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SHARMA-MITTAL ENTROPY PROPERTIES ON RECORD VALUES

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1. Introduction

In the context of equilibrium thermodynamics, physicists originally developed the notion of entropy, which was later extended through the development of statistical mechanics. Shannon (1948) introduced a generalization of Boltzmann-Gibbs entropy and later it was known as Shannon entropy or Shannon information measure. Shannon entropy represents an absolute limit on the best possible lossless compression of any communication. More generally the concept of entropy is a measure of uncertainty associated with a random variable. For a continuous random variable X with probability density function (pdf) f, the Shannon entropy is defined by

$$H(X) = -\int_{-\infty}^{\infty} f(x) \log f(x)\ \mathrm{dx}. \tag{1}$$

In continuous case, $H(X)$ is also referred to as the differential entropy. It is known that $H(X)$ measures the uniformity of f. When $H(X_1) > H(X_2)$, for any two random variables with pdf f_1 and f_2 respectively, then we conclude that it is more difficult to predict outcomes of X_1, as compared with predicting outcomes of X_2 (see, Zarezadeh and Asadi, 2010).

Sharma and Mittal (1975) introduced a two parameter entropy measure $H_{\alpha,\beta}(X)$ of a random variable X with pdf f as a generalization of the Shannon entropy measure and it is given by

$$H_{\alpha,\beta}(X) = \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^{\infty} \{f(x)\}^{\alpha}\,\mathrm{dx}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \tag{2}$$

with $\alpha, \beta > 0$, $\alpha \neq 1 \neq \beta$ and $\alpha \neq \beta$. It is clear to be note that if we take limit $\beta \to 1$ in (2) then Sharma-Mittal entropy becomes Rényi entropy (1961) which

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is given by

$$H_{\alpha,1}(X) = \frac{1}{1-\alpha} \log \int \{f(x)\}^{\alpha} \,\mathrm{dx}. \tag{3}$$

If we take limit as $\beta \to \alpha$, in (2), then the resulting expression is Tsallis entropy (1988) and is given by

$$H_{\alpha,\alpha}(X) = \frac{1}{1-\alpha} \left\{ \int_{-\infty}^{\infty} \{f(x)\}^{\alpha} \,\mathrm{dx} - 1 \right\}. \tag{4}$$

In the limiting case when both parameters approach 1, we recover the ordinary Shannon entropy (1948) as given in (1).

One may observe several applications of Sharma-Mittal entropy from the available literature. Frank and Daffertshofer (2000) have established the relation between anomalous diffusion process and Sharma-Mittal entropy. Masi (2005) explained how this entropy measure unifies Rényi and Tsallis entropies. For more details on the applications of this entropy see, Aktürk *et al.* (2008) and Kosztołowicz and Lewandowska (2012). Nielsen and Nock (2012) obtained a closed form formula for the Sharma-Mittal entropy of any distribution belonging to the exponential family of distributions.

Let $\{X_i, i \geq 1\}$, be a sequence of independent and identically distributed (*iid*) continuous random variables with common cumulative distribution function (cdf) $F(x)$ and pdf $f(x)$. An observation X_j will be called an upper record value if its value is greater than that of all previously realized observations. Thus X_j is an upper record value if $X_j > X_i$ for all $i < j$. If we construct the sequence of upper record values of the sequence $\{X_i, i \geq 1\}$ then the *nth* member of this sequence is denoted by $X_{U(n)}$. If we write $f_{X_{U(n)}}(x)$ to denote the pdf of $X_{U(n)}$, then from Arnold *et al.* (1998) we have

$$f_{X_{U(n)}}(x) = \frac{[-\log(1-F(x))]^{n-1}}{(n-1)!} f(x). \tag{5}$$

An analogous definition can be given for lower record values as well. If we write $X_{L(n)}$ to denote the *nth* lower record value and $f_{X_{L(n)}}(x)$ to denote its pdf then we have

$$f_{X_{L(n)}}(x) = \frac{[-\log(F(x))]^{n-1}}{(n-1)!} f(x). \tag{6}$$

Chandler (1952) first introduced the probabilistic study of record statistics. Record values arise naturally in problems such as industrial stress testing, meteorological analysis, hydrology, sporting, stock markets, athletic events and seismology. For more details on applications of record values see, Arnold *et al.* (1998), Nevzorov (2001) and the references there in.

Anderson *et al.* (2004) have attributed some connection between record statistics and the strain released in quakes. Majumdar and Ziff (2008) have enlisted the detailed involvement of record theory in its multiple applications in spin glasses, adaptive process, evolutionary models of biological population and so on. See also Sibani and Henrik (2009) for some record dynamics arising in some physical systems.

Recently much attention is focused with a generalized version of record values which are called generalized(k)record values (kth records) in which if we put

$k = 1$, then they became classical records. For some recent treatment on these generalized record values see, Madadi and Tata (2014), Paul and Thomas (2013, 2015a), Minimol and Thomas (2013, 2014) and Thomas and Paul (2014).

Of late several articles have been published on various information measures associated with record values. Baratpour *et al.* (2007) studied some information properties of records based on Shannon entropy. Abbasnejad and Arghami (2011) studied the Rényi entropy properties of records and compared the same information with that of the *iid* observations. Baratpour *et al.* (2007), Ahmadi and Fashandi (2012) and Paul and Thomas (2013, 2015b) have obtained some characterization results based on Shannon, Rényi and Tsallis entropies of record values. Shannon information in k-records was studied by Madadi and Tata (2014).

The rest of this paper is organized as follows. In section 2 we express the Sharma-Mittal entropy of *nth* upper record arising from an arbitrary distribution in terms of Sharma-Mittal entropy of *nth* upper record arising from standard exponential distribution. Section 3 provides bounds for Sharma-Mittal entropy of records. In section 4, we characterize exponential distribution by maximizing Sharma-Mittal entropy of record values arising from a specified class of distributions. Section 5 contains expressions for some measures associated with Sharma-Mittal entropy on records and concomitants of records. In subsection 5.1, it is shown that the Sharma-Mittal divergence information between record value and the parent distribution is distribution free. Section 5.2 contains the representation of Sharma-Mittal entropy of concomitants of record values arising from Farlie-Gumbel-Morgenstern family of bivariate distributions. In section 5.3 we provide an expression for the residual Sharma-Mittal entropy of *nth* upper record arising from an arbitrary distribution in terms of the corresponding expressions for the *nth* upper record arising from standard uniform distribution.

2. Sharma-Mittal Entropy of Record Values

In this section, we describe some properties of Sharma-Mittal entropy of record values. In the following theorem, we express Sharma-Mittal entropy of *nth* upper record arising from an arbitrary distribution in terms of Sharma-Mittal entropy of *nth* upper record arising from standard exponential distribution. In the theorem and in the remaining part of this paper we use the notation $G(a,b)$ to denote the well known gamma distribution with pdf

$$g_{a,b}(x) = \frac{a^b}{\Gamma(b)} e^{-ax} x^{b-1}, \;\; a > 0, \;\; b > 0, \;\; x > 0.$$

Theorem 1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables from a distribution with cdf $F(x)$, pdf $f(x)$ and quantile function $F^{-1}(.)$. Let $\{X_{U(n)}\}$ be the associated sequence of upper record values. Then the Sharma-Mittal entropy of $X_{U(n)}$ can be expressed as*

$$H_{\alpha,\beta}(X_{U(n)}) = \frac{1}{1-\beta} \left\{ \left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}} E_{g_{1,(n-1)\alpha+1}} \left[\left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \right] \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right\}, \tag{7}$$

where $E_{g_{a,b}}(.)$ denotes the expected value of Gamma distribution with parameter $a = 1$ and $b = (n-1)\alpha+1$ and U is a random variable, with $G(1,(n-1)\alpha+1)$ distribution.

Proof. The Sharma-Mittal entropy of *nth* record value is given by

$$H_{\alpha,\,\beta}(X_{U(n)}) = \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^{\infty}\left[\frac{\{-\log(1-F(x))\}^{n-1}}{(n-1)!}f(x)\right]^{\alpha}\mathrm{dx}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}.$$

On putting $u = -\log[1-F(x)]$, $x = \left[F^{-1}(1-e^{-u})\right]$ and $\mathrm{d}u = \frac{f(x)}{1-F(x)}\mathrm{d}x$ we get

$$\begin{aligned}
H_{\alpha,\beta}(X_{U(n)}) =& \frac{1}{1-\beta}\left\{\left(\int_0^{\infty}\frac{e^{-u}u^{(n-1)\alpha}}{[(n-1)!]^{\alpha}}\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}\\
=& \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}}\int_0^{\infty}\frac{e^{-u}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\right.\right.\\
&\left.\left.\times\ \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}}-1\right\} \qquad (8)\\
=& \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}}E_{g_{1,(n-1)\alpha+1}}\right.\right.\\
&\left.\left.\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \qquad (9)
\end{aligned}$$

Now we state the following theorem without proof as the proof is just similar to the proof of theorem 1.

Theorem 2. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables with common cdf F(x), pdf f(x) and quantile function $F^{-1}(.)$. Let $\{X_{L(n)}\}$ be the associated sequence of lower record values. Then the Sharma-Mittal entropy of $X_{L(n)}$ can be expressed as*

$$\begin{aligned}
H_{\alpha,\,\beta}(X_{L(n)}) =& \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}}E_{g_{1,(n-1)\alpha+1}}\right.\right.\\
&\left.\left.\left[\left\{f\{F^{-1}(e^{-U})\}\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}, \qquad (10)
\end{aligned}$$

where U is a random variable with $G(1,\,(n-1)\alpha+1)$ distribution.

The following is a corollary to theorem 1.

Corollary 3. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables arising from standard exponential distribution. Let $\{X^*_{U(n)}\}$ be the associated sequence of upper record values. Then the Sharma-Mittal entropy of $X_{U(n)}$ can be expressed as*

$$H_{\alpha,\,\beta}(X^*_{U(n)}) \ = \ \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}\alpha^{(n-1)\alpha+1}}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \qquad (11)$$

The following theorem follows from theorems 1 and 2 as a consequence of corollary 3.

THEOREM 4. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables having a common cdf $F(x)$, pdf $f(x)$ and quantile function $F^{-1}(.)$. Let $\{X_{U(n)}\}$ and $\{X_{L(n)}\}$ be the associated sequences of upper and lower record values respectively. Then the Sharma-Mittal entropy of $X_{U(n)}$ and $X_{L(n)}$ can be expressed as*

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)}) &= \Bigg\{ \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta} \right) \Big(\alpha^{(n-1)\alpha+1} E_{g_{1,(n-1)\alpha+1}} \\ &\times \left[\left\{ f\left(F^{-1}(1-e^{-U})\right) \right\}^{\alpha-1} \right] \Big)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \Bigg\} \end{aligned} \tag{12}$$

$$\begin{aligned} H_{\alpha,\beta}(X_{L(n)}) &= \Bigg\{ \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta} \right) \Big(\alpha^{(n-1)\alpha+1} E_{g_{1,(n-1)\alpha+1}} \\ &\times \left[\left\{ f\left(F^{-1}(e^{-U})\right) \right\}^{\alpha-1} \right] \Big)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \Bigg\}, \end{aligned} \tag{13}$$

where $X^*_{U(n)}$ denotes the nth upper record value arising from the standard exponential distribution and U is a random variable, with $G(1,(n-1)\alpha+1)$ distribution.

3. Bounds for Sharma-Mittal Entropy of Record Values

Baratpour *et al.* (2007) and Abbasnejad and Arghami (2011) have obtained bounds for Shannon entropy of records and Rényi entropy of records respectively. In this section, we use the relation (7) for deriving some bounds on Sharma-Mittal entropy of upper record values.

THEOREM 5. *If X has pdf $f(x)$ and the Sharma-Mittal entropy $H_{\alpha,\beta}(X_{U(n)})$ of $X_{U(n)}$ arising from $f(x)$ is such that $H_{\alpha,\beta}(X_{U(n)}) < \infty$ then we have*

(a) for all $\alpha > 1$ and $0 < \beta < 1$,
$H_{\alpha,\beta}(X_{U(n)}) \leq \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta} \right) \left(\alpha^{(n-1)\alpha+1} B_n S_\alpha(f) \right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$,
and

(b) for $0 < \alpha < 1$ and $\beta > 1$,
$H_{\alpha,\beta}(X_{U(n)}) \geq \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta} \right) \left(\alpha^{(n-1)\alpha+1} B_n S_\alpha(f) \right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$,
where,

(i) $X^*_{U(n)}$ denotes the nth upper record value arising from the standard exponential distribution

(ii) $B_n = \frac{e^{-((n-1)\alpha)}\{(n-1)\alpha\}^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}$ and

(iii) $S_\alpha(f) = \int_{-\infty}^{\infty} \lambda_F(x) \{f(x)\}^{\alpha-1}\, dx$, where $\lambda_F(x)$ is the hazard function of X.

PROOF. The Sharma-Mittal entropy of *nth* upper record value is given by

$$H_{\alpha,\beta}(X_{U(n)}) = \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\Big(\alpha^{(n-1)\alpha+1} E_{g_{1,(n-1)\alpha+1}}$$
$$\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\Big)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$$
$$= \left(H_{\alpha,\,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1} \int_0^\infty \frac{e^{-u}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\right.$$
$$\left.\times \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta},$$

where $g_{1,(n-1)\alpha+1}$ is the pdf corresponding to the $G(1,\,(n-1)\alpha+1)$ distribution. Since the mode of the distribution with pdf $g_{1,(n-1)\alpha+1}$ is $m_n = (n-1)\alpha$ we have

$$g_{1,(n-1)\alpha+1}(m_n) = \frac{e^{-(n-1)\alpha}[(n-1)\alpha]^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)} = B_n.$$

Hence we have $g_{1,(n-1)\alpha+1}(u) \leq B_n$. Now for $\alpha > 1$ and $0 < \beta < 1$ the entropy is

$$H_{\alpha,\beta}(X_{U(n)}) = \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1} \int_0^\infty g_{1,(n-1)\alpha+1}(u)\right.$$
$$\left.\times \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$$
$$\leq \left(H_{\alpha,\,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)$$
$$\times \left(\alpha^{(n-1)\alpha+1} B_n \int_0^\infty \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$$
$$= \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)$$
$$\times \left(\alpha^{(n-1)\alpha+1} B_n \int_{-\infty}^\infty \lambda_F(y)\left\{f(y)\right\}^{\alpha-1} \mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}$$
$$= \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1} B_n S_\alpha(f)\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}.$$

For $0 < \alpha < 1$ and $\beta > 1$ the proof is similar.

4. CHARACTERIZATION PROPERTY BY THE SHARMA-MITTAL ENTROPY OF RECORDS

In this section, we derive exponential distribution as the distribution that maximizes the Sharma-Mittal entropy of record values under some information constraints. Let C be a class of all distributions with cdf $F(x)$ over the support set $\mathbb{R}^+$ with $F(0) = 0$ such that

(i) $\lambda_F(x,\theta) = a(\theta)b(x)$

(ii) $b(x) \leq M$, where M is a positive real constant with $b(x) = B^{'}(x)$ such that $b(x)$ and $a(\theta)$ are non-negative functions of x and θ receptively.

Now we prove the following theorem.

THEOREM 6. *Under the conditions described above Sharma-Mittal entropy $H_{\alpha,\,\beta}(X_{U(n)})$ arising from the distribution $F(x)$ is maximum in C, if and only if $F(x;\theta) = 1 - e^{-Ma(\theta)x}$.*

PROOF. Let $X_{U(n)}$ be the nth upper record value arising from the cdf $F(x;\theta) \in C$. Then by (7) we have

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)}) &= \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right) \\ &\quad\times \left(\alpha^{(n-1)\alpha+1} E_{g_{1,(n-1)\alpha+1}}\left[\left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} \\ &\quad- \frac{1}{1-\beta} \\ &= \left(H_{\alpha,\,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1}\int\limits_0^\infty \frac{e^{-u}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\right. \\ &\quad\left.\times \left\{f\left(F^{-1}(1-e^{-U})\right)\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &= \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1}\int\limits_0^\infty \frac{e^{-u}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\right. \\ &\quad\left.\times \left\{a(\theta)b\left[B^{-1}\left\{\frac{u}{a(\theta)}\right\}\right] e^{-a(\theta)BB^{-1}\left\{\frac{u}{a(\theta)}\right\}}\right\}^{\alpha-1} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &= \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left(\alpha^{(n-1)\alpha+1}\int\limits_0^\infty \frac{e^{-u\alpha}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\right. \\ &\quad\left.\times [a(\theta)]^{\alpha-1} b^{\alpha-1}\left[B^{-1}\left\{\frac{u}{a(\theta)}\right\}\right] \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}. \end{aligned} \tag{14}$$

Noting that $b(x) \leq M$ we have

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)}) &\leq \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left([a(\theta)]^{\alpha-1} M^{\alpha-1}\alpha^{(n-1)\alpha+1}\right. \\ &\quad\left.\times \int\limits_0^\infty \frac{e^{-u\alpha}u^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \\ &\leq \left(H_{\alpha,\beta}(X^*_{U(n)}) + \frac{1}{1-\beta}\right)\left\{[a(\theta)]^{\alpha-1} M^{\alpha-1}\right\}^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}. \end{aligned} \tag{15}$$

Then clearly

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)}) &\leq \frac{1}{1-\beta}\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}\alpha^{(n-1)\alpha+1}}\left\{[a(\theta)]\,M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}}-\frac{1}{1-\beta} \\ &\leq \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}\alpha^{(n-1)\alpha+1}}\left\{[a(\theta)]\,M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \quad (16) \end{aligned}$$

This proves the necessary part of the theorem.

On the other hand, suppose the nth upper record value arising from $F(x;\theta) = 1 - e^{-Ma(\theta)x}$ has maximum Sharma-Mittal entropy in class C. Then we have

$$H_{\alpha,\beta}(X_{U(n)}) = \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1)}{\{\Gamma(n)\}^{\alpha}\alpha^{(n-1)\alpha+1}}\left\{[a(\theta)]\,M\right\}^{\alpha-1}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \quad (17)$$

It is clear to be note that the maximum entropy of nth upper record value $X_{U(n)}$ arising from any arbitrary distribution under conditions (i) and (ii) will holds the inequality (16). As (17) is the expression in the right side of (16), it then follows that exponential distribution attains the maximum Sharma-Mittal entropy in the class C.

5. Some Properties of Sharma-Mittal Entropy on record Values

In this section we provide exact expressions for the Sharma-Mittal divergence measure on record values. Further in this section we derive expressions for Sharma-Mittal entropy of concomitants of both upper and lower record values arising from Farlie-Gumbel-Morgenstern family. In the last part of this section we derive an expression for residual Sharma-Mittal entropy of record values arising from an arbitrary distribution.

5.1. Sharma-Mittal Divergence Measure on Record Values

Divergent measures deals with the distance between two probability distributions or the dissimilarity between two distributions. In recent years these measures play key role in theoretical and applied statistical inference and data processing problems, such as estimation, classification, comparison etc. Sharma and Mittal in 1977 introduced a two parameter divergent measure viz. Shrma-Mittal divergence measure denoted by $D_{\alpha,\ \beta}(f:g)$, between two distributions $f(x)$ and $g(x)$ and is defined by

$$D_{\alpha,\ \beta}(f:g) = \frac{1}{\beta-1}\left\{\left(\int_{-\infty}^{\infty}\left(\frac{f(x)}{g(x)}\right)^{\alpha-1}f(x)\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}, \quad \forall\alpha>0,\ \alpha\neq1\neq\beta. \quad (18)$$

Aktürk *et al.* (2007) shown that, most of the widely used divergence measures such as Rényi, Tsallis, Bhattacharya and Kullback-Liabler divergences are special cases of Sharma-Mittal divergence measure.

In this section we study the Sharma-Mittal divergence between the probability distribution of *nth* upper record value and the parent distribution from which it arises.

THEOREM 7. *The Sharma-Mittal divergence between the nth upper record and the parent distribution is given by the following representation*

$$D_{\alpha,\ \beta}(f_{U(n)}, f) \quad = \quad \frac{1}{\beta - 1}\left\{\left(\frac{\Gamma((n-1)\alpha + 1)}{(\Gamma(n))^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \tag{19}$$

PROOF. The Sharma-Mittal information between the *nth* upper record and the parent distribution is given by

$$D_{\alpha,\beta}(f_{U(n)}, f) = \frac{1}{\beta - 1}\left\{\left(\int_{-\infty}^{\infty} \frac{\left[\{-\log[1 - F(x)]\}^{n-1}\right]^{\alpha}}{((n-1)!)^{\alpha}} f(x)\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

On putting $u = -\log[1 - F(x)]$, we get $x = \left[F^{-1}(1 - e^{-u})\right]$, $\mathrm{d}u = \frac{f(x)}{1-F(x)}dx$ and hence we have

$$\begin{aligned} D_{\alpha,\ \beta}(f_{U(n)}, f) &= \frac{1}{\beta - 1}\left\{\left(\int_{0}^{\infty} \frac{e^{-u}u^{(n-1)\alpha}}{((n-1)!)^{\alpha}}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \qquad (20) \\ &= \frac{1}{\beta - 1}\left\{\left(\frac{\Gamma((n-1)\alpha + 1)}{(\Gamma(n))^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \end{aligned}$$

Hence the theorem.

NOTE 1. *The Sharma-Mittal divergence between the nth upper record and the parent distribution can also be represented as*

$$D_{\alpha,\ \beta}(f_{U(n)}, f) = \alpha^{\frac{((n-1)\alpha+1)(1-\beta)}{1-\alpha}}\left\{\frac{1}{\beta - 1} - H_{\alpha,\beta}(X^{*}_{U(n)})\right\} - \frac{1}{\beta - 1} \tag{21}$$

where, $X^{*}_{U(n)}$ denotes the nth upper record value arising from the standard exponential distribution.

REMARK 8. *The Sharma-Mittal information between the nth record value $X_{U(n)}$ and the parent distribution as given by 19 and 21 establishes that this information is a distribution free information measure.*

5.2. *Sharma-Mittal Entropy of Concomitants of Records from Farlie-Gumbel-Morgenstern (FGM) family of Distributions*

Let X and Y be two random variables with cdf's given by $F_X(\mathrm{x})$ and $F_Y(y)$ respectively with corresponding pdf's $f_X(x)$ and $f_Y(y)$ and jointly distributed with cdf $F(x, y)$ given by, (see, Johnson *et al.*, 2002).

$$F(x, y) = F_X(x)F_Y(y)\left\{1 + \gamma(1 - F_X(x))(1 - F_Y(y))\right\},\ \ -1 \leq \gamma \leq 1, \tag{22}$$

where γ is known as association parameter. Then the family of distributions having the above form of cdf's is called Farlie-Gumbel-Morgenstern (FGM) family of distributions. It is obvious that (22) includes the case of independence as well when $\gamma = 0$. The joint pdf corresponding to the cdf defined in (22) is given by,

$$f(x,y) = f_X(x) f_Y(y) \left\{1 + \gamma(1 - 2F_X(x))(1 - 2F_Y(y))\right\}, \quad -1 \leq \gamma \leq 1. \tag{23}$$

For a sequence (X_i, Y_i), $i = 1, 2, \ldots$ of iid random variables, if we construct the sequence $\{X_{U(n)}\}$ of upper record values from the sequence $\{X_i\}$, then the Y value associated with the $X_{U(n)}$ is called the concomitant of the nth upper record value and is denoted by $Y_{U[n]}$. Similarly the concomitant of nth lower record value $X_{L(n)}$ may be denoted by $Y_{L[n]}$. Then the pdf of $Y_{U[n]}$ is denoted by $f_{Y_{U[n]}}$ and is given by (for details see, Arnold *et al.*, 1998,p. 274)

$$f_{Y_{U[n]}}(y) = \int f_{Y|X}(y|x)\; f_{X_{U(n)}}(x)\, dx = f_Y(y)\left\{1 - \gamma_n(1 - 2F_Y(y))\right\}, \tag{24}$$

where $\gamma_n = (1 - \frac{1}{2^n})\gamma$. Using (2) and (24) we can represent the Sharma-Mittal entropy of concomitant of nth record value as follows:

$$\begin{aligned} H_{\alpha,\beta}(Y_{U[n]}) &= \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^{\infty} (f_Y(y)\left\{1 - \gamma_n(1 - 2F_Y(y))\right\})^{\alpha} \mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \\ &= \frac{1}{1-\beta}\left\{\left(\int_{-\infty}^{\infty} \left\{f_Y(y)\right\}^{\alpha} (\left\{1 - \gamma_n(1 - 2F_Y(y))\right\})^{\alpha} \mathrm{d}y\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}. \end{aligned}$$

On putting $F_Y(y) = u$, $y = F_y^{-1}(u)$ and $f_y(y)\mathrm{d}y = \mathrm{d}u$, we get

$$\begin{aligned} H_{\alpha,\beta}(Y_{U[n]}) &= \frac{1}{1-\beta}\left\{\left(\int_0^1 \left\{f_Y(F_y^{-1}(u))\right\}^{\alpha-1} \left\{1 - \gamma_n(1 - 2u)\right\}^{\alpha} \mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \\ &= \frac{1}{1-\beta}\left\{\left(E_U\left[\left\{f_Y(F_y^{-1}(U))\right\}^{\alpha-1} \left\{1 - \gamma_n(1 - 2U)\right\}^{\alpha}\right]\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}, \end{aligned}$$

where U is a uniformly distributed random variable over $(0, 1)$. Similarly the Sharma-Mittal entropy of concomitant of nth lower record can be represented by

$$H_{\alpha,\beta}(Y_{L[n]}) = \frac{1}{1-\beta}\left\{\left(E_U\left[\left\{f_Y(F_y^{-1}(1-U))\right\}^{\alpha-1} \left\{1 + \gamma_n(1 - 2U)\right\}^{\alpha}\right]\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

5.3. The Residual Sharma-Mittal Entropy of Record Values

Suppose X represents the life time of a unit with pdf $f(.)$, then $H_{\alpha,\,\beta}(X)$ as defined in (2) is useful for measuring the associated uncertainty. If a component is known to have survived up to an age t, then information about the remaining life time is an important characteristic required for analysis of data arising from areas such as reliability, survival studies, economics, business etc. However for the analysis of uncertainty about remaining life time of the unit, we will consider residual Sharma-Mittal entropy and is defined by

$$H_{\alpha,\ \beta}(X;t)=\frac{1}{1-\beta}\left\{\left(\int_t^{\infty}\left\{\frac{f(x)}{\bar{F}(t)}\right\}^{\alpha}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}, \tag{25}$$

where $H_{\alpha,\ \beta}(X;t)$ measures the expected uncertainty contained in the conditional density of $X-t$ given $X>t$ and $\bar{F}(t)=1-F(t)$. In this section we derive a closed form representation for the residual Sharma-Mittal entropy of record values in terms of residual Sharma-Mittal entropy of uniform distribution over $[0,1]$. The survival function of the nth upper record value can be written as $\bar{F}_{X_{U(n)}}(x)$, and is given by

$$\bar{F}_{X_{U(n)}}(x)=\sum_{j=0}^{n-1}\frac{[-\log\bar{F}(x)]^j}{j!}\bar{F}(x)=\frac{\Gamma(n;-\log\bar{F}(x))}{\Gamma(n)}, \tag{26}$$

where $\Gamma(a;x)$ denotes the incomplete Gamma function and is defined by

$$\Gamma(a;x)=\int_x^{\infty}e^{-u}u^{a-1}du,\ \ a,\,x>0.$$

LEMMA 9. *Let $Z_{U(n)}$ denote the nth upper record value from a sequence of observations from $U(0,1)$. Then*

$$H_{\alpha,\beta}(Z_{U(n)};t)=\frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1;-\log(1-t))}{\{\Gamma(n;-\log(1-t))\}^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\} \tag{27}$$

PROOF. By considering (5), (25) and (26), the residual Sharma-Mittal entropy of $Z_{U(n)}$ is given by

$$H_{\alpha,\beta}(Z_{U(n)};t)\ =\ \frac{1}{1-\beta}\left\{\left(\int_t^{\infty}\frac{[-\log(1-x)]^{(n-1)\alpha}}{\{\Gamma(n;-\log(1-t))\}^{\alpha}}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}.$$

On putting $-\log(1-x)=u$, $x=1-e^{-u}$ and $\mathrm{d}x=e^{-u}\mathrm{d}u$.

$$\begin{aligned}H_{\alpha,\beta}(Z_{U(n)};t)\ &=\ \frac{1}{1-\beta}\left\{\left(\int_{-\log(1-t)}^{\infty}\frac{u^{(n-1)\alpha}e^{-u}}{\{\Gamma(n;-\log(1-t))\}^{\alpha}}\mathrm{d}u\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}\\ &=\ \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1;-\log(1-t))}{\{\Gamma(n;-\log(1-t))\}^{\alpha}}\right)^{\frac{1-\beta}{1-\alpha}}-1\right\}. \qquad (28)\end{aligned}$$

Hence the lemma.

If we define $\Gamma_t(a;\lambda)$ as the pdf of truncated Gamma distribution $G(\alpha;\lambda)$ as below

$$\Gamma_t(a;\lambda)=\frac{\lambda^a}{\Gamma(a;t)}x^{a-1}e^{-\lambda x},\quad x\ >\ t\ >\ 0,$$

where $a>0$ and $\lambda>0$, then we have the following theorem.

THEOREM 10. *The residual Sharma-Mittal entropy of $X_{U(n)}$ arising from an arbitrary distribution can be written in terms of the residual Sharma-Mittal entropy of $Z_{U(n)}$ as follows*

$$H_{\alpha,\beta}(X_{U(n)};t) = \left\{H_{\alpha,\beta}(Z_{U(n)};t) + \frac{1}{1-\beta}\right\} \times \left(E_V\left[\left\{f\left(F^{-1}(1-e^{-V})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta} \tag{29}$$

where $V \sim \Gamma_{-\log(1-F(t))}((n-1)\alpha+1; -\log(1-F(t)))$.

PROOF. The residual Sharma-Mittal entropy of $X_{U(n)}$ is given by

$$H_{\alpha,\beta}(X_{U(n)};t) = \frac{1}{1-\beta}\left\{\left(\int_t^\infty \frac{[-\log(1-F(x))]^{(n-1)\alpha}}{\{\Gamma(n;-\log(1-F(t)))\}^\alpha}\mathrm{d}x\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\}.$$

On putting $v = -\log[1-F(x)]$, $x = \left[F^{-1}(1-e^{-v})\right]$ and $\mathrm{dv} = \frac{f(x)}{1-F(x)}\mathrm{d}x$ we get

$$\begin{aligned} H_{\alpha,\beta}(X_{U(n)};t) &= \frac{1}{1-\beta}\left\{\left(\int_{-\log(1-F(t))}^\infty \frac{v^{(n-1)\alpha}e^{-v}}{\{\Gamma(n;-\log(1-F(t)))\}^\alpha}\right.\right. \\ &\quad \left.\left. \times \left\{f\left(F^{-1}(1-e^{-v})\right)\right\}^{\alpha-1}\mathrm{d}v\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \\ &= \frac{1}{1-\beta}\left\{\left(\frac{\Gamma((n-1)\alpha+1;-\log(1-F(t)))}{\{\Gamma(n;-\log(1-F(t)))\}^\alpha}\right.\right. \\ &\quad \times \int_{-\log(1-F(t))}^\infty \frac{v^{(n-1)\alpha}e^{-v}}{\Gamma((n-1)\alpha+1;-\log(1-F(t)))} \\ &\quad \left.\left. \times \left\{f\left(F^{-1}(1-e^{-v})\right)\right\}^{\alpha-1}\mathrm{d}v\right)^{\frac{1-\beta}{1-\alpha}} - 1\right\} \\ &= \left\{H_{\alpha,\,\beta}(Z_{U(n)};t) + \frac{1}{1-\beta}\right\}\left(E_V\left[\left\{f\left(F^{-1}(1-e^{-v})\right)\right\}^{\alpha-1}\right]\right)^{\frac{1-\beta}{1-\alpha}} - \frac{1}{1-\beta}. \end{aligned}$$

Hence the theorem.

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Summary

In this paper we derive Sharma-Mittal entropy of record values and analyse some of its important properties. We establish some bounds for the Sharma-Mittal entropy of record values. We generate a characterization result based on the properties of Sharma-Mittal entropy of record values for exponential distribution. We further establish some distribution free properties of Sharma-Mittal divergence information between distribution of a record value and the parent distribution. We extend the concept of Sharma-Mittal entropy to the concomitants of record values arising from a Farlie-Gumbel-Morgenstern (FGM) bivariate distribution. Also we consider residual Sharma-Mittal Entropy and used it to describe some properties of record values.

Keywords: Record values; Sharma-Mittal entropy; Maximum entropy principle; Characterization; Concomitants of record values; Residual Sharma-Mittal entropy.