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ANALYSIS OF SIMPLE STEP-STRESS ACCELERATED LIFE TEST DATA FROM LINDLEY DISTRIBUTION UNDER TYPE-I CENSORING

Sharon Varghese A.
Department of Statistics, Pondicherry University, Puducherry-605 014, India
V. S. Vaidyanathan[1]
Department of Statistics, Pondicherry University, Puducherry-605 014, India

1. Introduction

Technological advances in engineering has resulted in products having high mean time to failure (MTTF). However, prolonged time to failure makes the study of lifetime characteristics difficult. To overcome this, the technique of accelerated life test (ALT) is used rather than usual life test. ALT is a technique to fasten the failure of products in order to obtain quick information about life characteristics. In ALT, products are exposed to higher levels of stress factors like temperature, pressure, humidity, voltage etc. to get quick failures and the data thus obtained is properly analyzed to infer the life characteristics under normal use. Based on stress loading there are three types of ALT's namely, constant stress ALT (CSALT), step-stress ALT (SSALT) and progressive stress ALT(PSALT).

In CSALT, only one level of higher stress is used. Sometimes it may be difficult to run at a higher stress for too long and CSALT may not produce enough quick failures. In PSALT, a test unit is subjected continuously to increasing stress. One major drawback of PSALT is that the progressive stress cannot be controlled accurately enough for long time in order to produce enough number of failures. In SSALT, a test unit is subjected to a specified level of stress for a prefixed period of time. If it does not fail during that period of time, then the stress level is increased for further prefixed period of time. This process is continued till all test units fail or some termination criteria is met. SSALT with two levels of stress is known as simple SSALT. SSALT yields quick failures when compared to CSALT and PSALT. Also it provides reliable estimates for life characteristics. Han and Ng (2014) have described the advantage of using SSALT over CSALT. For more details about ALT one may refer to Nelson (1990).

SSALT has attracted great attention in reliability literature. SSALT models the lifetime distribution of a test unit as a function of increased stress levels. There are three fundamental models for accommodating the effect of increased stress levels on the lifetime distribution. These are tampered random variable

[1] Corresponding Author e-mail: vaidya.stats@gmail.com

model, tampered hazard model and cumulative exposure model. Out of these, the most common fundamental model is cumulative exposure model, which is proposed by Sedyakin (1966) and generalized by Bagdonavicius (1978) and Nelson (1980). It assumes that remaining life of test unit under a stress level depends only on the current stress level and the current cumulative fraction failed.

Using cumulative exposure model, Miller and Nelson (1983) studied optimum plans for simple SSALT for exponential distribution. Bai *et al.* (1989) extended the case to type-I censoring. Bai and Hun (1991) derived optimum plans under competing cause of failure framework under exponential lifetime distribution. A Bayesian approach to SSALT under exponential distribution is discussed in Dorp *et al.* (1996).

From the available literature, it is seen that a lot of work has been done in the case of exponential distribution because of its mathematical tractability and having to deal with single parameter. It can be used to model lifetime of product that fails at a constant rate, regardless of the time it has survived. Although this property simplifies the analysis, it makes the model inappropriate for the reliability analysis of products that dont fail at constant rate. In this scenario, use of sophisticated models that take into account conditions such as increasing failure rate, is required. One such model is Lindley distribution, introduced by Lindley (1958), having one parameter. Ghitany *et al.* (2008) studied its properties and showed through numerical study that Lindley distribution is a better fit to lifetime data than exponential distribution. As of now, no work has been done in ALT models under Lindley distribution. This motivates the present work to introduce SSALT model and estimate the parameters involved under Lindley distribution.

In this paper, inference on simple SSALT data from Lindley distribution under type-I censoring with cumulative exposure model is considered. The paper is organized as follows. Section 2 describes the underlying model with assumptions. In section 3, estimation of parameters by maximum likelihood method is discussed. Numerical illustration based on both simulated data and real life data is carried out in section 4 using Newton-Raphson algorithm and parametric bootstrap procedures. Concluding remarks are given in section 5.

2. Model and Assumptions

Let x_1 and x_2 be two stress levels, with x_0 being the stress under normal use. Consider n identical units that are subjected to simple SSALT with initial stress level x_1. At prefixed time period τ, stress level is changed to x_2 and the test is continued until the censoring time τ_k. When all units fail before τ_k, it would result in complete data. In the proposed model, following assumptions will hold:-

i. For any stress level x_i, lifetime of a test unit follows Lindley distribution with cumulative distribution function (c.d.f)

$$F_i(t) = 1 - \frac{1+\theta_i+\theta_i t}{1+\theta_i} e^{-\theta_i t}, \ t \geq 0, \ \theta_i > 0, \ i = 1, 2. \tag{1}$$

and with corresponding probability density function (p.d.f)

$$f_i(t) = \frac{\theta_i^2}{1+\theta_i}(1+t)e^{-\theta_i t}, \ t \geq 0, \ \theta_i > 0, \ i = 1, 2. \tag{2}$$

ii. A cumulative exposure model holds. Under cumulative exposure model assumption, c.d.f of a test unit under simple SSALT is given by

$$G(t) = \begin{cases} F_1(t), & 0 \le t \le \tau \\ F_2(s + t - \tau), & \tau \le t \le \infty \end{cases} \tag{3}$$

where s is the solution of the equation

$$F_2(s) = F_1(\tau) \tag{4}$$

The corresponding p.d.f is given by

$$g(t) = \begin{cases} f_1(t), & 0 \le t \le \tau \\ f_2(s + t - \tau), & \tau \le t \le \infty \end{cases} \tag{5}$$

Under assumptions i and ii, c.d.f of a test unit is obtained as

$$G(t) = \begin{cases} 1 - \frac{1+\theta_1+\theta_1 t}{1+\theta_1} e^{-\theta_1 t}, & 0 \le t \le \tau \\ 1 - \frac{(1+\theta_2+\theta_2(s+t-\tau))}{1+\theta_2} e^{-\theta_2(s+t-\tau)}, & \tau \le t \le \infty \end{cases} \tag{6}$$

and the corresponding p.d.f is given by

$$g(t) = \begin{cases} \frac{\theta_1^2}{1+\theta_1} e^{-\theta_1 t}(1 + t), & 0 \le t \le \tau \\ \frac{\theta_2^2}{1+\theta_2} e^{-\theta_2(s+t-\tau)}(1 + s + t - \tau), & \tau \le t \le \infty \end{cases} \tag{7}$$

where s is obtained using equations (1) and (4) as

$$s = \frac{-\left[1 + \theta_2 + W\left(\frac{-(1+\theta_2)}{1+\theta_1} e^{-1-\theta_2-\theta_1\tau}(1 + \theta_1 + \theta_1\tau)\right)\right]}{\theta_2} \tag{8}$$

Here $W(.)$ denotes the Lambert's W function. For details about Lambert's W function one may refer to Corless *et al.* (1996) .

3. Maximum likelihood estimation

In this section, likelihood function corresponding to the model given in equation (7) is formulated based on type-I censoring and estimation of parameters θ_1 and θ_2 by maximum likelihood (ML) estimation method is discussed. In type-I censoring scheme, units are tested for a pre-specified period of time known as censoring time and the corresponding failure time is noted. After censoring time, unfailed units are removed from the life test and their lifetime is taken to be that of censoring time. For more details about type-I censoring and the corresponding likelihood function, one may refer to Meeker and Escobar (1998).

Consider a random sample of n units with lifetime having Lindley distribution that are subjected to simple SSALT with type-I censoring. Let n_1 and n_2 denote respectively, the number of failures that occur before and after stress level change time τ. Let N be the total number of failed units among n units. Test is terminated when all products fail before the censoring time or when censoring time τ_k is attained. Let $t_{i:n}$ denote the failure time of i-th failed unit

$i = 1, 2, ..., n$. With the above test scheme, the following failure times will be observed:

$$0 < t_{1:n} < t_{2:n} < ... < t_{n_1:n} \leq \tau < t_{n_1+1:n} < ... < t_{n_1+n_2:n} \leq \tau_k$$

The Likelihood function (L) for simple SSALT under cumulative exposure model with type-I censoring is given by (Meeker and Escobar, 1998)

$$L = \frac{n!}{(n-N)!} \left[\prod_{i=1}^{n_1} f_1(t_{i:n}) \right] \left[\prod_{i=n_1+1}^{N} f_2(s + t_{i:n} - \tau) \right] [1 - F_2(s + \tau_k - \tau)]^{n-N} \tag{9}$$

Using equations (6) and (7) in equation (9) and taking natural logarithm on both sides we obtain log-likelihood function as

$$\begin{aligned}
\log L = & \log\big(\frac{n!}{(n-N)!}\big) + \sum_{i=1}^{n_1} \big[2\log\theta_1 - \theta_1 t_{i:n} + \log(1 + t_{i:n}) - \log(1+\theta_1)\big] \\
& + \sum_{i=n_1+1}^{N} \big[2\log\theta_2 - \theta_2(t_{i:n} - \tau + s) + \log(1 + t_{i:n} - \tau + s) - \log(1+\theta_2)\big] \\
& + (n-N)\big[\log\big(1 + \theta_2 + \theta_2(\tau_k - \tau + s)\big) - \theta_2(\tau_k - \tau + s) - \log(1+\theta_2)\big]
\end{aligned} \tag{10}$$

Differentiating equation (10) with respect to θ_1 and θ_2 and equating to zero, we get the corresponding log-likelihood equations as

$$\begin{aligned}
\frac{\partial \log L}{\theta_1} = & \sum_{i=1}^{n_1} \big[\frac{2}{\theta_1} - t_{i:n} - \frac{1}{1+\theta_1}\big] + \sum_{i=n_1+1}^{N} \big[-\theta_2 s'_{\theta_1} + \frac{s'_{\theta_1}}{1 + t_{i:n} - \tau + s}\big] \\
& + (n-N)\big[\frac{\theta_2 s'_{\theta_1}}{1 + \theta_2 + \theta_2(\tau_k - \tau + s)} - \theta_2 s'_{\theta_1}\big] = 0
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\partial \log L}{\theta_2} = & \sum_{i=n_1}^{N} \big[\frac{2}{\theta_2} - (\theta_2 - s'_{\theta_2} + t_{i:n} - \tau + s) + \frac{1 + t_{i:n} - \tau + \theta_2 s'_{\theta_2} + s}{1 + \theta_2 + \theta_2(t_{i:n})} \\
& - \frac{1}{1+\theta_2}\big] + (n-N)\big[\frac{1 + \tau_k - \tau + \theta_2 s'\theta_2 + s}{1 + \theta_2 + \theta_2(\tau_k - \tau + s)} - \theta_2 s'_{\theta_2} \\
& - \tau_k + \tau - s - \frac{1}{1+\theta_2}\big] = 0
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
s'_{\theta_1} = & \big[\frac{W\big(\frac{-(1+\theta_2)}{(1+\theta_1)}(1 + \theta_1 + \theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\big)}{\big(1 + W\big(\frac{-(1+\theta_2)}{(1+\theta_1)}(1 + \theta_1 + \theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\big)\big)} \\
& \frac{1}{\big(\frac{(1+\theta_2)}{(1+\theta_1)}(1 + \theta_1 + \theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\big)\theta_2}\big] \\
& \times \frac{\theta_1(\theta_2+1)\,\tau e^{-(\theta_1\tau+1+\theta_2)}\,(\theta_1(\tau+1) + \tau + 2)}{(\theta_1+1)^2}
\end{aligned}$$

and

$$s'_{\theta_2} = -\,\theta_2\Big[\frac{W\big(\frac{-(1+\theta_2)}{(1+\theta_1)}(1+\theta_1+\theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\big)}{-(1+\theta_2)\big(1+W\big(\frac{-(1+\theta_2)}{(1+\theta_1)}(1+\theta_1+\theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\big)\big)}\theta_2+1\Big] \\ +W\Big(\frac{-(1+\theta_2)}{(1+\theta_1)}(1+\theta_1+\theta_1\tau)e^{-(\theta_1\tau+1+\theta_2)}\Big)$$

It can be seen that equations (11) and (12) are non-linear in θ_1 and θ_2. However, one may use numerical methods or gradient algorithms to obtain ML estimates (MLE) $\widehat{\theta_1}$ and $\widehat{\theta_2}$ of θ_1 and θ_2 respectively. Also, an estimate of the covariance matrix of $\widehat{\theta_1}$ and $\widehat{\theta_2}$ can be obtained using

$$\widehat{V} = \widehat{F}^{-1} \tag{13}$$

where

$$\widehat{F} = \left(\widehat{F_{ij}}\right)_{2\times 2} \tag{14}$$

with

$$\widehat{F_{ij}} = -\left(\frac{\partial^2 \log L}{\partial\theta_i\theta_j}\right)|\widehat{\theta}_i, \widehat{\theta}_j,\ i,j = 1,2.$$

4. Numerical illustration

4.1. Simulated data set

In this section, the performance of ML estimates is evaluated based on simulated data in terms of mean squared error (MSE) and relative absolute bias (RAB). Also, parametric bootstrap confidence intervals (CI) are constructed for the unknown parameters θ_1 and θ_2. To generate samples from ALT model given in (6) and to determine ML estimates, we proceed as follows:

1. Generate a random sample of size n from Uniform (0,1) distribution, and arrange them in ascending order to obtain order statistics $(U_{1:n}, ..., U_{n:n})$.
2. For given values of stress change time and parameters τ, θ_1 and θ_2, find n_1 such that
$$U_{n_1:n} \le 1 - \frac{1+\theta_1+\theta_1\tau}{1+\theta_1}e^{-\theta_1\tau} \le U_{n_1+1:n}$$
3. For given censoring time τ_k, find n_2 such that
$$U_{n_2:n-n_1} \le 1 - \frac{1+\theta_2+\theta_2(s+\tau_k-\tau)}{1+\theta_2}e^{-\theta_2(s+\tau_k-\tau)} \le U_{n_2+1:n-n_1}$$
4. The ordered observations $t_{1:n} \le ... \le t_{n_1:n} \le t_{n_1+1:n} \le ... \le t_{n_1+n_2:n} \le \tau_k$ are obtained as follows:
$$t_{i:n} = \begin{cases} \frac{-\left[W\left((U_{i:n}-1)(1+\theta_1)e^{-(1+\theta_1)}\right)+1+\theta_1\right]}{\theta_1}, & i = 1,2,...,n_1 \\ \frac{-\left[W\left((U_{i:n}-1)(1+\theta_2)e^{-(1+\theta_2)}\right)+1+\theta_2\right]}{\theta_2} - s + \tau, & i = n_1+1,...,N \end{cases}$$

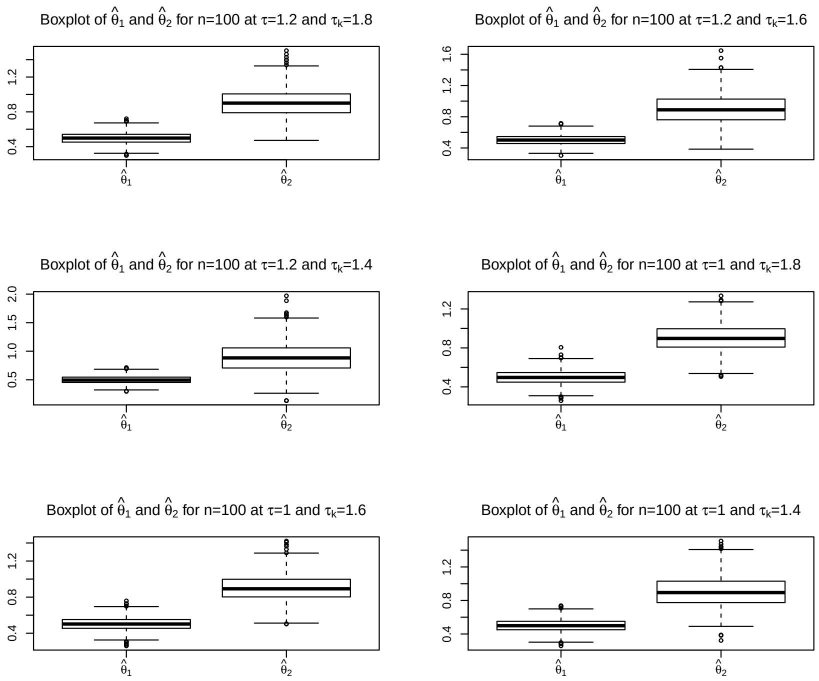

Figure 1 – Boxplots of $\widehat{\theta_1}$ and $\widehat{\theta_2}$ for 1000 simulations

5. Using the values of n_1,n_2,τ,τ_k and the ordered observations obtained in step 4, ML estimates of θ_1 and θ_2 is obtained by solving equations (11) and (12) using Newton-Raphson algorithm. Newton-Raphson algorithm is implemented by using maxLik package in R.

In this simulation study, values for θ_1 and θ_2 are taken as 0.5 and 0.9 respectively. Samples of sizes $n = 25, 50$ and 100 with stress level change times, τ=1, 1.2, 1.4 and censoring times τ_k=1.4, 1.6 and 1.8 are generated. For each case, the number of Monte Carlo (MC) runs is taken as 1000. Let $\widehat{\theta_{ik}}$ be the MLE of θ_i, $i = 1, 2$ based on k-th MC run, $k = 1, 2, ..., 1000$. The average ML estimates and respective MSEs and RABs are computed as follows and the results obtained are presented in Table 1.

$$\overline{\widehat{\theta_i}} = \frac{1}{r} \sum_{k=1}^{r} \widehat{\theta_{ik}}, i = 1, 2 \tag{15}$$

$$MSE(\widehat{\theta_i}) = \frac{1}{r} \sum_{k=1}^{r} \left(\widehat{\theta_{ik}} - \theta_i\right)^2, i = 1, 2 \tag{16}$$

$$RAB(\widehat{\theta_i}) = \frac{|\overline{\widehat{\theta_i}} - \theta_i|}{\theta_i}, i = 1, 2 \tag{17}$$

From Table 1, it is observed that as sample size n increases, MSE and RAB of ML estimates tend to zero. Further, as stress level change time increases, number

TABLE 1

MLEs of θ_1 and θ_2 with corresponding MSEs and RABs for different values of n, τ and τ_k based on 1000 simulations.

	τ	1			1.2			1.4	
	τ_k	1.4	1.6	1.8	1.4	1.6	1.8	1.6	1.8
$n = 25$	$\overline{\widehat{\theta}_1}$	0.4944	0.5038	0.4910	0.4989	0.5028	0.5022	0.4949	0.5060
	$\overline{\widehat{\theta}_2}$	0.9082	0.9102	0.8984	1.0174	0.8974	0.9168	1.0505	0.9157
	$\text{MSE}\left(\widehat{\theta}_1\right)$	0.0201	0.0235	0.0224	0.0179	0.0178	0.0211	0.0164	0.0153
	$\text{MSE}\left(\widehat{\theta}_2\right)$	0.1292	0.0985	0.0743	0.2286	0.1379	0.1072	0.2589	0.1509
	$\text{RAB}\left(\widehat{\theta}_1\right)$	0.0112	0.0076	0.0179	0.0020	0.0055	0.0043	0.0103	0.0120
	$\text{RAB}\left(\widehat{\theta}_2\right)$	0.0091	0.0113	0.0017	0.1304	0.0029	0.0187	0.1672	0.0174
$n = 50$	$\overline{\widehat{\theta}_1}$	0.4933	0.5041	0.5001	0.4999	0.4997	0.4946	0.4974	0.4993
	$\overline{\widehat{\theta}_2}$	0.9092	0.9084	0.8982	0.8921	0.8912	0.8919	0.9159	0.8850
	$\text{MSE}\left(\widehat{\theta}_1\right)$	0.0110	0.0113	0.0109	0.0100	0.0094	0.0091	0.0079	0.0079
	$\text{MSE}\left(\widehat{\theta}_2\right)$	0.0703	0.0494	0.0372	0.1281	0.0691	0.0519	0.1392	0.0828
	$\text{RAB}\left(\widehat{\theta}_1\right)$	0.0135	0.0082	0.0003	0.0003	0.0005	0.0109	0.0051	0.0013
	$\text{RAB}\left(\widehat{\theta}_2\right)$	0.0103	0.0094	0.0021	0.0088	0.0098	0.0091	0.0176	0.0167
$n = 100$	$\overline{\widehat{\theta}_1}$	0.5019	0.5039	0.4976	0.4998	0.5009	0.4989	0.5037	0.5005
	$\overline{\widehat{\theta}_2}$	0.9040	0.9016	0.9003	0.8854	0.8960	0.9001	0.8923	0.9049
	$\text{MSE}\left(\widehat{\theta}_1\right)$	0.0056	0.0054	0.0056	0.0045	0.0043	0.0046	0.0037	0.0037
	$\text{MSE}\left(\widehat{\theta}_2\right)$	0.0332	0.0230	0.0191	0.0745	0.0363	0.0267	0.0830	0.0419
	$\text{RAB}\left(\widehat{\theta}_1\right)$	0.0038	0.0079	0.0047	0.0003	0.0018	0.0022	0.0075	0.0010
	$\text{RAB}\left(\widehat{\theta}_2\right)$	0.0045	0.0018	0.0004	0.0161	0.0044	0.0001	0.0085	0.0054

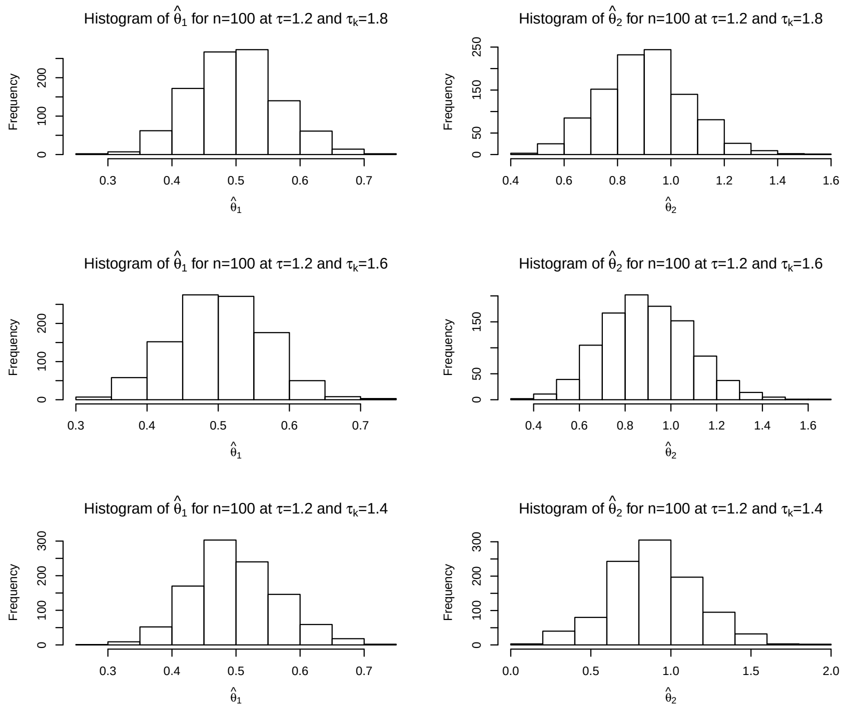

Figure 2 – Histograms of $\widehat{\theta_1}$ and $\widehat{\theta_2}$ for 1000 simulations

failures in lower stress level increases, resulting in lesser MSE and RAB for MLEs of θ_1. Similarly, as duration between stress level change time and censoring time increases, number of failures in higher stress level increases, resulting in lesser MSE and RAB for MLEs of θ_2. Also, from Figures 1 and 2, it is observed that estimates of θ_1 and θ_2 do not vary much from their original values for 1000 simulations.

Since exact distribution for MLEs of θ_1 and θ_2 cannot be found, it is not possible to construct exact CIs for θ_1 and θ_2. However, confidence intervals can be constructed using bootstrap approach. Towards this, parametric bootstrap confidence intervals namely, percentile bootstrap CI (PBCI) and bootstrap-t CI (BTCI) are considered. For an elaborate discussion on bootstrap CI, one may refer to Efron and Tibshirani (1993). The following steps are used to generate r^* bootstrap samples:

1. Using equation (15), obtain $\overline{\widehat{\theta}}_i$, $i = 1, 2$.

2. Generate a random sample of size n^* from Uniform (0,1) distribution, and arrange them in ascending order to obtain order statistics $(U^*_{1:n}, ..., U^*_{n:n})$.

3. For given value of stress change time τ, find n_1^* such that

$$U_{n_1^*:n^*} \leq 1 - \frac{1 + \overline{\widehat{\theta}}_1 + \overline{\widehat{\theta}}_1 \tau}{1 + \overline{\widehat{\theta}}_1} e^{-\overline{\widehat{\theta}}_1 \tau} \leq U_{n_1^*+1:n^*}$$

4. For given censoring time τ_k, find n_2^* such that

$$U_{n_2^*:n^*-n_1^*} \leq 1 - \frac{1 + \overline{\widehat{\theta}}_2 + \overline{\widehat{\theta}}_2 (s + \tau_k - \tau)}{1 + \overline{\widehat{\theta}}_2} e^{-\overline{\widehat{\theta}}_2 (s+\tau_k-\tau} \leq U_{n_2^*+1:n^*-n_1^*}$$

5. The ordered observations $t^*_{1:n^*} \leq ... \leq t^*_{n_1^*:n^*} \leq t^*_{n_1^*+1:n^*} \leq ... \leq t^*_{n_1^*+n_2^*:n^*}$ $\leq \tau_k$ are calculated as follows:

$$t^*_{i:n^*} = \begin{cases} \dfrac{-\left[W\left((U_{i:n^*}-1)(1+\overline{\widehat{\theta}}_1)e^{-\left(1+\overline{\widehat{\theta}}_1\right)}\right)+1+\overline{\widehat{\theta}}_1\right]}{\overline{\widehat{\theta}}_1}, & i = 1, 2, ..., n_1^* \\ \dfrac{-\left[W\left((U_{i:n^*}-1)(1+\overline{\widehat{\theta}}_2)e^{-\left(1+\overline{\widehat{\theta}}_2\right)}\right)+1+\overline{\widehat{\theta}}_2\right]}{\overline{\widehat{\theta}}_2} - s + \tau, & i = n_1^* + 1, ..., n_1^* + n_2^* \end{cases}$$

6. Using the values n_1^*, n_2^*,τ and τ_k and the ordered observations obtained in step 5, bootstrap MLEs of θ_1 and θ_2 namely, θ_1^* and θ_2^* are obtained by solving equations (11) and (12) using Newton-Raphson algorithm.

7. Repeat steps 2 to 6 r^* times to obtain r^* bootstrap samples

$$\left(\theta_i^{*[1]}, \theta_i^{*[2]}, ..., \theta_i^{*[r^*]}\right), i = 1, 2.$$

Bootstrap-t confidence interval

The steps involved in constructing Bootstrap-t CIs for θ_i, $i = 1, 2$ are as follows:

1. Find the ordered statistics $\left(\psi_i^{[1]}, ..., \psi_i^{[r^*]}\right)$, where

$$\psi_i^{[l]} = \frac{\theta_i^{*^{[l]}} - \overline{\widehat{\theta_i}}}{S.E\left(\overline{\widehat{\theta_i}}\right)}, i = 1, 2, l = 1, 2, ..., r^*$$

   Here

$$S.E\left(\overline{\widehat{\theta_i}}\right) = \sqrt{\frac{1}{r^2}\sum_{k=1}^{r} V(\widehat{\theta_{ik}})} \tag{18}$$

   with $V(\widehat{\theta_{ik}})$ obtained based on its estimate from equation (13).

2. For given $\alpha \in (0, 1)$, let $\widehat{z}^*_{\alpha/2}$ and $\widehat{z}^*_{1-\alpha/2}$ denote the $(r^*\alpha/2)$-th and $r^*\left((1-\alpha/2)\right)$-th largest values in ordered $\psi_i^{[l]}$s obtained in step 1.

3. Two-sided $100(1-\alpha)\%$ bootstrap-t CI is

$$\left(\overline{\widehat{\theta_i}} - \widehat{z}^*_{1-\alpha/2} S.E(\overline{\widehat{\theta_i}}), \overline{\widehat{\theta_i}} - \widehat{z}^*_{\alpha/2} S.E(\overline{\widehat{\theta_i}})\right), i = 1, 2.$$

Percentile bootstrap confidence interval

To obtain percentile bootstrap CI, the following steps are used

1. Arrange bootstrap estimates in ascending order to obtain

$$\left(\theta_i^{*^{[1]}}, \theta_i^{*^{[2]}}, ..., \theta_i^{*^{[r^*]}}\right), i = 1, 2.$$

2. Let $\theta^*_{i(\alpha/2)}$ and $\theta^*_{i(1-\alpha/2)}$ be the $(r^*\alpha/2)$-th and $(r^*(1-\alpha/2))$-th values in the ordered arrangement obtained in above step.

3. Two-sided $100(1-\alpha)\%$ percentile bootstrap CI is obtained as

$$\left(\theta^*_{i(\alpha/2)}, \theta^*_{i(1-\alpha/2)}\right).$$

For ease of computation, the values of n and n^* in the present study are taken to be equal. Number of bootstrap replications, r^* is taken as 500. The limits of BTCI and PBCI obtained for various values of τ taking $\alpha = 0.05$ are given in Table 2, Table 3 and Table 4.

From Tables 2 to 4, it is seen that the width of the CIs decreases as sample size increases.

TABLE 2
Bootstrap CIs with $\tau = 1$

τ_k		1.4	1.6	1.8
$n = 25$	$\widehat{\overline{\theta_1}}$	0.4944	0.5038	0.491
	BTCI	(0.4877,0.5077)	(0.4963,0.5132)	(0.4839, 0.5042)
	Width	0.02	0.0169	0.0203
	PBCI	(0.1883,0.7585)	(0.2728,0.8008)	(0.1881,0.7766)
	Width	0.5702	0.528	0.5885
	$\widehat{\overline{\theta_2}}$	0.9082	0.9102	0.8984
	BTCI	(0.8886,0.9387)	(0.8942,0.9364)	(0.8843,0.9197)
	Width	0.0501	0.0422	0.0354
	PBCI	(0.2665,1.7142)	(0.3457,1.5745)	(0.3946,1.4814)
	Width	1.4477	1.2288	1.0868
$n = 50$	$\widehat{\overline{\theta_1}}$	0.4933	0.5041	0.5001
	BTCI	(0.4876,0.5012)	(0.4981, 0.5113)	(0.4949,0.5084)
	Width	0.0136	0.0132	0.0135
	PBCI	(0.2878,0.7042)	(0.3176,0.7293)	(0.2872,0.6889)
	Width	0.4164	0.4117	0.4017
	$\widehat{\overline{\theta_2}}$	0.9092	0.9085	0.8982
	BTCI	(0.8963, 0.9264)	(0.8969,0.9235)	(0.8868,0.9123)
	Width	0.0301	0.0266	0.0255
	PBCI	(0.4911,1.4139)	(0.5155,1.3607)	(0.5260,1.3154)
	Width	0.9228	0.8452	0.7894
$n = 100$	$\widehat{\overline{\theta_1}}$	0.5019	0.504	0.4976
	BTCI	(0.4978,0.5068)	(0.5002,0.5089)	(0.4933,0.5023)
	Width	0.009	0.0087	0.009
	PBCI	(0.3640,0.6483)	(0.3664,0.6357)	(0.3651,0.6495)
	Width	0.2843	0.2693	0.2844
	$\widehat{\overline{\theta_2}}$	0.9041	0.9016	0.9003
	BTCI	(0.8940,0.9182)	(0.8938,0.9136)	(0.8932, 0.9099)
	Width	0.0242	0.0198	0.0167
	PBCI	(0.5467,1.2729)	(0.5795,1.1799)	(0.6419,1.1590)
	Width	0.7262	0.6004	0.5171

TABLE 3
Bootstrap CIs with $\tau = 1.2$

τ_k		1.4	1.6	1.8
	$\overline{\widehat{\theta_1}}$	0.499	0.5028	0.5022
	BTCI	(0.4916,0.5097)	(0.4962, 0.5135)	(0.4954,0.5130)
	Width	0.0181	0.0173	0.0176
	PBCI	(0.2438,0.7952)	(0.2464,0.7672)	(0.2448,0.7802)
	Width	0.5514	0.5208	0.5354
	$\overline{\widehat{\theta_2}}$	1.0174	0.8974	0.9168
$n = 25$	BTCI	(0.9924,1.0400)	(0.8778,0.9285)	(0.8987,0.9436)
	Width	0.0476	0.0507	0.0449
	PBCI	(0.4915,2.1132)	(0.2660,1.7012)	(0.3438,1.6801)
	Width	1.6217	1.4352	1.3363
	$\overline{\widehat{\theta_1}}$	0.4999	0.4997	0.4946
	BTCI	(0.4952,0.5066)	(0.4944,0.5062)	(0.4896,0.5014)
	Width	0.0114	0.0118	0.0118
	PBCI	(0.3212,0.6716)	(0.3274,0.6982)	(0.3152,0.6773)
	Width	0.3504	0.3708	0.3621
	$\overline{\widehat{\theta_2}}$	0.8921	0.8912	0.8918
$n = 50$	BTCI	(0.8734,0.9220)	(0.8759, 0.9124)	(0.8787,0.9093)
	Width	0.0486	0.0365	0.0306
	PBCI	(0.2641,1.6359)	(0.3889,1.4617)	(0.4467,1.3957)
	Width	1.3718	1.0728	0.949
	$\overline{\widehat{\theta_1}}$	0.4998	0.5009	0.4989
	BTCI	(0.4961,0.5051)	(0.4971,0.5053)	(0.4953,0.5037)
	Width	0.009	0.0082	0.0084
	PBCI	(0.3541,0.6315)	(0.3775, 0.6372)	(0.3665,0.6259)
	Width	0.2774	0.2597	0.2594
	$\overline{\widehat{\theta_2}}$	0.8855	0.896	0.9001
$n = 100$	BTCI	(0.8715,0.9064)	(0.8852,0.9110)	(0.8912,0.9104)
	Width	0.0349	0.0258	0.0192
	PBCI	(0.3827,1.4314)	(0.5060,1.3028)	(0.6192,1.2148)
	Width	1.0487	0.7968	0.5956

TABLE 4
Bootstrap CIs with $\tau = 1.4$

τ_k		1.6	1.8
$n = 25$	$\overline{\widehat{\theta_1}}$	0.4949	0.506
	BTCI	(0.4882,0.5061)	(0.4995,0.5155)
	Width	0.0179	0.016
	PBCI	(0.2285,0.7596)	(0.2765,0.7647)
	Width	0.5311	0.4882
	$\overline{\widehat{\theta_2}}$	1.0505	0.9157
	BTCI	(1.0207,1.0754)	(0.8973,0.9487)
	Width	0.0547	0.0514
	PBCI	(0.4914,2.3491)	(0.2658,1.6965)
	Width	1.8577	1.4307
$n = 50$	$\overline{\widehat{\theta_1}}$	0.4974	0.4993
	BTCI	(0.4923,0.5044)	(0.4943,0.5057)
	Width	0.0121	0.0114
	PBCI	(0.3169,0.6939)	(0.3322,0.6891)
	Width	0.377	0.3569
	$\overline{\widehat{\theta_2}}$	0.9159	0.885
	BTCI	(0.8974,0.9481)	(0.8724,0.9069)
	Width	0.0507	0.0345
	PBCI	(0.2653,1.6858)	(0.3853,1.3576)
	Width	1.4205	0.9723
$n = 100$	$\overline{\widehat{\theta_1}}$	0.5038	0.5005
	BTCI	(0.5003,0.5083)	(0.4968,0.5045)
	Width	0.008	0.0077
	PBCI	(0.3757,0.6244)	(0.3870,0.6321)
	Width	0.2487	0.2451
	$\overline{\widehat{\theta_2}}$	0.8924	0.9049
	BTCI	(0.8784,0.9133)	(0.8950,0.9196)
	Width	0.0349	0.0246
	PBCI	(0.3897, 1.4115)	(0.5191,1.2636)
	Width	1.0218	0.7445

TABLE 5
MLEs and Bootstrap CIs for real life data

Parameter	MLE	BTCI	Width	PBCI	Width
θ_1	0.2684	(0.1984, 0.3640)	0.1656	(0.1848,0.3543)	0.1695
θ_2	2.3070	(1.778, 2.843)	1.065	(1.802, 2.852)	1.05

4.2. Real life data set

In this section, a real life data from Han and Kundu (2015) is used to illustrate the proposed model. Data set consists of lifetimes (measured in hundred hours) of 31 failed solar lightning devices out of 35 devices subjected to a simple step stress test with temperature as stress. In this experiment, temparature was changed from $293K$ to $353K$ at stress change time (τ) 5 and the experiment was stopped at censoring time (τ_k) 6. It is assumed that at each temperature level, lifetime of the device follows Lindley distribution. From the data, it is observed that $n_1 = 16$ and $n_2 = 15$. The model parameters are estimated by solving equations (11) and (12). Bootstrap CIs are also found and results obtained are presented in the Table 5.

5. Concluding remarks

A simple step stress model with Lindley lifetime under type-I censoring scheme is introduced in this paper. Under the assumption of cumulative exposure model, point estimation of parameters by the method of maximum likelihood is discussed. Parametric bootstrap confidence intervals for unknown parameters are constructed using bootstrap-t CI approach and percentile bootstrap CI approach. The proposed model is illustrated with both real life and simulated data sets. From simulation study, it is found that as sample size increases, MSEs decrease and RABs tend to zero for both the model parameters. It is also found that precision of MLEs and CIs depends on duration between stress level change time and censoring time. As duration between stress level change time and censoring time increases, number of failures in higher stress level increases. This results in smaller MSE and RAB for MLEs and shorter CIs.

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Summary

This article introduces simple step-stress accelerated life time model with Lindley lifetime under type-I censoring. The corresponding likelihood function is developed and parameter estimation by maximum likelihood approach is discussed. Also, parametric bootstrap confidence intervals are constructed for the unknown parameters. Further, performance of the estimates of the proposed model is evaluated through simulation study as well as real life data set.

Keywords: Accelerated life testing; Bootstrap; Cumulative exposure model; Lindley distribution; Maximum likelihood estimation; Type-I censoring