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ON DYNAMIC GENERALIZED MEASURES OF INACCURACY

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1. Introduction

Let X and Y be two absolutely continuous non-negative random variables (rv's), which may be assumed as lifetimes of two components of a system. The probability density function (pdf), cumulative distribution function (cdf) and survival function (sf) of X are respectively denoted by f, F and $\bar{F}$; and that of Y by g, G and $\bar{G}$. Let $\eta_X = \frac{f}{\bar{F}}$ and $\eta_Y = \frac{g}{\bar{G}}$ be the hazard rate functions of X and Y, respectively; and $\xi_X = \frac{f}{F}$ and $\xi_Y = \frac{g}{G}$ be their corresponding reversed hazard rate functions. Kerridge (1961) proposed a measure of inaccuracy between two pdf's f and g, given by

$$K_{X,Y} = -E_f(\ln g(X)) = -\int_0^\infty (\ln g(x)) f(x) dx = I^{KL}_{X,Y} + I^S_X, \tag{1}$$

where $I^{KL}_{X,Y} = E_f\left(\ln \frac{f(X)}{g(X)}\right) = \int_0^\infty \left(\ln \frac{f(x)}{g(x)}\right) f(x) dx$ denotes the Kullback-Leibler divergence (see Kullback and Leibler, 1951), a popular measure of discrimination between X and Y; and $I^S_X = -E_f(\ln f(X)) = -\int_0^\infty (\ln f(x)) f(x) dx$ denotes the well-known Shannon (1948) entropy of X. The inaccuracy given in (1) measures the average information required to convey which of a number of possibilities is true, to someone who believes that the probability distribution of the possibilities is g when it is actually f. Measuring inaccuracy through (1) between two probability distributions involving current age is not an appropriate tool. To overcome this difficulty, Taneja *et al.* (2009) proposed a measure of inaccuracy between X and Y at time t (> 0) as

$$K_{X,Y}(t) = K_{X_t,Y_t} = -\int_t^\infty \left(\ln \frac{g(x)}{\bar{G}(t)}\right) \frac{f(x)}{\bar{F}(t)} dx \tag{2}$$

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and studied its properties, where $X_t = (X - t|X > t)$ and $Y_t = (Y - t|Y > t)$ denote the residual lifetime rv's. $K_{X,Y}(t)$ in (2) is known as the measure of residual inaccuracy between two rv's X_t and Y_t. Note that $K_{X,Y}(t)$ reduces to residual entropy due to Muliere *et al.* (1993) $I_X^S(t) = -\int_t^\infty \left(\ln \frac{f(x)}{\bar{F}(t)}\right)\frac{f(x)}{\bar{F}(t)}dx$, when $f = g$. Some situations arise in real life where inaccuracy relies on the past. Based on this idea, Kumar *et al.* (2011) defined a measure of inaccuracy between past lifetime distributions of X and Y as

$$\bar{K}_{X,Y}(t) = K_{X^t,Y^t} = -\int_0^t \left(\ln \frac{g(x)}{G(t)}\right)\frac{f(x)}{F(t)}dx, \tag{3}$$

where $X^t = (t - X|X < t)$ and $Y^t = (t - Y|Y < t)$ denote the past lifetime rv's. $\bar{K}_{X,Y}(t)$ in (3) is known as the measure of past inaccuracy between X and Y. Kundu and Nanda (2015) obtained several characterization results based on the measure of inaccuracy for truncated rv's.

Note that the measure (1) is based on the Kullback-Leibler discrimination and the Shannon entropy. There have been several attempts on generalizations of these two measures in the literature. For more on this direction, we refer to Rényi (1961), Varma (1966), Tsallis (1988), Kapur and Kesavan (1992) and Kapur (1994). These generalized information measures have many important properties such as smoothness, large dynamic range with respect to certain conditions that make them applicable in practice. Pharwaha and Singh (2009) showed that non-Shannon measures can be used to determine the randomness of mammograms because of having higher dynamic range than Shannon's entropy over a variety of scattering conditions. Non-Shannon measures are also applicable in estimating scatter density and regularity (see Smolikova *et al.*, 2002). After the seminal work by Kerridge (1961), several authors devoted their attention to generalize inaccuracy measure in the discrete domain. They studied their properties and characterizations. These measures are useful in different areas of science and technology, particularly in coding theory. For detail, we refer to Nath (1968), Rathie and Kannappan (1973), Sharma and Autar (1973), Sharma and Gupta (1976), Taneja and Gupta (1978), Taneja and Tuteja (1986), Bhatia and Taneja (1991), Bhatia (1999) and the references therein. Motivated by their work, based on the Renyi entropy and its discrimination, in this paper, we propose dynamic generalized measures of inaccuracy of order $\alpha\ (\neq 1) > 0$. A generalization of Kerridge's measure of inaccuracy (1) is given by

$$K_{X,Y}^R = \frac{1}{\alpha - 1}\ln\left(\frac{\int_0^\infty (f(x))^\alpha (g(x))^{1-\alpha}dx}{\int_0^\infty (f(x))^\alpha dx}\right) = I_{X,Y}^R + I_X^R, \tag{4}$$

where

$$I_{X,Y}^R = \frac{1}{\alpha-1}\ln\left(E_f\left(\frac{f(X)}{g(X)}\right)^{\alpha-1}\right) = \frac{1}{\alpha-1}\ln\left(\int_0^\infty (f(x))^\alpha (g(x))^{1-\alpha}dx\right)$$

and

$$I_X^R = \frac{1}{1-\alpha}\ln\left(E_f(f(X)^{\alpha-1}\right) = \frac{1}{1-\alpha}\ln\left(\int_0^\infty (f(x))^\alpha dx\right)$$

are the Renyi's discrimination measure between X and Y Asadi *et al.* (2005), Gil (2011); and Renyi's entropy measure of X Rényi (1961), respectively. Note that when $\alpha \to 1$, then (4) reduces to (1). Also for $f = g$, (4) coincides with the Renyi entropy I_X^R. In analogy to (2) and (3), generalized measures of inaccuracy of order α between the residual lifetime distributions; and the past lifetime distributions are respectively defined as

$$K_{X,Y}^R(t) = \frac{1}{\alpha-1}\ln\left(\frac{\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^\alpha \left(\frac{g(x)}{\bar{G}(t)}\right)^{1-\alpha} dx}{\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^\alpha dx}\right) \tag{5}$$

$$= I_{X,Y}^R(t) + I_X^R(t) \tag{6}$$

and

$$\bar{K}_{X,Y}^R(t) = \frac{1}{\alpha-1}\ln\left(\frac{\int_0^t \left(\frac{f(x)}{F(t)}\right)^\alpha \left(\frac{g(x)}{G(t)}\right)^{1-\alpha} dx}{\int_0^t \left(\frac{f(x)}{F(t)}\right)^\alpha dx}\right) \tag{7}$$

$$= \bar{I}_{X,Y}^R(t) + \bar{I}_X^R(t). \tag{8}$$

In Equations (6) and (8), $I_{X,Y}^R(t) = \frac{1}{\alpha-1}\ln\left(\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^\alpha \left(\frac{g(x)}{\bar{G}(t)}\right)^{1-\alpha} dx\right)$ and $\bar{I}_{X,Y}^R(t) = \frac{1}{\alpha-1}\ln\left(\int_0^t \left(\frac{f(x)}{F(t)}\right)^\alpha \left(\frac{g(x)}{G(t)}\right)^{1-\alpha} dx\right)$ respectively represent residual and past Renyi discrimination measures of order α between X and Y, whereas

$$I_X^R(t) = \frac{1}{1-\alpha}\ln\left(\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right)^\alpha dx\right)$$

and

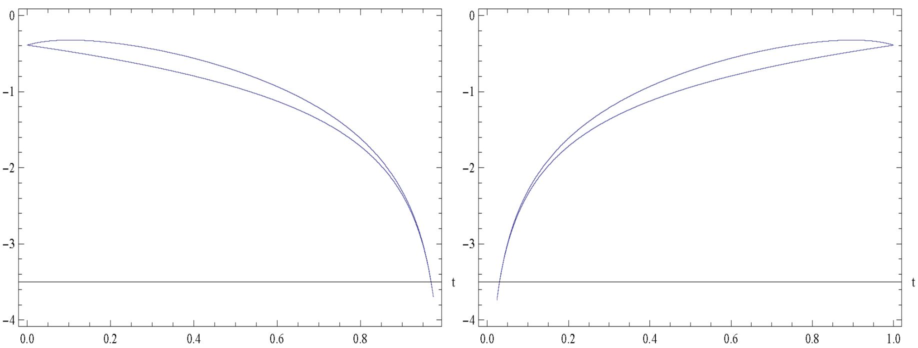
$$\bar{I}_X^R(t) = \frac{1}{1-\alpha}\ln\left(\int_0^t \left(\frac{f(x)}{F(t)}\right)^\alpha dx\right)$$

represent the residual and past Renyi entropy measures of order α. We call the measures given in (5) and (7) as measures of residual and past inaccuracy of order α, respectively. Henceforth, increasing and decreasing are used as non-strict sense. Throughout the paper, we also assume that $\alpha\ (\neq 1) > 0$.

In the following examples, we discuss the role of the generalized inaccuracy measures of order α between the residual and past lifetime distributions.

EXAMPLE 1. *Consider the rv's as described in Example* 1.1 *of Di Crescenzo and Longobardi (2004) with different notation. The pdf's of the rv's X and Y_β are given by $f(x) = 1$, $0 < x < 1$ and $g(x|\beta) = \beta\left(x - \frac{1}{2}\right) + 1$, $0 < x < 1$, $-2 \le \beta \le 2$, respectively. Note that when $x = \frac{1}{2}$, $K_{X,Y_\beta}^R = K_{X,Y_{-\beta}}^R$ for $\beta \in [-2,2]$, that is, the inaccuracy measure of order α between X and Y_β is equal to the inaccuracy measure of order α between X and $Y_{-\beta}$. Further, for $\beta \neq 0$, we have from* (5)

$$K_{X,Y_\beta}^R(t) = \frac{1}{\alpha-1}\ln\left(\frac{\int_t^1 \left(\beta\left(x-\frac{1}{2}\right)+1\right)^{1-\alpha} dx}{(1-t)\left(1-\frac{t(t-1)\beta}{2}-t\right)^{1-\alpha}}\right). \tag{9}$$



(a) (b)

Figure 1 – Figure (a) represents the plot of the measure (9) for $t \in (0,1)$, with $\beta = 1.7$ (top line) and $\beta = -1.7$ (bottom line), whereas Figure (b) represents the plot of the measure (10) for $t \in (0,1)$, with $\beta = 1.7$ (bottom line) and $\beta = -1.7$ (top line).

Moreover, from Figure $1(a)$ *we observe that* $K^R_{X,Y_\beta}(t) \neq K^R_{X,Y_{-\beta}}(t)$ *in general for all* $t \in (0,1)$. *Hence, we reach to the conclusion that though* $K^R_{X,Y_\beta} = K^R_{X,Y_{-\beta}}$, *the inaccuracy measure of order* α *of* X_t *and* Y_{β_t} *is different from that of* X_t *and* $Y_{-\beta_t}$.

EXAMPLE 2. *Note that under the assumptions of Example* 1, *we can show that* $K^R_{X,Y_\beta} = K^R_{X,Y_{-\beta}}$, *for* $x = \frac{1}{2}$, *where* $\beta \in [-2,2]$, *that is, the inaccuracy measure of order* α *between* X *and* Y_β *is equal to that of* X *and* $Y_{-\beta}$. *Further, for* $\beta \neq 0$, *we have from* (7)

$$\bar{K}^R_{X,Y_\beta}(t) = \frac{1}{\alpha - 1} \ln\left(\frac{\int_0^t \left(\beta\left(x - \frac{1}{2}\right) + 1\right)^{1-\alpha} dx}{t\left(\frac{t(t-1)\beta}{2} + t\right)^{1-\alpha}} \right), \tag{10}$$

which is plotted in Figure $1(b)$. *From the figure, we observe that* $\bar{K}^R_{X,Y_\beta}(t) \neq \bar{K}^R_{X,Y_{-\beta}}(t)$ *in general for all* $t \in (0,1)$. *Thus, we conclude that though* $K^R_{X,Y_\beta} = K^R_{X,Y_{-\beta}}$, *the inaccuracy measure of order* α *of* X^t *and* Y^{β_t} *is not equal to that of* X^t *and* $Y^{-\beta_t}$.

The remainder of the paper is arranged as follows. In Section 2, we study some properties and characterization results of the measure of inaccuracy of order α between residual lifetime distributions. Several bounds of $K^R_{X,Y_\beta}(t)$ are obtained. Results are extended to weighted distributions and examples are provided. Similar results are derived based on the measure of inaccuracy of order α between past lifetime distributions in Section 3. In Section 4, a nonparametric estimator for the residual inaccuracy of order α is proposed, and is validated through a numerical example. Finally, some concluding remarks have been added in Section 5.

2. SOME RESULTS ON $K^R_{X,Y}(t)$

As mentioned earlier, in this section, we study some properties and characterizations of $K^R_{X,Y}(t)$ given by (5). Our first theorem shows how $K^R_{X,Y}(t)$ is affected by a common increasing transformation of X and Y.

THEOREM 1. *Let ϕ be a strictly increasing function. Then*

$$K^R_{X,Y}(\phi^{-1}(t)) = K^R_{\phi(X),\phi(Y)}(t).$$

PROOF. From (5) we have

$$\begin{aligned} K^R_{\phi(X),\phi(Y)}(t) &= \frac{1}{\alpha-1}\ln\left(\frac{\int_t^\infty \frac{\left(f(\phi^{-1}(x))\right)^\alpha \left(g(\phi^{-1}(x))\right)^{1-\alpha} dx}{\left(\bar{F}(\phi^{-1}(t))\right)^\alpha \left(\bar{G}(\phi^{-1}(t))\right)^{1-\alpha} \phi'(\phi^{-1}(x))}}{\int_t^\infty \frac{(f(\phi^{-1}(x)))^\alpha dx}{\left(\bar{F}(\phi^{-1}(t))\right)^\alpha \phi'(\phi^{-1}(x))}}\right) \\ &= \frac{1}{\alpha-1}\ln\left(\frac{\int_{\phi^{-1}(t)}^\infty \frac{(f(z))^\alpha (g(z))^{1-\alpha} dz}{\left(\bar{F}(\phi^{-1}(t))\right)^\alpha \left(\bar{G}(\phi^{-1}(t))\right)^{1-\alpha}}}{\int_{\phi^{-1}(t)}^\infty \frac{(f(z))^\alpha dz}{\left(\bar{F}(\phi^{-1}(t))\right)^\alpha}}\right) \\ &= K^R_{X,Y}(\phi^{-1}(t)). \end{aligned}$$

This completes the proof.

As an application of Theorem 1, we consider the following example.

EXAMPLE 3. *Suppose that X_1 follows Pareto I (b_1, a) and X_2 follows Pareto I (b_2, a) such that $f_{X_1}(x) = \frac{b_1}{a}(\frac{x}{a})^{-(b_1+1)}$, $x > a$; $a, b_1 > 0$ and $f_{X_2}(x) = \frac{b_2}{a}(\frac{x}{a})^{-(b_2+1)}$, $x > a$; $a, b_2 > 0$. Then*

$$K^R_{X_1,X_2}(t) = \frac{1}{\alpha-1}\left[(1-\alpha)\ln b_2 - \ln t + \ln\left(\frac{\alpha(b_1+1)-1}{\alpha(b_1+1)+(1-\alpha)(b_2+1)-1}\right)\right].$$

Clearly $K^R_{X_1,X_2}(t)$ is a function of t. Now let $\phi(x) = x - a$. Then $\phi(X)$ follows Pareto II (Lomax) distribution. Note that $\phi(x)$ is a strictly increasing function. If $Y_1 = \phi(X_1)$ and $Y_2 = \phi(X_2)$ then Y_1 and Y_2 follow Pareto II distribution with common parameter a. Using Theorem 1, we can easily find the expression of $K^R_{Y_1,Y_2}(t)$, given by

$$K^R_{Y_1,Y_2}(t) = \frac{1}{\alpha-1}\left[(1-\alpha)\ln b_2 - \ln(t+a) + \ln\left(\frac{\alpha(b_1+1)-1}{\alpha(b_1+1)+(1-\alpha)(b_2+1)-1}\right)\right]$$

which otherwise not directly obtain.

Next, we obtain characterization of exponential distribution.

THEOREM 2. *Let X and Y be two non-negative rv's with cdf's F and G, respectively. Further, let $K^R_{X,Y}(t)$ be independent of t for all $t > 0$. Then G is exponential if F is exponential.*

PROOF. From (5) it is not hard to obtain the following relation

$$[K^R_{X,Y}(t)]' + \eta_Y(t) = \frac{\eta_X^\alpha(t) e^{(\alpha-1)I^R_X(t)}}{\alpha-1}\left[1 - \eta_Y^{1-\alpha}(t) e^{-(\alpha-1)K^R_{X,Y}(t)}\right]. \tag{11}$$

As F is exponential, we have $\eta_X(t)$ is constant. Also, from Theorem 3.1 of Abraham and Sankaran (2006), we know that if X is exponentially distributed then $I_X^R(t) =$ constant. Under the given assumptions, from (11)

$$A\eta_Y^{1-\alpha}(t)+\eta_Y(t)=B, \tag{12}$$

where A and B are arbitrary constants. Assume that $\eta_Y(t)$ is differentiable. Now by differentiating from (12) with respect to t, we get

$$\eta_Y'(t)[A(1-\alpha)\eta_Y^{-\alpha}(t)+1]=0,$$

implies $\eta_Y(t)=$ constant. Thus G is exponential. This completes the proof.

To estimate the effects of different covariates influencing the times to the failures of a system, Cox introduced the notion of proportional hazard rate model in 1972. This model has some useful applications in different areas of science and technology. We refer to Cox and Oakes (1984) for various applications of this model. Assume that the survival functions of the rv's X and Y are related by

$$\bar{F}(x)=(\bar{G}(x))^{\theta},\ x>0, \tag{13}$$

where $\theta>0$ is known as proportionality constant.

THEOREM 3. *Suppose that $K_{X,Y}^R(t)$ is independent of t. Also let the distribution functions of X and Y satisfy the proportional hazard rate model. Then*

$$I_X^R(t)=\ln\left(B-C\eta_X^{\alpha-1}(t)\right)^{\frac{1}{1-\alpha}},$$

where $B=\frac{\theta^\alpha}{A(1-\alpha)}$ and $C=\frac{\theta}{1-\alpha}$.

PROOF. Under the assumption that $K_{X,Y}^R(t)$ is independent of t, we have

$$A\bar{G}^{1-\alpha}(t)\int_t^\infty f^\alpha(x)dx=\int_t^\infty f^\alpha(x)g^{1-\alpha}(x)dx, \tag{14}$$

where A is a constant independent of t. Differentiating (14) with respect to t and rearranging the terms, we obtain

$$\int_t^\infty f^\alpha(x)dx=\frac{f^\alpha(t)}{A(1-\alpha)}\Big[\frac{1}{\eta_Y^\alpha(t)}-\frac{A}{\eta_Y(t)}\Big]. \tag{15}$$

Now using the proportional hazard rate model (13), we have $\eta_X(t)=\theta\eta_Y(t)$. Thus from (15), the required result follows.

The following remark is an immediate consequence of the Theorem 3.

REMARK 4. *Let X be an exponentially distributed rv. Then, $K_{X,Y}^R(t)$ is independent of t if and only if X and Y satisfy the proportional hazard rate model.*

EXAMPLE 4. *Consider a series system with n components having lifetimes $X_i, i = 1, \ldots, n$. Assume that X_i's are independent and identically distributed with a common pdf $f(x|\sigma) = \sigma e^{-\sigma x}$, $x > 0$, $\sigma > 0$. The lifetime of the system is $Z = min\{X_1, \ldots, X_n\}$. Moreover, it is not difficult to show that $\bar{F}_Z(x) = (\bar{F}_{X_i}(x))^n$, where $\bar{F}_Z(x)$ and $\bar{F}_{X_i}(x)$ are sf's of Z and X_i, $i = 1, \ldots, n$, respectively. This implies that Z and X_i satisfy the proportional hazard rate model. For $i = 1, \ldots, n$, we obtain*

$$K^R_{X_i,Z}(t) = \frac{1}{\alpha - 1} \ln\left[\frac{(\sigma/n)^{\alpha-1}}{\alpha(n + \alpha - n\alpha)}\right],$$

provided $n + \alpha - n\alpha \neq 0$, independent of t.

In our next two consecutive theorems we provide relation between $K^R_{X,Y}(t)$ and $K^R_{X,Y}$. The following definition is useful in this regard.

DEFINITION 5. *A non-negative rv X is said to have*

(i) increasing (decreasing) failure rate (IFR (DFR)) if $\eta_X(t)$ is increasing (decreasing) in $t > 0$.

(ii) new better (worse) than used (NBU (NWU)) if $\bar{F}(x + t) \leq (\geq) \bar{F}(x)\bar{F}(t)$ for all $x, t > 0$.

Also,

$$IFR \Rightarrow NBU \text{ and } DFR \Rightarrow NWU.$$

THEOREM 6. *For the rv's X and Y, if*

(i) $\frac{\eta_X(t)}{\eta_Y(t)}$ is increasing (decreasing) in t,

(ii) both X and Y have increasing (decreasing) failure rate, then for $\alpha < 1$

$$K^R_{X,Y}(t) \geq (\leq) K^R_{X,Y}.$$

PROOF. Note that $K^R_{X,Y}(t)$ is the sum of the measures $I^R_{X,Y}(t)$ and $I^R_X(t)$. Also, it is easy to show that $I^R_{X,Y}(t)$ can be expressed as

$$I^R_{X,Y}(t) = \frac{1}{\alpha - 1} \ln \int_0^1 \frac{f_t^{\alpha-1}(F_t^{-1}(y))}{g_t^{\alpha-1}(G_t^{-1}(y))} dy, \tag{16}$$

where f_t (g_t) is the pdf of X_t (Y_t) and $F_t(G_t)$ is the cdf of X_t (Y_t). Moreover, along the arguments used in the proof of the Theorem 2.2 of Ebrahimi and Kirmani (1996) and for $\alpha < 1$, we obtain

$$\begin{aligned}
\frac{f_t^{\alpha-1}(F_t^{-1}(y))}{g_t^{\alpha-1}(F_t^{-1}(y))} &= \frac{\eta_X^{\alpha-1}(F^{-1}(1-(1-y)\bar{F}(t)))}{\eta_Y^{\alpha-1}(F^{-1}(1-(1-y)\bar{G}(t)))} \frac{(1-y)^{\alpha-1}}{\bar{G}_t^{\alpha-1}(F_t^{-1}(y))} \\
&\leq (\geq) \frac{\eta_X^{\alpha-1}(F^{-1}(y))}{\eta_Y^{\alpha-1}(F^{-1}(y))} \frac{(1-y)^{\alpha-1}}{\bar{G}_t^{\alpha-1}(F_t^{-1}(y))} \\
&\leq (\geq) \frac{\eta_X^{\alpha-1}(F^{-1}(y))}{\eta_Y^{\alpha-1}(F^{-1}(y))} \frac{(1-y)^{\alpha-1}}{\bar{G}^{\alpha-1}(F^{-1}(y))} = \frac{f^{\alpha-1}(F^{-1}(y))}{g^{\alpha-1}(F^{-1}(y))},
\end{aligned}$$

where the first and second inequalities are due to the conditions stated in (i) and (ii), respectively. Therefore, for $\alpha < 1$ we have

$$I^R_{X,Y}(t) \geq (\leq) \frac{1}{\alpha - 1} \ln \int_0^1 \frac{f^{\alpha-1}(F^{-1}(y))}{g^{\alpha-1}(F^{-1}(y))} dy = I^R_{X,Y}. \tag{17}$$

Thus, from (17) and the Corollary 4.1 of Abraham and Sankaran (2006), the proof completes.

REMARK 7. *From the Theorem* 6, *we conclude that under the assumptions made, the inaccuracy between two systems of age t is never smaller (larger) than the inaccuracy when those systems were new for $\alpha < 1$.*

REMARK 8. *For two absolutely continuous non-negative random variables X and Y, if (i) $\frac{\eta_X(t)}{\eta_Y(t)}$ is increasing (decreasing) in t, (ii) both X and Y are NBU (NWU) and (iii) X has DFR (IFR), then for $\alpha > 1$, we obtain $K^R_{X,Y}(t) \leq (\geq) K^R_{X,Y}$. However, if X is both NBU and DFR (NWU and IFR), then X follows exponential.*

Bounds of probability measures are useful when either the measure does not have a closed form or it is difficult to compute. The following theorems provide some upper and lower bounds of $K^R_{X,Y}(t)$, which are functions of hazard rate and Renyi's residual entropy.

DEFINITION 9. *Let X and Y be two non-negative rv's with pdf's f and g, respectively. Then X is said to be less than or equal to Y in likelihood ratio ordering, denoted by $X \overset{lr}{\leq} Y$, if $\frac{f(t)}{g(t)}$ is decreasing in $t > 0$.*

THEOREM 10. *Let $X \overset{lr}{\leq} Y$. Then*

$$(a)\; K^R_{X,Y}(t) \leq \frac{\alpha}{\alpha-1} \ln\left(\frac{\eta_X(t)}{\eta_Y(t)}\right) + I^R_X(t),\; if\;\; \alpha > 1$$

and

$$(b)\; K^R_{X,Y}(t) \geq \frac{\alpha}{\alpha-1} \ln\left(\frac{\eta_X(t)}{\eta_Y(t)}\right) + I^R_X(t),\; if\;\; \alpha < 1.$$

PROOF. (a) Making use of $X \overset{lr}{\leq} Y$ and $x > t$ in (5), we obtain

$$\begin{aligned} K^R_{X,Y}(t) &\leq \frac{1}{\alpha-1} \ln \int_t^\infty \frac{f^\alpha(t) g(x)}{\bar{F}^\alpha(t) \bar{G}^{1-\alpha}(t) g^\alpha(t)} dx + I^R_X(t) \\ &= \frac{\alpha}{\alpha-1} \ln\left(\frac{\eta_X(t)}{\eta_Y(t)}\right) + I^R_X(t) \end{aligned}$$

as $\left(\frac{f(x)}{g(x)}\right)^\alpha \leq \left(\frac{f(t)}{g(t)}\right)^\alpha$ for $\alpha > 1$. Similarly, for $\alpha < 1$, it is not hard to obtain the inequality given in Part (b). This completes the proof.

Let the pdf of X be f, and w be a non-negative function with $\mu_w = E(w(X)) < \infty$. Also, let f_w, F_w and $\bar{F}_w$, respectively be the pdf, cdf and sf of a weighted rv X_w, where $f_w = wf/\mu_w$, $F_w = E(w(X)|X < t)F/\mu_w$ and $\bar{F}_w = E(w(X)|X > t)\bar{F}/\mu_w$. The next corollary is a consequence of the Theorem 10, since

$$\frac{\eta_X(t)}{\eta_{X_w}(t)} = \frac{E(w(X)|X > t)}{w(t)}.$$

We omit the proof here. Note that the corollary provides the bounds of the dynamic measure of inaccuracy of order α for residual lifetime distributions of X and X_w. Let w be increasing. Then it is easy to show that $\frac{f}{f_w}$ is decreasing. Hence, $X \overset{lr}{\leq} X_w$.

COROLLARY 11. *Let w be increasing. Then*

$$(a)\, K^R_{X,X_w}(t) \leq \frac{\alpha}{\alpha-1}\ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) + I^R_X(t),\ if\ \alpha > 1$$

and

$$(b)\, K^R_{X,X_w}(t) \ \geq \ \frac{\alpha}{\alpha-1}\ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) + I^R_X(t),\ if\ \alpha < 1.$$

We consider the following example which illustrates the Corollary 11.

EXAMPLE 5. *Let a rv X follow Pareto $I(a,b)$, where $b > 0$ and $a > 1$. Consider the weight function $w(x) = x$. Therefore, it can be easily established that $X \overset{lr}{\leq} X_w$. Moreover,*

$$K^R_{X,X_w}(t) = \frac{\alpha}{\alpha-1}\ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) + I^R_X(t) + \frac{1}{\alpha-1}\ln\Big(\frac{a-1}{a+\alpha-1}\Big). \tag{18}$$

Thus from (18), *the inequalities in Corollary* 11 *follow.*

REMARK 12. *For all α ($\neq 1$),*

$$K^R_{X,Y}(t) \leq \ln\Big(\frac{\eta_X(t)}{\eta_Y(t)}\Big) + I^R_X(t),\ if\ X \overset{lr}{\leq} Y$$

and

$$K^R_{X,X_w}(t) \leq \ln\Big(\frac{E(w(X)|X > t)}{w(t)}\Big) + I^R_X(t),\ if\ X \overset{lr}{\leq} X_w.$$

Our next theorem provides lower bound of $K^R_{X,Y}(t)$ in terms of the hazard rate function.

THEOREM 13. *Suppose that $g(x)$ is decreasing in x. Then*

$$K^R_{X,Y}(t) \geq -\ln(\eta_Y(t))$$

for $\alpha \neq 1$.

PROOF. We have $g(x) \leq g(t)$, as $g(x)$ is decreasing in x and $x > t$. Using this we have from (5)

$$K^R_{X,Y}(t) \quad \geq \quad \frac{1}{\alpha-1}\ln\int_t^\infty \frac{f^\alpha(x)}{\bar{F}^\alpha(t)}\frac{g^{1-\alpha}(t)}{\bar{G}^{1-\alpha}(t)}dx + I^R_X(t) = -\ln(\eta_Y(t)).$$

This completes the proof.

Note that the hazard rate function of X_w can be written as $\eta_{X_w}(t) = \frac{w(t)}{E(w(X)|X>t)}\eta_X(t)$. Therefore, Theorem 13 leads to the following corollary.

COROLLARY 14. *Suppose $f_w(x)$ is decreasing in x. Then*

$$K^R_{X,X_w}(t) \geq -\ln\Big(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\Big)$$

for $\alpha \neq 1$.

EXAMPLE 6. *Consider a rv X and the weight function as in Example 5. Then $K^R_{X,X_w}(t)$ in (18) can be written further as the following form*

$$K^R_{X,X_w}(t) = -\ln\Big(\frac{w(t)\eta_X(t)}{E(w(X)|X>t)}\Big) + \frac{1}{\alpha-1}\ln\Big(\frac{a\alpha+\alpha-1}{a+\alpha-1}\Big). \tag{19}$$

In the following theorem we obtain upper bound of $K^R_{X,X_w}(t)$, in terms of hazard rate and Renyi's residual entropy.

THEOREM 15. *Let the weight function $w(x)$ be increasing in x. Then for $\alpha \neq 1$,*

$$K^R_{X,X_w}(t) \leq I^R_X(t) + \ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big).$$

PROOF. Given that $w(x)$ is increasing. Therefore, from (5) we have

$$\begin{aligned} K^R_{X,X_w}(t) \quad &\leq \quad I^R_X(t) + \frac{1}{\alpha-1}\ln\Big[\frac{1}{\bar{F}(t)}\int_t^\infty f^\alpha(x)\Big(\frac{w(t)f(x)}{E(w(X)|X>t)}\Big)^{1-\alpha}\Big]dx \\ &= \quad I^R_X(t) + \ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big). \end{aligned}$$

This completes the proof.

EXAMPLE 7. *Consider the rv X as described in Example 5. Then $K^R_{X,X_w}(t)$, obtained in (18) can be written as*

$$K^R_{X,X_w}(t) \quad = \quad I^R_X(t) + \ln\Big(\frac{E(w(X)|X>t)}{w(t)}\Big) + \frac{1}{\alpha-1}\ln\Big(\frac{a}{a+\alpha-1}\Big). \tag{20}$$

Therefore, from (20) it is not hard to verify the Theorem 15.

In the following theorem, we consider three non-negative rv's X_1, X_2 and X_3 and obtain bounds of $K^R_{X_1,X_3}(t) - K^R_{X_2,X_3}(t)$.

THEOREM 16. *Suppose that the rv's* X_1, X_2 *and* X_3 *have pdf's* f_1, f_2, f_3; *sf's* $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$ *and hazard rate functions* η_1, η_2, η_3, *respectively. Assume* $X_1 \overset{lr}{\leq} X_2$, *that is,* $\frac{f_2}{f_1}$ *is increasing in* x. *Then*

$$(a)\; K^R_{X_1,X_3}(t) - K^R_{X_2,X_3}(t) \leq \frac{\alpha}{\alpha-1}\ln\left(\frac{\eta_1(t)}{\eta_2(t)}\right) + I^R_{X_1}(t) - I^R_{X_2}(t),\; if\ \alpha > 1,$$

and

$$(b)\; K^R_{X_1,X_3}(t) - K^R_{X_2,X_3}(t) \geq \frac{\alpha}{\alpha-1}\ln\left(\frac{\eta_1(t)}{\eta_2(t)}\right) + I^R_{X_1}(t) - I^R_{X_2}(t),\; if\ \alpha < 1.$$

PROOF. (a) Under the given hypothesis, we have $\frac{f_1(x)}{f_2(x)} \leq \frac{f_1(t)}{f_2(t)}$. Therefore, for $\alpha > 1$, from (5) we deduce that

$$\begin{aligned} K^R_{X_1,X_3}(t) &\leq \frac{1}{\alpha-1}\ln\int_t^\infty \frac{f_2^\alpha(x) f_1^\alpha(t)}{f_2^\alpha(t)\bar{F}_1^\alpha(t)} \frac{f_3^{1-\alpha}(x)}{\bar{F}_3^{1-\alpha}(t)}dx + I^R_{X_1}(t) \\ &= K^R_{X_2,X_3}(t) + \frac{\alpha}{\alpha-1}\ln\left(\frac{\eta_1(t)}{\eta_2(t)}\right) + I^R_{X_1}(t) - I^R_{X_2}(t). \end{aligned}$$

This completes the proof of Part (a). Part (b) follows similarly.

EXAMPLE 8. *Suppose* X_i *follows exponential distribution with parameter* $\sigma_i > 0$, $i = 1,2$. *Assume that* X_1 *and* X_2 *are independently distributed and* $\sigma_1 > \sigma_2$. *Hence,* $X_1 \overset{lr}{\leq} X_2$. *Consider another rv* $X_3 = \min\{X_1, X_2\}$. *Then*

$$\begin{aligned} K^R_{X_1,X_3}(t) - K^R_{X_2,X_3}(t) = &\frac{\alpha}{\alpha-1}\ln\left(\frac{\eta_1(t)}{\eta_2(t)}\right) + I^R_{X_1}(t) - I^R_{X_2}(t) \\ &+ \frac{1}{\alpha-1}\ln\left(\frac{\sigma_1+\sigma_2-\sigma_1\alpha}{\sigma_1+\sigma_2-\sigma_2\alpha}\right), \end{aligned} \tag{21}$$

provided $\sigma_1+\sigma_2-\sigma_1\alpha > 0$ *and* $\sigma_1+\sigma_2-\sigma_2\alpha > 0$, *proves the inequality in Theorem* 16.

THEOREM 17. *Consider three rv's* X_1, X_2 *and* X_3 *as described in Theorem* 16. *Further, assume* $X_2 \overset{lr}{\leq} X_3$, *that is,* $\frac{f_3}{f_2}$ *is increasing in* $x > 0$. *Then for* $\alpha \neq 1$,

$$K^R_{X_1,X_2}(t) - K^R_{X_1,X_3}(t) \geq -\ln\left(\frac{\eta_2(t)}{\eta_3(t)}\right).$$

PROOF. Under the given conditions, we have $\frac{f_2(x)}{f_3(x)} \leq \frac{f_2(t)}{f_3(t)}$. Therefore, from (5) for $\alpha \neq 1$, we obtain

$$\begin{aligned} K^R_{X_1,X_2}(t) &\geq \frac{1}{\alpha-1}\ln\int_t^\infty \frac{f_1^\alpha(x) f_2^{1-\alpha}(t)}{f_3^{1-\alpha}(t)\bar{F}_1^\alpha(t)} \frac{f_3^{1-\alpha}(x)}{\bar{F}_2^{1-\alpha}(t)}dx + I^R_{X_1}(t) \\ &= K^R_{X_1,X_3}(t) - \ln\left(\frac{\eta_2(t)}{\eta_3(t)}\right). \end{aligned}$$

Hence the result follows.

EXAMPLE 9. *Let the rv's X_2 and X_3 follow Pareto I (a_2, b_2) and Pareto I (a_3, b_3), respectively, where $a_2, b_2, a_3, b_3 > 0$ and $b_2 > b_3$. Also, assume that X_2 and X_3 are independently distributed. It can be shown that $X_2 \overset{lr}{\leq} X_3$. Consider another rv $X_1 = \min\{X_2, X_3\}$. Then*

$$K^R_{X_1,X_2}(t) - K^R_{X_1,X_3}(t) = -\ln\left(\frac{\eta_2(t)}{\eta_3(t)}\right) + \frac{1}{\alpha-1}\ln\left(\frac{a_2\alpha+a_3}{a_3\alpha+a_2}\right). \tag{22}$$

Therefore, from (22) *the Theorem* 17 *can be easily verified.*

3. SOME RESULTS ON $\bar{K}^R_{X,Y}(t)$

In this section we consider time dependent measure of inaccuracy of order α between past lifetime distributions $\bar{K}^R_{X,Y}(t)$ defined in (7). Even if the past lifetime information divergence measures appears to be a dual of its residual version (5), Di Crescenzo and Longobardi (2004) have identified some importance of Kullback-Leibler divergence measure for past lifetime rv's. Since $\bar{K}^R_{X,Y}(t)$ is a generalized divergence, thus a separate study of the same is also worthwhile. However, as most of the results are parallel to its residual version, the statements of the results in past lifetime are omitted. The following theorem shows how $\bar{K}^R_{X,Y}(t)$ is affected by an increasing transformation of X and Y.

THEOREM 18. *Let ϕ be a strictly increasing function. Then*

$$\bar{K}^R_{X,Y}(\phi^{-1}(t)) = \bar{K}^R_{\phi(X),\phi(Y)}(t).$$

EXAMPLE 10. *Consider two rv's X_1 and X_2 following exponential distributions with mean σ_1 and σ_2, respectively. Here, $\bar{K}^R_{X_1,X_2}(t)$ can be obtained as*

$$\bar{K}^R_{X_1,X_2}(t) = \frac{\alpha\left[1-e^{-t\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}\right]}{\sigma_1\sigma_2^\alpha\left(1-e^{-\frac{\alpha t}{\sigma_1}}\right)\left(1-e^{-\frac{t}{\sigma_2}}\right)^\alpha\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}. \tag{23}$$

Further, let $\phi(x) = x^{1/\gamma}$, $x > 0$, $\gamma > 0$. Note that ϕ is a strictly increasing function. It can be showed that $Y_1 = \phi(X_1)$ and $Y_2 = \phi(X_2)$ follow Weibull distribution with a common shape parameter γ, and scale parameters σ_1 and σ_2, respectively. Then, in order to derive the expression of $\bar{K}^R_{Y_1,Y_2}(t)$, one has to use the result of Theorem 18 *which otherwise not able to obtain directly. Note that $\bar{K}^R_{Y_1,Y_2}(t)$ can be obtained by substituting $\phi^{-1}(t) = t^\gamma$ in place of t in* (22), *given by*

$$\bar{K}^R_{Y_1,Y_2}(t) = \frac{\alpha\left[1-e^{-t^\gamma\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}\right]}{\sigma_1\sigma_2^\alpha\left(1-e^{-\frac{\alpha t^\gamma}{\sigma_1}}\right)\left(1-e^{-\frac{t^\gamma}{\sigma_2}}\right)^\alpha\left(\frac{\alpha}{\sigma_1}+\frac{1-\alpha}{\sigma_2}\right)}.$$

4. Numerical Example

In this section, we propose a nonparametric kernel-based density estimator for $K^R_{X,Y}(t)$ and is validated through the simulated random samples. We use kernel density estimator for f and empirical estimator for the survival function $\bar{F}$, and g is assumed to be a known density.

TABLE 1
Simulation results.

		$n=10$		$n=25$				$n=10$		$n=25$	
α	t	Bias	MSE	Bias	MSE	α	t	Bias	MSE	Bias	MSE
	1	-1.10	1.34	-0.93	0.91	2	1	0.69	0.96	0.33	0.21
0.5	1.2	-1.32	1.90	-0.90	0.87		1.2	0.14	0.57	-0.07	0.16
	1.4	-1.63	2.83	1.37	1.99		1.4	0.02	0.44	0.02	0.38
	1.5	-1.92	3.87	-1.41	2.10		1.5	0.06	0.52	0.16	0.41
	1.6	-1.94	3.91	-1.56	2.60		1.6	-0.15	0.63	0.03	0.53
	1	-0.47	0.39	-0.33	0.20		1	0.67	0.97	1.17	3.23
0.75	1.2	-0.94	1.13	-0.64	0.46	2.5	1.2	0.26	0.83	0.27	0.80
	1.4	-1.16	1.51	-0.75	0.65		1.4	0.33	0.75	0.20	0.83
	1.5	-1.23	1.72	-0.87	0.86		1.5	0.01	0.91	0.26	1.03
	1.6	-1.18	1.67	0.77	0.74		1.6	0.09	0.87	-0.06	0.49
	1	0.46	0.80	0.30	0.22						
1.5	1.2	0.06	0.48	0.01	0.14						
	1.4	-0.20	0.28	-0.15	0.29						
	1.5	-0.33	0.44	-0.31	0.29						
	1.6	-0.16	0.78	-0.15	0.36						

Let $(X_1, X_2, \ldots, X_n)$ be a random sample taken from a population with probability density function f and survival function $\bar{F}$. Then a nonparametric estimator of $K^R_{X,Y}(t)$ is defined by

$$\begin{aligned} \hat{K}^R_{X,Y}(t) = & \frac{1}{\alpha-1}\log\int_t^\infty \left(\frac{f_n(x)}{\bar{F}_n(t)}\right)^\alpha \left(\frac{g(x)}{\bar{G}(t)}\right)^{1-\alpha} dx \\ & + \frac{1}{1-\alpha}\log\int_t^\infty \left(\frac{f_n(x)}{\bar{F}_n(t)}\right)^\alpha dx, \end{aligned} \tag{24}$$

where $f_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right)$ denotes the kernel density estimator of f, $\bar{F}_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i > t)$ the empirical estimator of $\bar{F}$, and $I(X_i > t)$ the indicator variable. We assume that Y follows Pareto I (b,a) distribution with probability density function

$$g(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{-b-1}; x > a;\ a, b > 0. \tag{25}$$

For the simulation under complete sample, we have generated samples from Pareto I density g in (25) with $a = 1$ and $b = 3$ respectively. $\hat{K}^R_{X,Y}(t)$ for various values of t

and sample sizes $n = 10$ and $n = 25$ are calculated. The kernel function is taken to be Gaussian kernel, $K(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}$, with bandwidth $h_n = n^{-\frac{1}{2}}$ and 50 iterations are carried out. To find the bias, we further assume that f is also Pareto I with the form (25), and parameters $a = 0.5$ and $b = 2.5$ respectively. The bias and mean squared error (MSE) of $\hat{K}^R_{X,Y}(t)$ for different values of α and t are computed and given in Table 1. It is evident from Table 1 that MSE of $\hat{K}^R_{X,Y}(t)$ decreases as the sample size increases and in terms of bias and MSE, the estimator is more ideal when $\alpha = 2$. Since $\hat{K}^R_{X,Y}(t)$ is obtained as a plug-in estimator in the nonparametric setup, the asymptotic normality of (24) is direct using the standard nonparametric estimation theory. A numerical study of $\bar{K}^R_{X,Y}(t)$ is similar and hence omitted.

5. Conclusion

In this paper, based on the Renyi entropy, we proposed some dynamic generalized measures of inaccuracy of order $\alpha(\neq 1) > 0$ between two probability distributions. Note that as α tends to 1, these measures reduce to the dynamic inaccuracy measures due to Taneja *et al.* (2009) and Kumar *et al.* (2011). Some characterization results and bounds of the proposed measure in residual time are obtained. Numerical example using a nonparametric estimator is given to illustrate the usefulness of the generalized residual measure of inaccuracy of order α.

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Summary

Generalized information measures play an important role in the measurement of uncertainty of certain random variables, where the standard practice of applying ordinary uncertainty measures fails to fit. Based on the Renyi entropy and its divergence, we propose a generalized measure of inaccuracy of order α $(\neq 1) > 0$ between two residual and past lifetime distributions of a system. We study some important properties and characterizations of these measures. A numerical example is given to illustrate the usefulness of the proposed measure.

Keywords: Kerridge inaccuracy measure; Renyi entropy; Reliability measures; Characterization.