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AN EXTENDED VERSION OF KUMARASWAMY INVERSE WEIBULL DISTRIBUTION AND ITS PROPERTIES

C. Satheesh Kumar [1]
Department of Statistics, University of Kerala, Karyavattom, Trivandrum, India
Subha R. Nair
HHMSPB NSS College for Women, Trivandrum, Kerala, India

1. Introduction

The inverse Weibull distribution (IWD) and its modified versions have been used frequently in reliability studies due to its flexibility in handling survival data. The distribution finds wide applications in fields of engineering and non-engineering sciences, especially in certain areas like ecology, medicine, pharmacy etc. The distribution was first studied by Keller *et al.* (1985) while investigating failures of mechanical components subject to degradation. The cumulative distribution function (cdf) of the IWD has the following form, for any $x > 0$, $\alpha > 0$ and $\beta > 0$.

$$F_1(x) = \exp[-(\frac{\alpha}{x})^{\beta}]. \tag{1}$$

Many generalizations of the IWD have been studied by authors like Khan and Pasha (2009), Jazi *et al.* (2010), Khan *et al.* (2008), Jing (2010), de Gusmao *et al.* (2011), Shahbaz *et al.* (2012), Pararai *et al.* (2014), and Aryal and Elbatal (2015). Jiang *et al.* (2001) introduced a multiplicative model of the IWD distribution using the following cdf,

$$F_1^*(x) = \exp\left\{-\left[(\frac{\alpha_1}{x})^{\beta_1} + (\frac{\alpha_2}{x})^{\beta_2}\right]\right\}, \tag{2}$$

where $x > 0$, $\alpha_1 > 0$, $\alpha_1 > 0$, $\beta_1 > 0$ and $\beta_2 > 0$. We call the distribution with cdf Eq. (2) as the "inverse Weibull multiplicative model($IWMM(\alpha_1, \alpha_2, \beta_1, \beta_2)$)". Kumaraswamy (1980), considered a class of continuous distributions whose cdf takes the form $F_2(x) = 1 - (1 - x^a)^b$, for $x > 0$, $a > 0$ and $b > 0$.
Shahbaz *et al.* (2012) studied a class of distributions namely "the Kumaraswamy inverse Weibull distribution($KIWD$)" through the following cdf, for $x > 0$,

$$F_3(x) = 1 - [1 - exp(-a\beta x^{-\gamma})]^b, \tag{3}$$

[1] Corresponding Author e-mail: drcsatheeshkumar@gmail.com

in which $a > 0$, $b > 0$, $\beta > 0$ and $\gamma > 0$. Aryal and Elbatal (2015) obtained a modified version of the $KIWD$ namely "the Kumaraswamy modified inverse Weibull distribution ($KMIWD$)" through the following cdf.

$$F_4(x) = 1 - \left[1 - \exp\left(-a\left(\frac{\lambda}{x} + \frac{\theta}{x^{\beta}}\right)\right)\right]^b, \tag{4}$$

where $x > 0$, $\beta > 0, a > 0, b > 0, \lambda > 0$ and $\theta > 0$.

It is important to note that most of the distributions like the $IWMM$, the $KIWD$, the $KMIWD$ etc. have limited shapes for its hazard rate function and does not incorporate the decreasing or non-increasing hazard rate which is frequently observed in data sets arising from areas of medicine, engineering etc. In order to rectify this drawback, through this paper we develop a class of distributions which we name as "the extended Kumaraswamy inverse Weibull distribution ($EKIWD$)". Further the $EKIWD$ contains several important classes of distributions as cited in Table 2.1 of this paper, and as such the proposed model supports a wide variety of shapes in terms of its probability density function plots as well as hazard rate function plots. This flexible nature of the proposed class of distributions can be expected to have extensive utility in modelling data sets from various fields of scientific research and has motivated us to investigate many useful properties of the distribution. We have illustrated this merit of the $EKIWD$ over other existing models in section 6 by considering three types of real life data sets. Out of these, the first two data sets have increasing hazard rate while the last data set has a decreasing hazard rate function. Moreover it can be observed from Table 8.1. that the $EKIWD$ shows relatively better fit to both the types of data, which is an evidence of its flexibility in modelling data sets from various fields of medical, industrial and scientific research. The significance of the additional parameter is tested based on the three data sets using the likelihood-ratio test procedure and it can be seen from Table 8.2 that the new parameter is significant. The paper is organized as follows. In section 2, we present the definition and important properties of the $EKIWD$ along with a list of its important special cases. In section 3, distribution and moments of the order statistics of $EKIWD$ are obtained and in section 4 certain reliability aspects are discussed. Section 5 contains the maximum likelihood estimation of the parameters of the distribution along with the Fisher information matrix. In section 6 the usefulness of the model and the significance of the additional parameter is illustrated using three real life data sets and in section 7 the asymptotic behaviour of the $EKIWD$ is examined with the help of simulated data sets.

Now we present the following series representations, that are needed in the sequel.

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} B(j,k) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} B(j,k-j) \tag{5}$$

$$x^n = \sum_{k=0}^{n} S(n,k) x^{(k)}, \tag{6}$$

where $S(n,k)$is the Stirling numbers of the second kind and for any $x \in \Re = (-\infty, \infty)$, $x^{(k)} = x(x-1)\ldots(x-k+1)$, for $k \geq 1$ with $x^{(0)} = x$. Also,

$$(1+x)^a = \sum_{j=0}^{\infty} (a+1-j)_j \frac{x^j}{j!}, \tag{7}$$

for any $a \in \Re$ and $(x)_k = x(x+1)\ldots(x+k-1)$ for $k \geq 1$ with $(x)_0 = 1$. For any $c \in \Re$ and for any $j$and k positive integers,

$$(c-j)_j = (-1)^j (1-c)_j, \tag{8}$$

$$(c)_{(j-k)} = \frac{(-1)^k (c)_j}{(1-c-j)_k} \tag{9}$$

and

$$(c-j)_k = \frac{(1-c)_j (c)_k}{(1-c-k)_j}. \tag{10}$$

2. The Extended Kumaraswamy inverse Weibull distribution

In this section we present the definition of the $EKIWD$ and discuss some of its distributional properties.

Definition 1. *A continuous random variable X is said to have an extended Kumaraswamy inverse Weibull distribution [$EKIWD(\alpha, \beta, \rho, \sigma, \delta)$] if its cdf is of the following form, for $x \geq 0$, $\alpha > 0$, $\beta > 0$, $\delta > 0$, $\rho \geq 0$ and $\sigma \geq 0$ such that $(\rho, \sigma) \neq (0, 0)$.*

$$G(x) = 1 - [1 - \psi(\underline{\theta})]^\delta, \tag{11}$$

where

$$\psi(\underline{\theta}) = \psi(\alpha, \beta, \rho, \sigma) = \exp\left[-\left(\frac{\rho}{x^\alpha} + \frac{\sigma}{x^\beta}\right)\right]. \tag{12}$$

Some important special cases of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ corresponding to particular choices of its parameters are listed in Table 1.

Theorem 2. *For $x > 0$, the probability density function (pdf) $g(x)$, the survival function $S(x)$, and the hazard rate function $h(x)$ of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ are given by*

$$g(x) = \delta \left(\frac{\alpha\rho}{x^{\alpha+1}} + \frac{\beta\sigma}{x^{\beta+1}} \right) \psi(\underline{\theta}) [1 - \psi(\underline{\theta})]^{\delta-1}, \tag{13}$$

$$S(x) = [1 - \psi(\underline{\theta})]^\delta \tag{14}$$

and

$$h(x) = \delta \left(\frac{\alpha\rho}{x^{\alpha+1}} + \frac{\beta\sigma}{x^{\beta+1}} \right) \psi(\underline{\theta}) [1 - \psi(\underline{\theta})]^{-1}, \tag{15}$$

where $\psi(\underline{\theta})$ is as defined in Eq. (12).

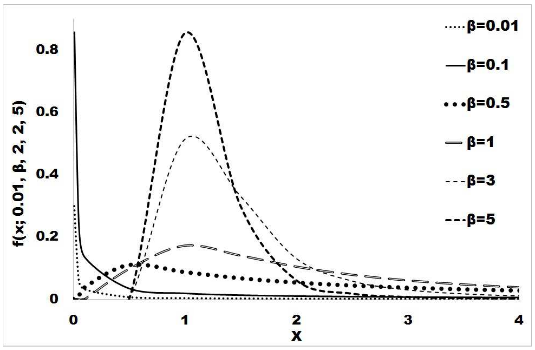

Figure 1 – Plots of pdf of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ for particular values of its parameters.

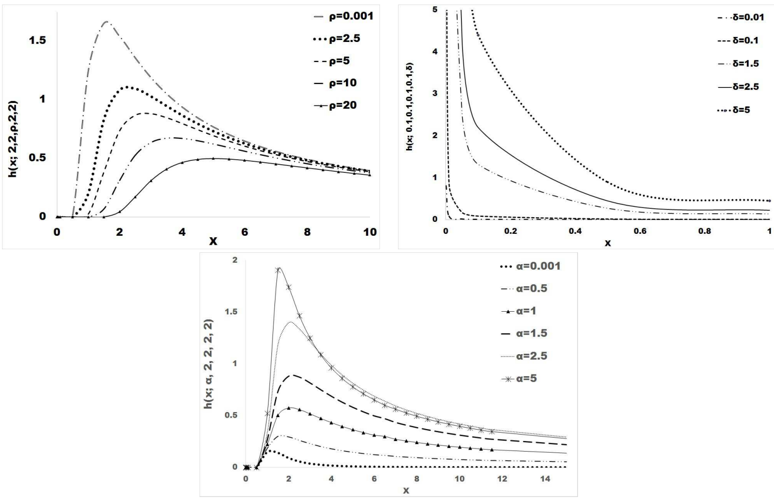

Figure 2 – Plots of the hazard rate function of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ for particular values of its parameters.

TABLE 1
Special Cases of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$

Sl. No:	Name of the distribution belonging to $EKIWD$	Particular choices of parameters
1	Kumaraswamy modified inverse Weibull ($KMIWD(a, \delta, \beta, \theta, \lambda)$) Aryal and Elbatal (2015)	$\rho = a\lambda, \sigma = a\theta, \alpha = 1$
2	Proportional inverse Weibull($PIWD(\alpha, \beta, \gamma)$) Oluyede and Yang (2014)	$\rho = \gamma\beta^{-\alpha}, \sigma = 0, \delta = 1$
3	Kumaraswamy proportional inverse Weibull ($KPIWD(\alpha, \beta, \gamma, \lambda, \delta)$) Oluyede and Yang (2014)	$\rho = \gamma\lambda\alpha^{\beta}, \sigma = 0$ or $\rho = 0, \sigma = \gamma\lambda\alpha^{\beta}$
4	Exponentiated modified inverted Weibull ($EMIWD(a, \beta, \lambda, \theta)$) Aryal and Elbatal (2015)	$\rho = a\lambda, \sigma = a\theta, \alpha, \delta = 1$
5	Inverse generalised Weibull($IGWD(\delta, \beta, \lambda)$) Jain *et al.* (2014)	$\rho = 0, \sigma = \lambda^{\beta}$ or $\rho = \lambda^{\beta}, \sigma = 0$
6	Generalized inverse generalised Weibull ($GIGWD(\delta, \beta, \gamma, \lambda)$) Jain *et al.* (2014)	$\rho = 0, \sigma = \gamma\lambda^{\beta}$ or $\rho = \gamma\lambda^{\beta}, \sigma = 0$
7	Kumaraswamy generalised inverse Weibull ($KGEWD(a, \beta, \lambda, \delta)$) Yang (2012)	$\rho = 0, \sigma = \lambda a^{-\beta}$
8	Kumaraswamy inverse Weibull ($KIWD(a, \delta, \beta, \theta,)$) Shahbaz *et al.* (2012)	$\rho = 0, \sigma = a\theta$
9	Modified inverse Weibull ($MIWD(\beta, \rho, \lambda)$) Khan and King (2012)	$\alpha = 1, \delta = 1$
10	Generalised inverse Weibull ($GIWD(\alpha, \beta, \delta)$) de Gusmao *et al.* (2011)	$\rho = 0, \sigma = \gamma\alpha^{\beta}, \delta = 1$
11	Inverse Weibull Khan *et al.* (2008)	$\rho = 0, \delta = 1$ or $\sigma = 0, \delta = 1$

PROOF. The proof is straightforward and hence omitted.

We present the plots of pdf and hazard rate function of the $EKIWD$ for particular values of its parameters in Figure 1 and Figure 2 respectively. Next, we obtain a series expansion of the cdf of $EKIWD$ through the following theorem.

THEOREM 3. *The cdf G(x) of the* $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ *can be represented as*

$$G(x) \quad = 1 - \sum_{k=0}^{\infty} (\delta + 1 - k)_k \frac{(-1)^k}{k!} F_1^*(x; k), \tag{16}$$

where $F_1^*(x; k)$ *is the cdf of the* $IWMM(k\rho,\ k\sigma,\ \alpha,\ \beta)$ *as given in Eq. (2).*

PROOF. The proof follows from Eq. (7) and Eq. (11).

On differentiating Eq. (16) with respect to x, we obtain the pdf of $EKIWD$ through the following corollary.

COROLLARY 4. *The pdf g(x) of* $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ *is*

$$g(x) = \sum_{k=0}^{\infty} (\delta + 1 - k)_k \frac{(-1)^{k+1}}{k!} f_1^*(x; k), \tag{17}$$

where $f_1^*(x; k)$ *is the pdf of the* $IWMM(k\rho,\ k\sigma,\ \alpha,\ \beta)$.

Certain structural properties of the $EKIWD$ are presented through the following theorems.

THEOREM 5. *For any* $\alpha > 0$, $\beta > 0$, $\rho \geq 0$, $\sigma \geq 0$ *and* $\delta > 0$, *the random variable* X *follows* $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ *with pdf Eq. (13) if and only if* $Y_1 = \left(\frac{\rho}{X^\alpha} + \frac{\sigma}{X^\beta}\right)$ *follows a particular form of exponentiated Weibull distribution(EWD) by Pal* et al. *(2006) with pdf*

$$g(y_1) = \delta \exp(-y) \left[1 - \exp(-y)\right]^\delta. \tag{18}$$

Proof is straight forward and hence omitted.

THEOREM 6. *For any* $\alpha > 0$, $\beta > 0$, $\rho \geq 0$, $\sigma \geq 0$ *and* $\delta > 0$, *the random variable* X *follows* $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ *with pdf Eq. (13) if and only if* $Y_2 = bX$, *for* $b \geq 0$ *follows* $EKIWD(\alpha, \beta, \rho b^\alpha, \sigma b^\beta, \delta)$.

Proof is straight forward and hence omitted.

THEOREM 7. *For any* $c > 0$, $\alpha > 0$, $\beta > 0$, $\rho \geq 0$, $\sigma \geq 0$ *and* $\delta > 0$, *a random variable* X *follows* $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ *with pdf Eq. (13) if and only if* $Y_3 = X^c$, *follows* $EKIWD(\alpha^*, \beta^*, \rho, \sigma, \delta)$ *with* $\alpha^* = \frac{\alpha}{c}$ *and* $\beta^* = \frac{\beta}{c}$.

Proof is straight forward and hence omitted.

THEOREM 8. *If X be any continuous random variable with cdf $F(x) > 0$, for every $x \in \Re^+ = (0, \infty)$ and if*
$E\left\{\ln\left(1-\exp[-(\frac{\rho}{X^\alpha}+\frac{\sigma}{X^\beta})]\right)/X>y\right\} = \ln\left(1-\exp[-(\frac{\rho}{y^\alpha}+\frac{\sigma}{y^\beta})]\right) - \frac{1}{\delta}$,
then X has the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$.

PROOF. The proof follows from Theorem 8 (Rinne (2008), p 262) with $h(x) = \ln\left\{1-\exp[-(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta})]\right\}$ and $d = \frac{-1}{\delta}$, since $E(h(X)) = \frac{-1}{\delta}$, $h(0) = 0$ and $\lim_{x\to\infty} h(x) = -\infty$, so that

$$\begin{aligned} F(x) \quad &= 1- \quad \exp\left[-\frac{1}{d}h(x)\right], \\ &= \quad 1-\left\{1-\exp\left[-(\rho x^\alpha+\sigma x^\beta)\right]\right\}^\delta, \end{aligned} \tag{19}$$

for $x \in \Re^+$, which is the cdf of $EKIWD$.

The raw moments of the $EKIWD$ can be calculated numerically using statistical soft-wares like MATHEMATICA and MATHCAD. The following theorem gives a theoretical expression for the r^{th} raw moment of the $EKIWD$.

THEOREM 9. *The r^{th} raw moment μ_r of the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is given by*

$$\begin{aligned} \mu_r \quad &= \quad \delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}\left[\frac{\sigma^j(1-\delta-j)_k}{(\delta)_j}\frac{m!}{k!}\binom{k}{j}\binom{k-j+1}{m}\right. \\ &\quad \left.\times\ S(j,m)\left(\alpha\rho\Psi_{r\alpha}+\beta\sigma\Psi_{r\beta}\right)\right], \end{aligned} \tag{20}$$

where

$$\Psi_{rc} \quad = \quad [(k-j+1)\rho]^{\left[-(c+m\beta-r)\alpha^{-1}\right]}\Gamma[(c+m\beta-r)\alpha^{-1}] \tag{21}$$

with $c = \alpha$ or $c = \beta$.

Proof of the theorem is given in Appendix-A.

COROLLARY 10. *The Mean of the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is given by*

$$\begin{aligned} \mu_1 \quad &= \quad \delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}\left[\sigma^j\frac{(1-\delta-j)_k}{(\delta)_j}\frac{m!}{k!}\binom{k}{j}\binom{k-j+1}{.m}\right. \\ &\quad \left.\times\ S(j,m)\left(\alpha\rho\Psi_{1\alpha}+\beta\sigma\Psi_{1\beta}\right)\right], \end{aligned} \tag{22}$$

where Ψ_{rc} is as defined in Eq. (21).

COROLLARY 11. *When $\alpha = \beta = \eta$, the r^{th} raw moment of the*

$$EKIWD(\eta, \eta, \rho, \sigma, \delta) is$$

$$\begin{aligned} \mu_r \quad &= \quad \delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{m=0}^{j}\left\{\frac{(1-\delta-j)_k}{(\delta)_j}\frac{m!}{k!}\binom{k}{j}\binom{k-j+1}{m}S(j,m)\rho^{[\eta^{-1}(r-m\eta)]-1}\right. \\ &\quad \left.\times\ \sigma^j(\rho+\sigma)(k-j+1)^{[\eta^{-1}(r-m\eta)]-1}\Gamma[(\eta^{-1}(m\eta-r))-1]\right\}. \end{aligned} \tag{23}$$

We obtain the characteristic function of the $EKIWD$ through the following theorem and its proof is included in Appendix - A.

THEOREM 12. *For $t \in \Re$ and $i = \sqrt{-1}$, the characteristic function of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is given by*

$$\Phi_x(t) \quad = \quad \delta \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\delta - k)_k \left[\xi(t; \alpha, \beta, \rho, \sigma) + \xi(t; \beta, \alpha, \sigma, \rho) \right], \qquad (24)$$

where

$$\xi(t; a, b, c, d) =$$
$$= \sum_{j=0}^{\infty} \sum_{m=0}^{j} \left\{ \frac{(-1)^j}{(j-m)!\ m!} (d + kd)^{j-m} c(it)^m \frac{\Gamma\left[a^{-1}[b(j-m) - m] + 1\right]}{(c + kc)^{[a^{-1}[b(j-m)-m]+1]}} \right\}.$$

The plots of mean, variance, skewness and kurtosis of the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ for particular values of its parameters are presented in Figure 3.

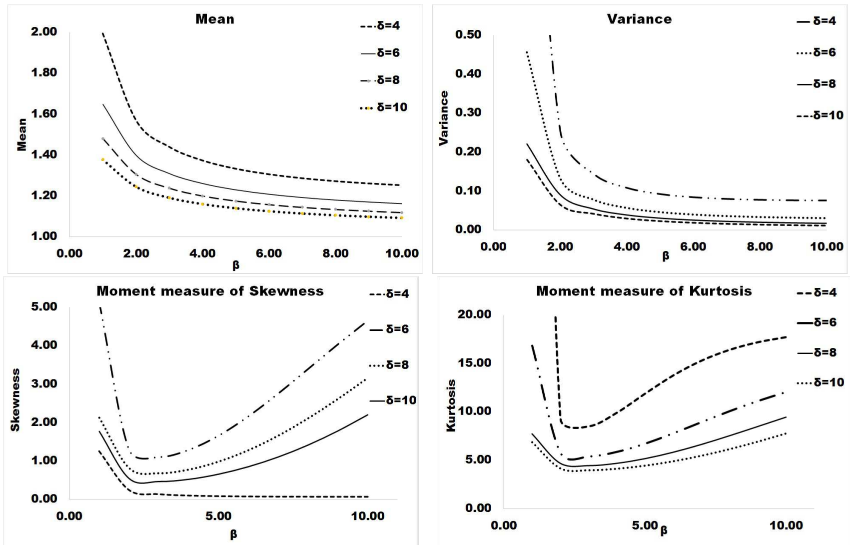

Figure 3 – Mean, variance, skewness and kurtosis for the $EKIWD(2, \beta, 2, 2, 2, \delta)$ for particular values of δ.

Some aspects regarding the quantiles and mode of the $EKIWD$ are presented through the following theorems. We have computed mode for various distinct values of α and β by using $MATHEMATICA$ software and it is observed that mode decreases as β increases for fixed α and vice versa.

THEOREM 13. *The quantile function of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is the solution of the equation:*

$$\rho q_u^\beta + \sigma q_u^\alpha + q_u^{\alpha+\beta} \ln \left[1 - (1-u)^{\delta^{-1}} \right] = 0. \qquad (25)$$

The proof immediately follows from the definition of quantile function, $x = q_u = G^{-1}(u)$.

COROLLARY 14. *When $\alpha = \beta = \eta$, the quantile function of*

$$EKIWD(\eta, \eta, \rho, \sigma, \delta) becomes$$

$$q_u = \left[\frac{-(\rho + \sigma)}{\ln[1 - (1 - u)]^{\delta^{-1}}} \right]^{\eta^{-1}}. \tag{26}$$

THEOREM 15. *The mode of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is the solution of the following equation.*

$$\alpha(\alpha + 1)\rho x^{\beta} + \beta(\beta + 1)\sigma x^{\alpha} - 2\alpha\beta\rho\sigma = 0. \tag{27}$$

The proof is given in Appendix - A.

COROLLARY 16. *When $\alpha = \beta = \eta$, the mode of $EKIWD(\eta, \eta, \rho, \sigma, \delta)$ is*

$$Mode = \left[\frac{2\rho\sigma\eta}{(\rho + \sigma)(1 + \eta)} \right]^{\eta^{-1}} \tag{28}$$

and as $\eta \to \infty$, the mode tends to unity.

3. ORDER STATISTICS AND MOMENTS OF ORDER STATISTICS

Let $X_{i:n}$ be the i^{th} order statistic based on a random sample $X_1, X_2, \ldots, X_n$ of size n from $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$, with pdf $g(x) = g(x; \delta)$ as given in Eq. (13) and let $\mu_r = \mu_r(\delta)$ be the r^{th} raw moment as given in Eq. (20). In this section we obtain the distribution and moments of the i^{th} order statistic of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$.

THEOREM 17. *For $x > 0$, the pdf of the i^{th} order statistics is given by*

$$g_{i:n}(x) = \sum_{k=0}^{i-1} \nu_{n:i:k} \; g(x; \delta^*), \tag{29}$$

where

$$\nu_{n:i:k} = \frac{\binom{n}{i}\binom{i}{k}(-1)^k (i - k)}{(n + k + 1 - i)}$$

and $\delta^* = \delta(n + k + 1 - i)$.

PROOF. Consider a random sample of size n from an $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$. The pdf of the i^{th} order statistic $X_{i:n}$ can be defined as

$$g_{i:n}(x) = \frac{n!}{(i - 1)! \; (n - i)!} [G(x)]^{i-1} [1 - G(x)]^{n-i} g(x). \tag{30}$$

By using Eq. (11) and Eq. (13) we have the following from Eq. (30).

$$\begin{aligned}
g_{i:n}(x) &= \delta \frac{n!}{(i-1)!\,(n-i)!}\left(\frac{\alpha\rho}{x^{\alpha+1}}+\frac{\beta\sigma}{x^{\beta+1}}\right)\exp\left[-\left(\frac{\rho}{x^{\alpha}}+\frac{\sigma}{x^{\beta}}\right)\right] \\
&\quad \times \left\{1-\exp\left[-\left(\frac{\rho}{x^{\alpha}}+\frac{\sigma}{x^{\beta}}\right)\right]\right\}^{\delta(n-i+1)-1} \\
&\qquad \left\{1-\left[1-\exp\left[-\left(\frac{\rho}{x^{\alpha}}+\frac{\sigma}{x^{\beta}}\right)\right]\right]^{\delta}\right\}^{i-1} \\
&= \sum_{k=0}^{i-1} \frac{(-1)^k (i-k)\binom{n}{i}\binom{i}{k}}{(n+k-i+1)}\delta(n+k-i+1)\left(\frac{\alpha\rho}{x^{\alpha+1}}+\frac{\beta\sigma}{x^{\beta+1}}\right) \\
&\quad \times \exp\left[-\left(\frac{\rho}{x^{\alpha}}+\frac{\sigma}{x^{\beta}}\right)\right].\left[1-\exp\left[-\left(\frac{\rho}{x^{\alpha}}+\frac{\sigma}{x^{\beta}}\right)\right]\right]^{\delta(n+k-i+1)-1},
\end{aligned}$$

by using binomial theorem. Thus, on simplification we have

$$g_{i:n}(x) = \sum_{k=0}^{i-1} \frac{(-1)^k\ (i-k)\binom{n}{i}\binom{i}{k}}{(n+k-i+1)} g(x;\delta^*),$$

which reduces to Eq. (29).

As a consequence of Theorem 17, we have the following corollaries.

Corollary 18. *For $x > 0$, the pdf of the largest order statistic $X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ is*

$$g_{n:n}(x) = \sum_{k=0}^{n-1} \binom{n}{k+1} (-1)^k g(x;\delta_1^*), \tag{31}$$

where $\delta_1^* = \delta(k+1)$.

Corollary 19. *For $x > 0$, the pdf of the smallest order statistic $X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ is*

$$g_{1:n}(x) = g(x;\delta_2^*), \tag{32}$$

where $\delta_2^* = n\delta$.

Corollary 20. *For $x > 0$, pdf of the median $X_{m+1:n}$, where n=2m+1, is the following; in which $\delta_3^* = \delta(m+k+1)$.*

$$g_{(m+1:n)}(x) = \sum_{k=0}^{m} \frac{(-1)^k (2m+1)\binom{2m}{m}\binom{m}{k}}{m+k+1} g(x;\delta_3^*). \tag{33}$$

The following theorem gives a characterization of the $EKIWD$ based on order statistics.

Theorem 21. *The smallest order statistic $X_{1:n}$ follows the $EKIWD(\alpha,\beta,\rho,\sigma,n\delta)$ if and only if X_1 follows $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$.*

Proof. By definition, the cdf of $X_{1:n}$ is

$$G_{1:n}(x) \quad = \quad 1-[1-G(x)]^n, \tag{34}$$

where G(x) is as defined in Eq. (11).
If X_1 follows $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$, clearly $X_{1:n}$ follows $EKIWD(\alpha,\beta,\rho,\sigma,n\delta)$ by Corollary 3.1.2.
Conversely, if $X_{1:n}$ has the distribution $EKIWD(\alpha,\beta,\rho,\sigma,n\delta)$, its cdf is

$$G_{1:n}(x) \quad = \quad 1-\left\{1-\exp\left[-\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right]\right\}^{n\delta}. \tag{35}$$

Now, the proof follows immediately from the comparison of Eq. (34) and Eq. (35).

Theorem 22. *For $r>0$, the r^{th} raw moment of the i^{th} order statistic $X_{i:n}$ of $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$ is the following, in which, $\nu_{n:i:k}$ and δ^* are as defined in Eq. (29).*

$$\mu_{r:i:n}(x) \quad = \quad \sum_{k=0}^{i-1}\nu_{n:i:k}\ \mu_r(\delta^*). \tag{36}$$

Proof follows from Theorems 9 and 17.

4. Certain Reliability Concepts

Stress-strength reliability measure is defined as the probability that a randomly selected device functions successfully. In this scenario, if X_1 is a random variable representing the stress that a device is subjected to, and X_2 is the strength that varies from device to device, then the stress-strength reliability measure is defined as $R=P(X_1<X_2)$. Through the following theorem, we obtain an expression for R in case of $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$ for known values of α, β, ρ and σ.

Theorem 23. *For i=1,2, let X_i be a random variable following $EKIWD(\alpha,\beta,\rho,\sigma,\delta_i)$ with pdf g(.) as defined in Eq. (13). Then $R=\dfrac{\delta_1}{\delta_1+\delta_2}$*

Proof.

$$\begin{aligned} R=&\int_0^\infty\left(\int_0^{x_2}g_1(x_1)dx_1\right)g_2(x_2)dx_2\\ =&1-\int_0^\infty\delta_2\left(\frac{\alpha\rho}{x_2^{\alpha+1}}+\frac{\beta\sigma}{x_2^{\beta+1}}\right).\exp\left(\frac{\rho}{x_2^\alpha}+\frac{\sigma}{x_2^\beta}\right)\\ &.\left[1-\exp\left(\frac{\rho}{x_2^\alpha}+\frac{\sigma}{x_2^\beta}\right)\right]^{\delta_1+\delta_2-1}dx_2=\frac{\delta_1}{\delta_1+\delta_2}, \end{aligned} \tag{37}$$

which shows that R depends only on the values of the parameter δ.

An expression for the mean residual life function of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is obtained through the following theorem.

THEOREM 24. *The mean residual life function M(.) of the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is the following for $x > 0$ and $i\alpha + \beta(j-i) \neq 1$.*

$$M(x) = \frac{1}{S(x)} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sum_{i=0}^{j} \left\{ \frac{(-1)^k (1+\delta+j-k)_{(k-j)} (k-j)^j \rho^i \sigma^{(j-i)} x^{1-i\alpha-\beta(j-i)}}{i!\,(j-i)!\,(k-j)![i\alpha+\beta(j-i)-1]} \right\}. \tag{38}$$

PROOF. By definition,

$$\begin{aligned} M(x) &= \frac{1}{S(x)} \int_x^{\infty} S(t)dt \qquad (39) \\ &= \frac{1}{S(x)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \left[(-1)^{j+k} \frac{(1+\delta-k)_k k^j}{k!\,j!} \int_x^{\infty} \left(\frac{\rho}{t^{\alpha}} + \frac{\sigma}{t^{\beta}} \right)^j dt \right], \end{aligned}$$

in the light of Eq. (7) and Eq. (14). Now applying binomial theorem to get

$$M(x) = \frac{1}{S(x)} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left[(-1)^{j+k} \frac{(1+\delta-k)_k k^j \binom{j}{i} \rho^i \sigma^{(j-i)}}{k!\,j!} \int_x^{\infty} t^{-(i\alpha+\beta(j-i))} dt \right],$$

which on simplification gives Eq. (38), by Eq. (5).

5. ESTIMATION

Here we discuss the maximum likelihood estimation of the parameters of

$$EKIWD(\alpha, \beta, \rho, \sigma, \delta)$$

based on a random sample $X_1, X_2, \ldots, X_n$ taken from the distribution. The log-likelihood function for the vector of parameters $\Theta = (\alpha, \beta, \rho, \sigma, \delta)$ is given by

$$\begin{aligned} L(\Theta) &= \sum_{i=1}^{n} \left\{ \ln(\delta) + \ln\left(\frac{\alpha\rho}{x_i^{\alpha+1}} + \frac{\beta\sigma}{x_i^{\beta+1}} \right) - \left(\frac{\rho}{x_i^{\alpha}} + \frac{\sigma}{x_i^{\beta}} \right) \right. \\ &\quad \left. + (\delta-1) \ln\left[1 - \exp\left(\frac{\rho}{x_i^{\alpha}} + \frac{\sigma}{x_i^{\beta}} \right) \right] \right\}. \qquad (40) \end{aligned}$$

On differentiating the log-likelihood function Eq. (40) with respect to the parameters $\alpha, \beta, \rho, \sigma$ and δ respectively, and equating to zero, we obtain the following likelihood equations.

$$\sum_{i=1}^{n}\left[u_i\ln(x_i)+\frac{u_i}{\epsilon_i}(1-\alpha\ln x_i)-\frac{\delta-1}{w_i-1}u_i\ln(x_i)\right] = 0, \quad (41)$$

$$\sum_{i=1}^{n}\left[v_i\ln(x_i)+\frac{v_i}{\epsilon_i}(1-\beta\ln(x_i))-\frac{\delta-1}{w_i-1}v_i\ln(x_i)\right] = 0, \quad (42)$$

$$\sum_{i=1}^{n}\left[\rho^{-1}u_i\left(\frac{\delta-1}{w_i-1}+\frac{\alpha}{\epsilon_i}-1\right)\right] = 0, \quad (43)$$

$$\sum_{i=1}^{n}\left[\sigma^{-1}v_i\left(\frac{\delta-1}{w_i-1}+\frac{\beta}{\epsilon_i}-1\right)\right] = 0 \quad (44)$$

and

$$\sum_{i=1}^{n}\left\{\delta^{-1}+\ln\left[1-\exp\left(-w_i\right)\right]\right\} = 0, \quad (45)$$

in which $u_i = \rho x_i^{-\alpha}$, $v_i = \sigma x_i^{-\beta}, w_i = \exp(u_i + v_i)$ and $\epsilon_i = \alpha u_i + \beta v_i$. The observed Fisher information matrix is derived as $I_\Theta = ((I_{ij}))$, where the elements of I_Θ are as obtained in Appendix B.

6. Applications

In this section the utility of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is demonstrated with the help of the following three data sets, of which the second data set is of biomedical origin while the other two are from industrial background. The first two data sets are examples of data with increasing hazard rate function and the third data set has a decreasing hazard rate function.

Data Set 1: Data on testing the tensile fatigue characterizations of a polyester /viscose yarn to study the problem of warp breakage during weaving, consisting of 100 yarn samples at 2.3 percent strain level . This data was initially studied by Quesenberry and Kent (1982).

Data Set 2: Data on survival of 40 patients suffering from Lukemmia, from the Ministry of Health Hospitals in Saudi Arabia taken from Abouammoh *et al.* (1994).

Data Set 3: Data on the failure times of the air conditioning system of an airplane consisting of 30 observations considered by Linhart and Zucchini (1986). We have obtained maximum likelihood estimators of the parameters of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ by using R software in the case of the above three data sets. For comparison we have considered the fitting of the following models - the $KMIWD$, the $IWMM$, the $EMIWD$, the $MIWD$ and the $GIWD$. We have computed certain information criteria such as AIC (Akaike information criterion), BIC (Bayean information criterion), AICC (second order Akaike information criterion) and CAIC (consistent Akaike information criterion) in case of each fitted models. The numerical results obtained are presented in Table 2. Further, for graphical comparison, we have obtained cumulative probability plots and the Weibull probability plots corresponding to each model as given in Figure 4 and Figure 5. From Table 2, Figure 4 and Figure 5, it can be observed that $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ gives relatively better fit to each of the data sets compared to existing models in both the cases of increasing and decreasing hazard rate. To test the significance of the additional parameter α of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$, we have adopted the following generalised likelihood ratio test procedure. The hypothesis to be tested is $H_0 : \alpha = 1$ against the alternative hypothesis $H_1 : \alpha \neq 1$. The test statistic used is

$$\Lambda = -2 \left\{ L_0(\Theta) - L_1(\Theta) \right\},$$

where $\Theta = (\alpha, \beta, \rho, \sigma, \delta)$, $L_0(\Theta)$ is the log-likelihood function of the $EKIWD$ under the null hypothesis H_0 and $L_1(\Theta)$ is the log-likelihood function of the $EKIWD$ under the alternative hypothesis H_1. Adopting the above test procedure, we have tested the significance of the parameter α in case of Data sets 1, 2 and 3 and the numerical results thus obtained is listed in Table 3. From Table 3 it can be seen that the parameter α is significant in case of all three data sets and hence the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ can be considered as a better model compared to other sub-models considered in the paper.

7. Simulation

For examining the performance of the maximum likelihood estimators (MLEs) of the parameters of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$, we carry out a simulation study by generating observations with the help of $MATHEMATICA$ for the following two sets of parameters: (1) $\alpha = 2, \beta = 2, \rho = 0.1, \sigma = 0.1, \delta = 2$ and (2) $\alpha = 0.1, \beta = 0.1, \rho = 0.1, \sigma = 0.1, \delta = 2.5$, corresponding to the two distinct shapes of the hazard rate function as seen in Figure 2. According to Efron (1991), a maximum of 200 bootstrap samples are required to obtain a good estimate of the variance of an estimator. Hence we have considered 200 bootstrap samples of sizes 25, 100, 500 and 1000 for comparing the performances of the different MLEs mainly with respect to their mean values and mean squared errors (MSEs). The average bias of estimates and average MSEs over 200 replications are calculated for different cases and the results are given in Table 4 and Table 5. From the results obtained, it can be observed that as sample size increases mean value of the estimators approach the original value of the respective parameters and MSEs of the estimators are in decreasing order.

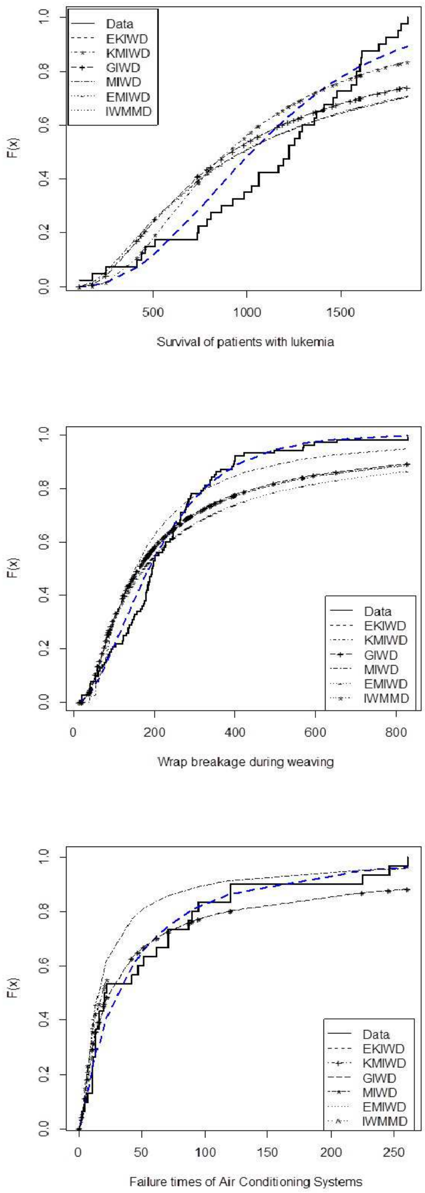

Figure 4 – Cumulative probability plots of $EKIWD$, $KMIWD$, $EMIWD$, $MIWD$, $GIWD$ and $IWMM$ corresponding to data sets 1, 2 and 3.

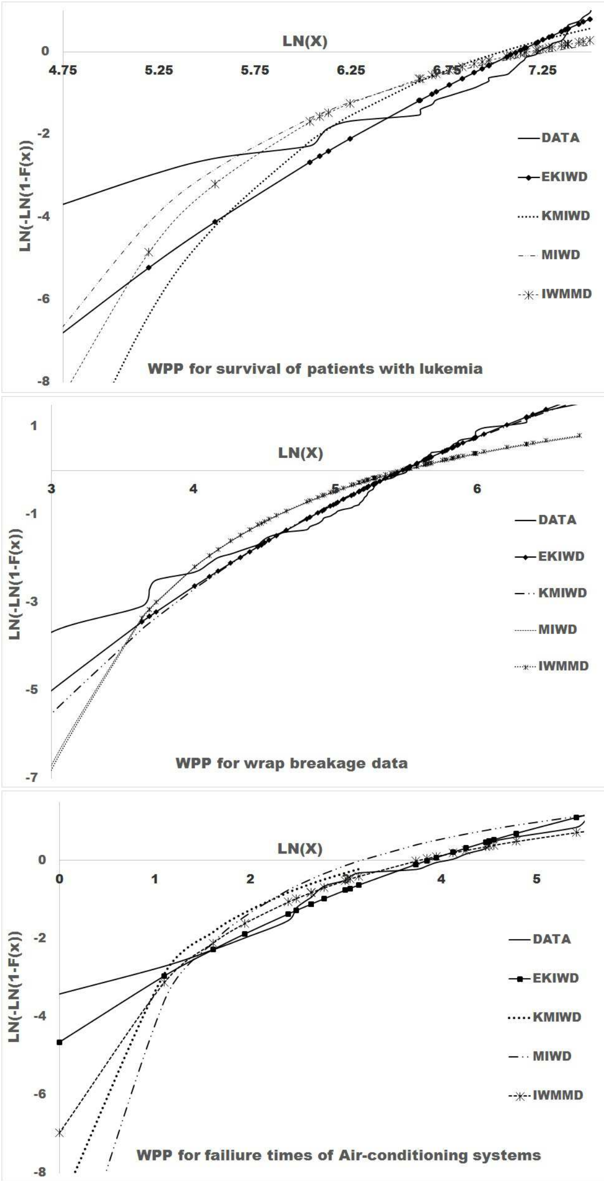

Figure 5 – Weibull probability plots of $EKIWD$, $KMIWD$, $EMIWD$, $MIWD$, $GIWD$ and $IWMM$ corresponding to data sets 1, 2 and 3.

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TABLE 2
Fitting $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ to data sets 1, 2 and 3.

Model	**Estimates of the parameters**					**Log-likelihood**	**AIC**	**BIC**	**AICc**	**CAIC**
Data Set 1										
EKIWD	α= 0.2206	β=0.2206	ρ= 21.875	σ= 1.213	δ=1005	-619.74	1249.48	1262.5	1250.12	1267.5
KMIWD	a=170.13	b=1.78	λ= 1.015	θ=7.368	α= 6.122	-640.35	1290.7	1303.73	1291.34	1308.72
GIWD	β=47.052	λ=3.447	θ=1.098			-645.751	1297.5	1305.32	1250.12	1308.32
MIWD	α= 1.103	θ=150.491	λ= 23.868			-652.5	1325.21	169.55	1316.17	1330.21
EMIWD	α= 19.109	θ=40.343	λ= 1.535	a=78.578		-653.293	1315	1326.80	1317.76	1331.80
IWMM	α_1= 59.877	α_2=62.215	β_1= 1.098	β_2=1.098		-655.75	1319.50	1329.92	1319.92	1333.92
Data Set 2										
EKIWD	α= 0.221	β= 0.221	ρ= 33.570	σ=14.211	δ= 20050	-299.46	608.91	617.36	610.68	622.36
KMIWD	a=35.506	b=2.71	λ= 37.579	θ= 2.213	α= 3.35	-312.51	635.01	645.23	636.19	650.23
GIWD	β=21.264	λ= 52.901	θ=1.199			-317.08	640.17	645.23	640.83	648.23
MIWD	α= 1.066	θ= 959.85	λ= 65.961			-317.74	641.49	646.56	642.16	649.56
EMIWD	α= 1.1621	θ= 41.634	λ= 8.677	a=30.282		-317.53	643.06	649.81	644.2	653.81
IWMM	α_1= 379.965	α_2=378.874	β_1= 1.198	β_2=1.198		-317.08	642.17	643.31	648.92	652.92
Data Set 3										
EKIWD	α= 0.19465	β= 0.19291	ρ= 7.66120	σ= 1.13121	δ=63.069	-151.37	312.74	319.75	315.24	324.75
KMIWD	a=1.19974	b=0.62404	λ= 6.255	θ=0.7694	α= 204.193	-154.8	319.6	326.61	322.1	331.61
GIWD	β=8.213	λ=0.7974	θ=4.398 0.7239			-155.2	316.22	320.42	317.14	323.42
MIWD	α= 70.701	θ=0.8335	λ=10.870			-158.04	322.08	326.28	323.00	329.28
EMIWD	α= 258.403	θ=0.0536	λ=0.6353	a=17.168		-156.86	321.72	323.32	327.32	331.32
IWMM	α_1= 7.387	α_2=3.978	β_1= 0.723	β_2=0.723		-155.11	318.23	323.83	319.83	327.83

TABLE 3

The values of log-likelihood functions and p values corresponding to the data set for testing $H_0 : \alpha = 1$ against the alternative hypothesis $H_1 : \alpha \neq 1$.

Source	Likelihood value under H_1(EKIWD)	Likelihood value (under H_0)	Λ	df	P value
Data Set 1	-619.74	-640.355	40.23	1	¡ 0.0001
Data Set 2	-299.46	-312.51	26.10	1	¡ 0.0001
Data Set 3	–151.37	-154.8	6.86	1	0.0088

TABLE 4

Average bias and mean squared errors(within brackets) of MLEs of the parameters of $EKIWD$ based on simulated data sets corresponding to $\alpha = 2$, $\beta = 2$, $\rho = 0.1$, $\sigma = 0.1$, $\delta = 2$

Sample Size	α	β	ρ	σ	δ
25	-0.5030	0.0948	-0.0127	-0.2281	-5.0194
	(2.53E-03)	(8.987E-05)	(1.625E-06)	(5.201E-04)	(2.519E-01)
100	-0.3122	0.0808	-0.0126	0.0188	-0.5137
	(9.747-04)	(6.525E-05)	(1.576E-06)	(3.542E-06)	(2.638E-03)
500	-0.2714	0.0211	0.0023	0.0166	-0.3997
	(7.365E-04)	(4.440E-06)	(5.111E-08)	(2.7701E-06)	(1.5974E-03)
1000	0.0560	0.0201	-0.0007	-0.0075	-0.1095
	(3.140E-05)	(4.0523E-06)	(5.269E-09)	(5.628E-07)	(1.1986E-04)

TABLE 5

Average bias and mean squared errors(within brackets) of MLEs of the parameters of $EKIWD$ based on simulated data sets corresponding to $\alpha = 0.1$, $\beta = 0.1$, $\rho = 0.1$, $\sigma = 0.1$, $\delta = 2.5$

Sample Size	α	β	ρ	σ	δ
25	0.0000	-0.0060	-0.0060	0.3845	-0.2148
	(0)	(4.568E-06)	(4.509E-06)	(1.847E-02)	(5.769E-03)
100	0.0000	0.0019	0.0008	-0.2049	0.0723
	(0)	(2.467E-07	(4.422E-08)	(2.798E-03)	(3.486E-03)
500	0.0000	0.0004	0.0004	-0.0678	0.0631
	(0)	(1.025E-08)	(1.146E-08)	(3.280E-04)	(2.846E-04)
1000	0.0000	0.0002	0.0001	-0.0376	0.0358
	(0)	(6.550E-08)	(6.326E-08)	(1.764E-04)	(1.604E-04)

Appendix A

1. Proof of Theorem 7: The r^{th} raw moment of the $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is

$$
\begin{aligned}
\mu_r &= \delta\int_0^\infty x^r\left(\frac{\alpha\rho}{x^{\alpha+1}}+\frac{\beta\sigma}{x^{\beta+1}}\right)\exp\left[-\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right].\left[1-\exp\left[-\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right]\right]^{\delta-1} \\
&= \delta\sum_{k=0}^\infty\frac{(-1)^k(\delta-k)_k}{k!}\left\{\alpha\rho\int_0^\infty\exp\left[-(k+1)\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right]x^{r-\alpha-1}dx\right. \\
&\quad \left.+\beta\sigma\int_0^\infty\exp\left[-(k+1)\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right]x^{r-\beta-1}dx\right\},
\end{aligned}
$$

by Eq. (7). Now on expanding the exponential term $\exp\left[-(k+1)\frac{\sigma}{x^\beta}\right]$ in the above integrals , we obtain

$$
\begin{aligned}
\mu_r &= \delta\sum_{k=0}^\infty\sum_{j=0}^\infty\frac{(-1)^{j+k}(\delta-k)_k(k+1)^j\sigma^j}{k!j!}\left\{\alpha\rho\int_0^\infty\exp\left[-(k+1)\left(\frac{\rho}{x^\alpha}\right)\right]x^{r-\alpha-j\beta-1}dx\right. \\
&\quad \left.+\beta\sigma\int_0^\infty\exp\left[-(k+1)\left(\frac{\rho}{x^\alpha}\right)\right]x^{r-\beta-j\beta-1}dx\right\} \\
&= \delta\sum_{k=0}^\infty\sum_{j=0}^k\sum_{m=0}^j\frac{(-1)^k m!}{k!}\binom{k}{j}\binom{k-j+1}{m}(\delta-k+j)_{(k-j)}S(j,m)\sigma^j\left(\alpha\rho I_{r\alpha}+\beta\sigma I_{r\beta}\right),
\end{aligned}
$$

by using Eq. (5) and Eq. (6), in which for $c=\alpha$ or $c=\beta$,

$$
I_{rc}=\int_0^\infty\left[\exp-((k-j+1)\rho)x^{-\alpha})\right]x^{r-c-j\beta-1}dx.
$$

By applying results Eq. (8) to Eq. (10), it can be further simplified into

$$
\mu_r = \delta\sum_{k=0}^\infty\sum_{j=0}^k\sum_{m=0}^j\left[\sigma^j\frac{(1-\delta-j)_k}{(\delta)_j}\frac{m!}{k!}\binom{k}{j}\binom{k-j+1}{m}S(j,m)\left(\alpha\rho I_{r\alpha}+\beta\sigma I_{r\beta}\right)\right].
$$

Evaluating the integrals $I_{r\alpha}$ and $I_{r\beta}$, using the substitution $u=x^{-\alpha}$, we obtain Eq. (20).

2. Proof of Theorem 8: By definition, the characteristic function of $EKIWD(\alpha, \beta, \rho, \sigma, \delta)$ is

$$
\begin{aligned}
\Phi_x(t) &= \delta\sum_{k=0}^\infty\frac{(-1)^k(\delta-k)_k}{k!}\int_0^\infty\left\{\exp(itx)\left(\frac{\alpha\rho}{x^{\alpha+1}}+\frac{\beta\sigma}{x^{\beta+1}}\right)\right. \\
&\quad \left.\times\exp\left[-(k+1)\left(\frac{\rho}{x^\alpha}+\frac{\sigma}{x^\beta}\right)\right]\right\}dx,
\end{aligned}
$$

in the light of Eq. (7) and Eq. (13). Splitting the integrals, we have,

$$\Phi_x(t) = \delta \sum_{k=0}^{\infty} \frac{(-1)^k (\delta-k)_k}{k!} \left[J(\alpha,\beta,\rho,\sigma) + J(\beta,\alpha,\sigma,\rho) \right],$$

where

$$J(\alpha,\beta,\gamma,\delta) = \int_0^{\infty} \alpha\gamma x^{-(\alpha+1)} \exp(itx) \exp\left[-(k+1)\gamma x^{-\alpha}\right] \exp\left[-(k+1)\delta x^{-\beta}\right] dx. \quad (A.1)$$

On expanding the exponential term $\exp\left[-(k+1)\delta x^{-\beta}\right]$ in (A.1) and integrating by using the substitution $u = x^{-\alpha}$ we get Eq. (24) in the light of Eq. (5).

3. Proof of Theorem 9: The pdf of $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$ can be written as

$$g(x) = h(x) \exp[-H(x)], \quad (A.2)$$

where H(x) is the cumulative hazard rate function. Modes of $EKIWD(\alpha,\beta,\rho,\sigma,\delta)$ are those values of x satisfying $f^{'}(x) = 0$. That is, $\left[h^{'}(x) - h^2(x)\right] \exp(-H(x)) = 0$, which gives the following in the light of (A.2).

$$g(x)\left[\frac{h^{'}(x)}{h(x)} - h(x)\right] = 0, \text{ which implies } \left[\frac{h^{'}(x)}{h(x)} - h(x)\right] = 0. \quad (A.3)$$

On differentiating h(x) given in Eq. (Eq. (15) with respect to x, we obtain

$$rh^{'}(x) = \delta h(x)\left[\frac{\alpha\rho}{x^{\alpha+1}} + \frac{\beta\sigma}{x^{\beta+1}} - \frac{\alpha(\alpha+1)\rho x^{\beta-1} + \beta(\beta)\sigma x^{\alpha-1}}{\alpha\rho x^{\beta} + \beta\sigma x^{\alpha}} + \frac{h(x)}{\delta}\right]. \quad (A.4)$$

Now (A.3) and (A.4) together gives

$$\frac{\alpha\rho}{x^{\alpha+1}} + \frac{\beta\sigma}{x^{\beta+1}} - \frac{\alpha(\alpha+1)\rho x^{\beta-1} + \beta(\beta)\sigma x^{\alpha-1}}{\alpha\rho x^{\beta} + \beta\sigma x^{\alpha}} = 0,$$

which on simplification reduces to Eq. (27).

Appendix B

Elements of the Information Matrix.

The elements of the information matrix $I(\Theta)$ are as given below, in which u_i, v_i, w_i and ϵ_i (for i $= 1, 2, \ldots, n$) are as defined in Eq. (41).

$$I_{11} = \frac{d^2 \ln(L(\Theta))}{d\alpha^2} = \sum_{i=1}^{n} \left\{ -u_i(\ln(x_i))^2 + \frac{(\delta-1)u_i(\ln(x_i))^2[w_i(1-u_i)-1]}{(w_i-1)^2} \right.$$
$$\left. + \frac{u_i}{\epsilon_i^2}\left[\beta v_i \ln(x_i)(\alpha \ln(x_i) - 2) - u_i\right] \right\},$$

$$I_{12} = \frac{d^2 \ln(L(\Theta))}{d\alpha d\beta} = \sum_{i=1}^{n} u_i v_i \left[\frac{(1-\delta)(\ln(x_i))^2 w_i}{(w_i-1)^2} - \frac{(1-\alpha\ln(x_i))(1-\beta\ln(x_i))}{\epsilon_i^2} \right],$$

$$I_{13} = \frac{d^2 \ln(L(\Theta))}{d\alpha d\rho} = \sum_{i=1}^{n} \left\{ \frac{u_i \ln(x_i)}{\rho} + \frac{u_i(\delta-1)\ln(x_i)}{\rho\,(w_i-1)^2}\left[u_i w_i - w_i + 1\right] \right.$$
$$\left. + \frac{u_i}{\rho\epsilon_i}\left[1 - \alpha\left(\ln(x_i) + \frac{u_i(1-\alpha\ln(x_i))}{\epsilon_i}\right)\right] \right\},$$

$$I_{14} = \frac{d^2 \ln(L(\Theta))}{d\alpha d\sigma} = \sum_{i=1}^{n} \left[\frac{(\delta-1)u_i v_i(\ln(x_i))w_i}{\sigma(w_i-1)^2} + \frac{\beta u_i v_i(\alpha\ln(x_i)-1)}{\sigma\epsilon_i^2} \right],$$

$$I_{15} = \frac{d^2 \ln(L(\Theta))}{d\alpha d\delta} = \sum_{i=1}^{n} \frac{u_i \ln(x_i)}{1-w_i},$$

$$I_{22} = \frac{d^2 \ln(L(\Theta))}{d\beta^2} = \sum_{i=1}^{n} \left\{ -v_i(\ln(x_i))^2 + \frac{(\delta-1)v_i(\ln(x_i))^2[w_i(1-v_i)-1]}{(w_i-1)^2} \right.$$
$$\left. + \frac{v_i}{\epsilon_i^2}\left[\alpha u_i \ln(x_i)(\beta \ln(x_i) - 2) - v_i\right] \right\},$$

$$I_{23} = \frac{d^2 \ln(L(\Theta))}{d\beta d\rho} = \sum_{i=1}^{n} \left[\frac{(\delta-1)u_i v_i(\ln(x_i))w_i}{\rho(w_i-1)^2} + \frac{\alpha u_i v_i(\beta\ln(x_i)-1)}{\rho\epsilon_i^2} \right],$$

$$I_{24} = \frac{d^2 \ln(L(\Theta))}{d\beta d\sigma} = \sum_{i=1}^{n} \left\{ \frac{v_i \ln(x_i)}{\sigma} + \frac{v_i(\delta-1)\ln(x_i)}{\sigma\,(w_i-1)^2}\left[v_i w_i - w_i + 1\right] \right.$$
$$\left. + \frac{v_i}{\sigma\epsilon_i}\left[1 - \beta\left(\ln(x_i) + \frac{v_i(1-\beta\ln(x_i))}{\epsilon_i}\right)\right] \right\},$$

$$I_{25} = \frac{d^2 \ln(L(\Theta))}{d\beta d\delta} = \sum_{i=1}^{n} \frac{v_i \ln(x_i)}{1-w_i},$$

$$I_{33} = \frac{d^2 \ln(L(\Theta))}{d\rho^2} = \sum_{i=1}^{n} \frac{u_i^2}{\rho^2}\left[\frac{w_i(1-\delta)}{(w_i-1)^2} - \frac{\alpha^2}{\epsilon_i^2} \right],$$

$$I_{34} = \frac{d^2 \ln(L(\Theta))}{d\rho d\sigma} = \sum_{i=1}^{n} \frac{u_i v_i}{\rho\sigma} \left[\frac{w_i(1-\delta)}{(w_i-1)^2} - \frac{\alpha\beta}{\epsilon_i^2} \right],$$

$$I_{44} = \frac{d^2 \ln(L(\Theta))}{d\sigma^2} = \sum_{i=1}^{n} \frac{v_i^2}{\sigma^2} \left[\frac{w_i(1-\delta)}{(w_i-1)^2} - \frac{\beta^2}{\epsilon_i^2} \right],$$

$$I_{45} = r\frac{d^2 \ln(L(\Theta))}{d\sigma d\delta} = \sum_{i=1}^{n} \frac{v_i}{\sigma(w_i-1)},$$

and

$$I_{55} = \frac{d^2 \ln(L(\Theta))}{d\delta^2} = \frac{-n}{\delta^2}.$$

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Summary

Here we consider an extended version of the Kumaraswamy modified inverse Weibull distribution and investigate some of its theoretical properties through deriving expressions for cumulative distribution function, reliability function, hazard rate function, quantile function, characteristic function, raw moments, median, mode etc. Certain reliability measures of the distribution are obtained along with the distribution and moments of its order statistics. The maximum likelihood estimation of the parameters of the distribution is discussed and certain real life data applications are given for illustrating the usefulness of the model. Further, with the help of simulated data sets it is shown that the average bias and mean square errors of the maximum likelihood estimators are in decreasing order as the sample size increases.

Keywords: Maximum likelihood estimation, Model selection, Moments, Order Statistics, Simulation