Gamma(\alpha+1)}e^{-\theta x};\ x>0,\ \alpha,\theta,\gamma>0$$

and

$$f_4(x;\alpha,\theta) = \frac{\theta^\alpha x^{\alpha-2}}{(\theta+1)\Gamma(\alpha)}(x+\alpha-1)e^{-\theta x}; x>0, \theta\geq 0, \alpha\geq 1.$$

Maximum likelihood estimates and moment estimates of the parameters of the distributions and their $\chi^2$ values are given in Table 1. They indicate that $NEGLD$ fits the data set better than the other distributions. Figure 1 shows the plots of the fitted densities.

*Figure 6* – **Fitted probability density function for the real data set**

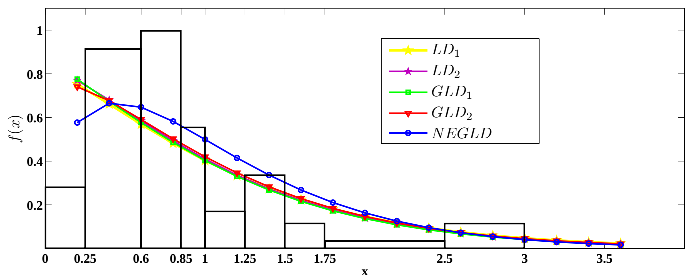



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*TABLE 1*

*Expected frequencies, Moment estimates, ML estimates and $\chi^2$ statistic obtained by fitting $LD_1$, $LD_2$, $GLD_1$, $GLD_2$ and $NEGLD$ to data*

| *Count* | *Observed* | Expected frequency by method of moments | | | | | Expected frequency by MLE | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $LD_1$ | $LD_2$ | $GLD_1$ | $GLD_2$ | $NEGLD$ | $LD_1$ | $LD_2$ | $GLD_1$ | $GLD_2$ | $NEGLD$ |
| 0-0.25 | 5 | 14 | 14 | 15 | 13 | 8 | 13 | 12 | 15 | 12 | 7 |
| 0.25-0.6 | 23 | 16 | 17 | 17 | 17 | 16 | 18 | 20 | 19 | 19 | 16 |
| 0.6-0.85 | 18 | 9 | 9 | 9 | 10 | 11 | 9 | 8 | 9 | 9 | 12 |
| 0.85-1.0 | 6 | 5 | 5 | 5 | 5 | 6 | 5 | 6 | 5 | 5 | 6 |
| 1.0-1.25 | 3 | 7 | 7 | 6 | 7 | 8 | 7 | 5 | 7 | 7 | 8 |
| 1.25-1.50 | 6 | 5 | 5 | 5 | 5 | 6 | 5 | 9 | 7 | 7 | 6 |
| 1.5-1.75 | 2 | 4 | 4 | 4 | 4 | 5 | 4 | 3 | 2 | 3 | 6 |
| 1.75-2.5 | 2 | 7 | 7 | 7 | 7 | 8 | 6 | 4 | 4 | 5 | 7 |
| 2.5-3 | 4 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 2 | 3 | 2 |
| 3-4 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| Total | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 72 | 72 |
| df | | 6 | 4 | 3 | 4 | 2 | 6 | 4 | 3 | 4 | 2 |
| Estimated values of parameters | | $\widehat{\theta}$=1.42 | $\widehat{\alpha}$= 1.21 | $\widehat{\alpha}$=0.9 | $\widehat{\alpha}$=2.1 | $\widehat{\alpha}$= 2.5 | $\widehat{\theta}$=1.58 | $\widehat{\alpha}$=1.342 | $\widehat{\alpha}$=1.231 | $\widehat{\alpha}$=2.314 | $\widehat{\alpha}$=2.72 |
| | | | $\widehat{\theta}$=1.51 | $\widehat{\gamma}$= 1.55 | $\widehat{\theta}$=1.51 | $\widehat{\theta}$=1.9 | | $\widehat{\theta}$=1.491 | $\widehat{\gamma}$=1.432 | $\widehat{\theta}$=1.74 | $\widehat{\theta}$=1.674 |
| | | | | $\widehat{\theta}$=1.5 | | $\widehat{k}$=2.8 | | | $\widehat{\theta}$=1.69 | | $\widehat{k}$=2.52 |
| | | | | | | $\widehat{\eta}$=2 | | | | | $\widehat{\eta}$=1.89 |
| | | | | | | $\widehat{\delta}$=2.5 | | | | | $\widehat{\delta}$=2.65 |
| $\chi^2$ | | 25.788 | 20.656 | 20.751 | 17.193 | 14.31 | 22.398 | 18.917 | 19.237 | 16.862 | 12.789 |

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Summary

In this paper, we introduce a new extended generalized Lindley distribution ($NEGLD$). Some statistical properties of the proposed distribution are explicitly derived. These include conditional moments, vitality function, geometric vitality function, mean inactivity time and various entropy measures. Maximum likelihood estimation, moment estimation and asymptotic confidence interval are used for estimating the parameters. The distribution has been fitted to a data set to test its goodness of fit and it has been found that this distribution gives better fit than the some other well-known existing distributions.

*Keywords*: Lindley distribution; Mathai-Haubold entropy; Maximum likelihood estimation; Asymptotic confidence interval; Likelihood ratio test.