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CONSIDERATIONS ON PROBABILITY: FROM GAMES OF CHANCE TO MODERN SCIENCE

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1. The origins of probability

The journey from mere intuition of uncertainty to its measurement took many centuries. The birth of probability as the measurement of chance events and thus the calculus of probability can be dated to 1654, when Blaise Pascal provided Fermat with the solution to two problems regarding how to divide the stakes fairly in the case of a game of chance being interrupted.

The opening words of *Recherches* (1837) by Poisson, one of the early eighteenth centurys greatest mathematicians, are famous: "A problem about games of chance proposed to an austere Jansenist by a man of the world was the origin of the calculus of probabilities". It is said that in around 1650 a certain Chevalier de Meré wrote a letter to Blaise Pascal in which he posed the famous logician two mathematical questions connected to games of chance. In the first he asked whether it was more likely to obtain at least one six by rolling a die four times or to obtain a double six by throwing two dice twenty-four times. In the second he posed a much more complex question: in a game which is won by reaching a set score, how should the stakes be divided fairly if the game is interrupted before the end? These were questions that had already been long debated with solutions that were not always correct, but which already sought to use combinatorics in order to construct the space of possibilities.

The anecdotes interest lies in the fact that problems of this kind were debated in the salons and intellectual coteries of the time, and not only among an inner circle of logicians and mathematicians.

Chevalier de Merés real name was Monsieur Antoine Gomboaud, a wise, honest man, a master of fine manners and the author of virtuous books under that pseudonym. He was by no means a frequenter of gambling dens; nor was Pascal, a strict Jansenist who led a secluded life. The purpose of his reply to the letter was not to win games or share stakes. On the contrary, its purpose was to learn to reason in a new way.

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Pascals solution came after a long exchange of letters with Fermat and permanently cleared the way for the establishment of probability theory. At the same time, other scholars arrived at similar solutions to different problems, almost as if to bear witness to the need for this new language. It is a language which revolutionised logic as much as it did mathematics, and one which offered modern science an array of extraordinary tools (Hacking, 1975).

Interest in this new way of dealing with uncertainty was further kindled by a parallel debate started by Pascal, who played an extremely important role within the Port-Royal school and contributed to *Logique de Port-Royal* (1662), Antoine Arnaulds textbook on logic. Even more famous, however, is his essay on “wager”, which at the time sparked a large number of debates and discussions and provided a strong impetus for the dissemination of the new logic represented specifically by the calculus of probabilities.

It was in combinatorial language, which is completely deterministic and mathematical, that Pascal also found the simplest language for helping the scientific world to understand the power of probability. It was a language which did not frighten scientists of the period, because combinatorial variability remained wholly governed and governable by man, it offered a moment of playful thought which had nothing to do with the reality of phenomena. It was entirely subordinated to what would emerge as Laplaces determinism, in which the concept of probability was limited to neutralising the effects of accidental errors of measurement on the way of seeking the “true” value of the observed variable.

The stage was set, and in the space of a few decades the greatest philosophers of the period, working from different assumptions and using different languages, succeeded in laying the foundations of probability theory and contributed to establishing the many meanings that the word “probability” has today. The seventeenth century alone produced to name but a few Leibniz, Huygens, de Witt, Wilkins, Arbuthnot, Fermat, Halley, Hobbes, Petty, and many other lesser known figures whose work led to the blossoming, in the centuries that followed, of the giants of probabilistic reasoning: first and foremost the Bernoullis, followed by Bayes with his famous theorem that provided a probabilistic solution to the debate surrounding induction, as well as Poisson, Gauss, Laplace, and others (I. Hacking, 1975).

In the “neutral” context of games of chance and combinatorial mathematics, the first probability theorists came to prove fundamental theorems: in this regard we need only mention De Moivre, Lagrange or the Bernoullis, through to Gauss.

These leading lights of modern thought, however, were not only mathematicians, but also and above all physicists and astronomers, and their philosophical speculations were strongly influenced by sense experience. Gauss (1809) derived his famous “bell curve” model through purely analytical means after imposing a number of preliminary formal conditions, which among other things he had assumed basing on empirical evidences obtained from repeated instrumental measurements of astronomical variables, following an absolutely circular logical process. This proof had already enabled Lagrange (1806) to propose the arithmetic mean of repeated measurements as the “most likely” value for an unknown quantity. And earlier still, the bold inversion of De Moivre’s theorem perhaps generated ambiguous confusion with the law of large numbers, which was resolved only with the advent of modern statistical inference.

For many years, probability was to remain a convenient expedient for dealing with uncertainty, that is for compensating for the limits of knowledge of the human mind, a mind which would never have been able to compete with the (wholly "metaphysical") infinite intelligence hypothesised by Laplace. Laplace's aim was not so much to seek the causes of phenomena as to measure uncertainty through the new language of probability. With the calculus of probabilities, efficient strategies could be constructed to minimise errors in forecasting uncertain events or estimating unknown quantities. The latter had always been a vital aim of astronomers between the seventeenth and nineteenth centuries, many of whom are known today specifically for their methodological and conceptual contribution to probability theory.

Nevertheless, the subtle, intense activity of this new logic and its language continued to expand the potential horizons of scientific thought. Once a phenomenon had been identified, it could be described by means of a probabilistic statistical model capable of interpreting its variability and, with it, the implicit rules that characterise the phenomenon itself in its uniqueness, in the manner of a game of chance, the rules of which need deciphering.

In applying the binomial law, which describes the results of repeated coin tosses, to human characteristics, Adolphe Quetelet (1869) was certainly not interested in playing heads or tails, but rather in seeking the "marvellous regularities" that nature set before his eyes. In studying certain anthropometric traits, such as height and chest measurements, he realised that the way in which they are distributed, from the smallest to the largest, could be admirably described by certain probabilistic models, specifically those derived from games of chance (Gini, 1964; Scardovi, 1982).

2. Probability, variability and uncertainty

Variability in natural and social phenomena and the resulting uncertainty of individual facts or events was the challenge faced by Galilean science. The central thread running through the development of statistical and probabilistic thought is the evolution of the semantic concept of variability. Two aspects, therefore, are inherent in tangible reality and its states of nature: the *variability* of phenomena and the *uncertainty* of events.

Variability is a characteristic of populations (of molecules, animals, plants, etc.) while uncertainty is a characteristic of events and their occurrence (if, how, when, etc.).

Variability and uncertainty describe two conceptually different aspects which humanity has always had to deal with by means of strategies which become established in terms of habitual behaviour or which are raised to the status of rational categories within the sphere of scientific research in order to seek the solution to complex problems.

Although the models of reasoning and the methods that derive from them have many aspects in common, they diverge conceptually in respect of these two terms variability and uncertainty and follow different rational processes.

Faced with variability, the human mind seeks out regularities that can lead to the discovery of natural laws or models of behaviour. Faced with uncertainty, it seeks out winning strategies in terms of the probability of events and measurement of risk.

The need to investigate real phenomena which express themselves as a plurality of results that differ from one another shifted the interest of scientific research from the individual case to cases taken together as a whole and shaped the probability of the individual case to its own ends. The search for laws of the whole considered in its entirety laid its theoretical foundations upon probabilistic models and upon the rules that generate them through combinations of events that could be compared with the results of games of pure chance.

A probabilistic model is a purely formal abstraction, and becomes a description of reality only when the probability that specifies it becomes a numerical value; that is to say, it becomes a statistical frequency, the frequency with which a real event occurs in a set of observations. In the language of mathematics, probability is a pure logical symbol, the parameter of a function, a conceptual category; in the language of science it is a number that must be sought in the concreteness of events and estimated on the basis of observations following a rational process that is rigorously statistical.

Let us take, for example, the sex ratio at birth, that is the proportion of male and female births in a natural population of births. This ratio, since it first began to be measured in the human species, converges on 0.51 for males and 0.49 for females, has remained constant across space, time and populations and has come to be assumed as the probability of a male or female birth.

The astonishing regularity of this ratio has been assumed to be an intrinsic characteristic of the species, that is to say the probability of a male or female birth in the human species, under natural conditions, without distortions caused by extraordinary factors. This means that in situations where a deviation from this ratio has emerged, the disturbance factors which generated the distortion have been sought and have always been found.

As a result, this number a simple statistical frequency thrown up by the observation of a plurality of individual events referring to just two categories is transformed into the probability of an uncertain event: male or female. While in terms of expectations we may state that out of 100 births, 51 will be male and 49 will be female, in reply to the question "what sex will the next birth be?" we may say that there is a probability of 0.51 that it will be male and 0.49 that it will be female. Nothing other than a virtual toss of an asymmetrical coin in favour of the male face (Gini, 1911).

Thus, if we were to bet on the sex of a newborn, in the absence of any other information, we would have more chance of winning by betting on male.

3. Probability. An evolved concept

In the history of science, the rationalisation of the concept of uncertainty and its measurement took centuries. In human development, from infancy through to maturity, when does awareness of uncertainty kick in and when does it start to be controlled?

Jean Piaget described the three stages in the development of the notion of chance in the mind of the child (Piaget, 1955). According to Piaget, the combination of chance and necessity appears to be the product of mature minds. At the first cognitive levels the mind does not distinguish the two concepts, and is only subsequently capable of separating the effects of causal factors from the effects of accidental factors in the comprehension of observed phenomena. Much later, by strength of reason it is able to combine the two factors into an explanatory model which finds its highest expression in statistical law, where chance and necessity blend in surprising regularities that make forecasts more stable in time and in space.

Extraordinarily, the evolution of scientific thought around the concepts of uncertainty and probability runs parallel to the acquisition of these concepts by the individual.

Not all cultures have attained the same scientific rationality, just as not all individuals cope with the uncertainty of everyday life with the same degree of rationality. It would thus seem that phylogenesis and ontogenesis follow the same evolutionary process.

It is interesting to consider how mankind has sought since ancient times to deal with the uncertainty of events and how it has even created situations for the sole purpose of entertainment which intrinsically contain chance (and therefore uncertainty) in their physical nature. I am referring, of course, to games of chance.

Emblematic is the history of die, whose earliest crude representation is provided by the astragalus, a small bone found in the foot of sheep or dogs which was transformed not only into a mere object of play but also the master of people's destinies: from caves to taverns, from gambling dens to the salons of the nobility and the bourgeoisie. The astragalus was very easy to handle and, if thrown, tended to land on either one of its four long, flat sides. The astragalus was gradually refined over time in order to ensure the quality of play and, because each side had a different weight from the others, players soon learned to take account of the "ease" with which each face would come up. It has thus been correctly identified as the precursor of the die, although for centuries it maintained its own independent existence in parallel with that of the more illustrious artefact, which has entered the history of science with honour, where it continues to play its role. (David, 1962; Monari, 2014).

Astragaluses and dice were not just used for games, as many chronicles recount. They also became tools for deciding the fate of men. In order to know their fate or the events of the world that surrounded them, ancient cultures asked for help from the oracles, who were seen as intermediaries between their doubts and the will of the gods, the creators of all things. The oracles, however, were often obscure (it could not be otherwise), while the priests were clearer and certainly more accessible to the people. One of the many strategies adopted by the ancient priests to satisfy inquirers was to seek signs in order to issue their response, basing it on the outcome of a random test such as throwing an object that could only fall in a limited number of different ways, with the implicit assumption that it was the gods that guided the result in a bizarre game of dice with the fate of men.

In order to provide a response to the uncertainty of an unknown event, a ritual was used which was itself intrinsically uncertain: the roll of a die. Through the game, construed as an oracular game, men believed that they could challenge the gods in order to discover their designs. Yet how is it possible to respond to uncertainty with uncertainty? What evolution in thought had taken place that substituted the oracle with a random experiment? To the variability of mundane things, the priest added the uncertainty generated by the instrument, which was kept in check by its physical symmetry, in other words through the acknowledgment of the additional component of variability that was induced.

What distinguishes the astragalus from the die? At a certain moment of human history, with a decisive leap of abstraction, astragalus came to be replaced by die, artefact through which man attempted to give shape to a highly sophisticated ideal concept: that of symmetry, the intention being to ensure the "equipossibility" of each of the six faces landing face up with each throw. The die became the symbolic representation of an immutable, perfect physical object which becomes variable at the moment it is deployed. Albeit in an irrational manner, mankind became the creator of variability and, upon creating it, challenged it.

The sophisticated intellectual process that led to the symmetrical die in keeping with Euclidean geometry was not grasped by the great thinkers of ancient times who instead went in search of certainty, that certainty which they could only find in mathematical-deductive reasoning but which did not provide answers to questions about the real world.

And in any case it is precisely gamblers, who invented dice, who are the least inclined to use probability as a rational tool for evaluating the risk of losing.

Cardano expounded the notion effectively in his autobiography, and it was reaffirmed by Laplace when in his *Essai philosophique sur les probabilitès* (1814) he wrote "It is principally at games of chance that a multitude of illusions support hope and sustain it against unfavorable chances. The majority of those who play at lotteries do not know how many chances are to their advantage, how many are contrary to them. They see only the possibility by a small stake of gaining a considerable sum, and the projects which their imagination brings forth, exaggerate to their eyes the probability of obtaining it; the poor man especially, excited by the desire of a better fate, risks at play his necessities by clinging to the most unfavorable combinations which promise him a great benefit. All would be without doubt surprised by the immense number of stakes lost if they could know of them; but one takes care on the contrary to give to the winnings a great publicity, which becomes a new cause of excitement for this funereal play".

As a keen judge of the disposition of gamblers, on which the success of lotteries is based, Laplace contemplated the fallacies of those who pursue "cold" numbers, in other words numbers that have not been drawn for a long time, or, alternatively, "hot" numbers, which have been drawn more frequently in recent draws. If the probability of each number being drawn remains constant and if the numbers are well mixed, "the past ought to have no influence upon the future". Again, Laplace: "The very frequent drawings of a number are only the anomalies of chance; I have submitted several of them to calculation and have constantly found that they are included within the limits which the supposition of an equal possibility of the drawing of all the numbers allows us to admit without improbability".

The gambler places his faith in fate and in the favourable signals that he believes he interprets. Thus probability is no longer the measurement of the possibility of an event occurring, but becomes a subjective evaluation, similar to a bet, and is only marginally related to the objective intrinsic structure of the game, that is the random experiment that is conducted in the game.

The event on which the gambler bets is no longer one of the many included in the relevant probability space, but is personalised by omens, memorable experiences, dreams, symbols, Neapolitan grimace, and so on.

4. Some considerations on probability and its measurement

Let us dwell for a moment on the concept of probability and on the debate regarding its nature that has sparked particularly in the twentieth century many disputes that have often been extremely heated. No single, incontrovertible definition of probability exists, even today, although all agree that it is a measurement of uncertainty. Yet what uncertainty, and what measurement? Does only one kind of uncertainty and therefore measurement exist?

The vaguely metaphysical connotation that the notion of uncertainty has carried since ancient times, torn between a destiny unknown to man and pure randomness over which he has no control, has placed its measurement that is, the probability of uncertain events in a sort of limbo. Almost always ignored by metrology, probability is dealt with separately even in the sacred theoretical texts on measurement, while logicians and philosophers have "fiddled" with it extensively, expatiating at length on possible definitions of it, almost always destroying them, without opening up prospects for a scientific or operational use of this extremely potent scientific tool. Yet scientific research has always gone on its own way, building its own tools in keeping with its own objectives and since its origins it has incorporated probability, gradually extending its role: from the theory of errors in measurement at the beginning of the nineteenth century, with Gauss and Laplace, to the hereditary nature of biological features with Mendel and population genetics with Fisher, Haldane and Wright, and, more recently, statistical mechanics, with Bohr, Heisenberg and Schroedinger (Scardovi, 1982).

Ever since the physical and natural sciences began to adopt scientific explanations for many otherwise inextricable phenomena in which physical constants change as changes occur in the relevant system and their measurements become variable, the only rational strategy has been found in the statistical distribution of their possible states of magnitude.

In the context of this new way of "doing" science, probability as the measurement of random events has become an essential element of statistical mechanics and population genetics. In the behavioural sciences, too, it has offered models of rational choice in conditions of uncertainty: we might mention to take just one example the reduction of economic strategy to game theory (von Neumann and Morgenstern, 1944).

It is clear, then, that probability has permeated every sector of research called upon to provide inductive or decision-based criteria, and has rightfully taken up its place in the most sophisticated systems where every uncertainty must be matched by a measurement. Why, then, do we still meet with so much reluctance to consider the probability of an event of the same epistemic nature as the estimation of a force, of a length or of a weight? And why must the measurement of probability spark so many more philosophical and methodological controversies than the measurement of a mass, sound, velocity or any other attribute?

Perhaps the answer is merely psychological and should be interpreted at the psychosensory level: uncertainty does not have the same substantial solidity as our work table.

A *dual concept*. Since its inception, the concept of probability has been dual: 1) the probability of random, repeatable events represented by games of chance and in some manner transferred to repeatable phenomena that can be reduced to classes and represented in the form of frequency distributions, and 2) opinions or conjectures (hypotheses) for measuring relative degrees of uncertainty, which well fitted the action of betting.

The subject is too complex to discuss in a few lines, so it is advisable to refer to the extensive literature on the subject (for example Carnap, 1950; Bunge, 1981; Hacking, 1975; De Finetti, 1931; Savage, 1954). Nevertheless, difficulties in defining probability, which have primarily occupied epistemologists, have not halted developments in statistical science and methodology which, with regard to the concept of probability, have constructed their theoretical framework.

Nowadays, the distinction introduced by Carnap between *Probability-1* and *Probability-2* (Carnap, 1950) is firmly established. The former, also termed epistemic probability, estimates the uncertainty of the opinion of an individual regarding an event occurring, or regarding the truthfulness of a hypothesis, more or less confirmed by empirical evidence. The latter measures the uncertainty of the occurrence of a random event generated by a random experiment or that of an event included in a set of events which are homogeneous in classificatory terms and virtually assumed to be repeatable.

Probability and frequency: an ambiguous duality. Over time, scientific and experimental research, more than philosophers' definitions, has sought the criteria with which the most appropriate measurement of the probability of events can be estimated, in order that it may be used effectively in its models, in checking hypotheses and in forecasting processes.

Probability thus emerges from games of chance to become the measurement of the occurrence of natural and social phenomena. Yet while in games of chance it was easy to grasp the simple criterion which measured probability as a ratio between the number of favourable outcomes of an event and the number of possible outcomes, within a combinatorial framework consisting of a finite number of possibilities, in real phenomena what measurement was to be adopted?

The answer emerged from modern research practice: measurement of the probability of a repeatable event (in the statistical and classificatory sense: the category-event) is given by its relative frequency, calculated over a sufficient number of observations, all other conditions being equal.

Pierre Simone de Laplace, one of the great founding fathers of probability theory and an extremely eminent astronomer and scientist, offered the first definition of probability outside any direct reference to games of chance, approaching statistical frequency and consisting in the ratio of the numerical size of the class in question to the total number of units observed. Laplace used games of chance as pure examples of calculation or as mental experiments.

In his philosophical essay on probability he took up a number of fundamental considerations set forth by Jacob Bernoulli in his *Ars Conjectandi* ( ), writing thus: "The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favourable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favourable cases and whose denominator is the number of all the cases possible. And: The preceding notion of probability suppose that, in increasing in the same ratio the number of favourable cases and that of all the cases possible, the probability remains the same".

Subsequently, still in keeping with Bernoulli's thinking, Laplace considers the hypothesis that the cases are equally possible and concedes that this is "one of the most delicate points of the theory".

Yet at what point can a statistical frequency be assumed to be an estimate of a probability? It is not enough for "equally possible cases" to be equivalent to category-events; the frequency must lose its erraticity as the size of the sets changes until it converges upon one number. This number, which ranges between 0 and 1, assumes a heuristic value, almost a synthetic *a priori* concept in the Kantian sense.

If we could toss a coin, which is perfect and immutable in its symmetry, thousands of times, we would see the appearance of heads and tails progressively balance each other around the frequency of 0.5: in thousands of tosses, heads would tend to appear in 50% of cases and tails in the other 50%. Is it a miraculous event? No, it is highly predictable event, given the premise of the coin's perfect symmetry. And it is even explained by fundamental theorems of convergence in probability theory.

In real phenomena, unfortunately, the premises are unknown, and science has easily yet successfully learned to reverse the deductive logical process according to which, if the probability of heads is 0.5, I expect the frequency of heads in repeated coin tosses to be approximately 0.5. The scientist, in contrast, says: if the frequency of heads in repeated tosses is 0.5, I expect the probability of heads to be approximately 0.5. It is the famous inversion of De Moivre's theorem, which is known as the "empirical postulate of chance" or the "law of large numbers". It is an inversion that may be admitted, under certain conditions, only statistically (Laplace, 1814; Gini, 1946).

Yet in so doing, science has overcome its challenges and has learned to play dice, winning.

Once again, Laplace writes: "Amid the variable and unknown causes which we comprehend under the name of chance, and which render uncertain and irregular the march of events, we see appearing, in the measure that they multiply, a striking regularity which seems to hold to a design and which has been considered as a proof of Providence. But reflecting upon this we soon recognize that this regularity is only the development of the respective possibilities of simple events which ought to present themselves more often when they are more probable".

The concept of statistical law, of phenomena which can be accurately described by probabilistic models in which it is not the single event but the set of events that is of interest, begins to make headway in science. In other words, it does not interest us whether the next toss will turn out heads, but rather whether over the series of tosses heads will come up 50% of times, ensuring the fairness of the coin theoretically expressible as p(C) = 0.5 and, alternatively, p(T) = 0.5.

What conditions are necessary for a statistical property, a regularity or a frequency distribution observed in a group to be freed from the group dimension to become probabilities, population parameters, scientific laws or theories? The law of large numbers had sought to provide an answer. It is not a mathematical theorem or a scientific discovery; it is the expression of the rational capacity of the human mind to find rules in the repetitiveness of its experiences and its perceptions.

The repetition of experiences and observations in the pursuit of regularity is a common feature of scientific research, whatever the factual or epistemological context in which it is conducted. The distinction between absolute laws and statistical laws which have stimulated such lively philosophical debate over the last few centuries is resolved statistically by distinguishing phenomena characterised by zero variability from phenomena consisting of discrete elementary events in which variability among the single elements cannot be eliminated (Hempel, 1952; Popper, 1934; Scardovi, 1988).

In absolute laws, each individual experience, each unique event represents in itself the category, the law: one single observation would be logically sufficient to represent the law. Replication of the tests only serves to convince the observer of the invariance of the observations. The discovery that a sterilised water molecule is composed of H_2O becomes a categorical, absolute proposition: all pure water molecules are made thus. Each molecule represents all of the others and becomes a scientific category, abstraction.

In statistical laws, on the other hand, replication of experiments is a necessary condition to bring to light the regularities that underlie the variability of individual observations. In order to recognise a statistical regularity, the size of the set of observations must be extended until the underlying law stabilises inertially (Hempel, 1966 Scardovi, 1988).

This size depends, first and foremost, on the variability of the phenomenon. The more variable a phenomenon, the larger the number of observations required for its overall characteristics to stabilise. Mendel (1866) wrote: "..true numerical laws can only result from a large number of individual cases; the larger their number, the more likely it is that chance irregularities will be eliminated." In the same way, Boltzmann describes his swarms of molecules: " an ever greater number must be considered so that from a certain point onwards increasing it further has no effect." (Boltzmann, 1905).

When the phenomena that interest science are "statistical", variability becomes the explanatory key and takes on its own specific semantic meaning. The macrophenomenon understood as the set of many elementary events is statistically regular as it is the result of many irregular microphenomena: as if several asymmetrical coins were tossed simultaneously, each one with a different probability of heads and therefore of tails, or many symmetrical coins which lose their symmetry by clashing with each other.

This means seeking a different perspective where order and regularity emerge from individual disorder.

It was specifically on these models that Mendel (1866) constructed the laws of genetic inheritance of biological features, which are expressed in their simplest, most schematic form of a diallelic gene, from the expansion of Newtons binomial $[p(A)+p(a)]^n$, where the exponent indicates the generations and A, a indicate the dominant and recessive alleles that determine the phenotype, while the numerical coefficients of the binomial define the numerical proportions of the genotypes and the phenotypes.

The theorematic schema at the basis of the representation of the process of inheritance demonstrated experimentally by Mendel, and by those who would follow him and code molecular genetics and population genetics, is the elementary one of the coin toss which is founded on the aggregation of microstates (the possible combinatorial outcomes) into macrostates (the set of combinations which produce one same expected result).

Combinatorial calculation and the variability that results from it become the interpretative key in the most recent scientific research which has found in ancient games of chance the most appropriate language for giving semantic and at the same time formal content to the explanation of its processes.

5. Probability and gambling: challenging chance

When the event is not repeatable from a classificatory perspective and there is in any case no experimental context that can provide the information necessary to reunite the previous two criteria, measurement is purely subjective and its numerical determination finds support in the so-called criterion of gambling. In this latter context, measurement is influenced by the (expert) subject that produces it and therefore may change as the expert changes: the subject is one of the most topical, complex ones in statistical methodology and draws upon Bayesian inductive logic in its assumptions.

In gambling, nevertheless, we unfailingly find the language of games, which still remains the analogic model of reference, almost a return to the origins.

When at the end of the seventeenth century, in a tavern of the Tower of London, Edward Lloyd began "insuring" merchant ships setting sail for the East Indies, the premium and the claim became the tangible estimate of a bet undertaken by the ship's owner and the insurer about the outcome of the voyage based on little information and on an "experiment" that had nothing to learn from combinatorial games and did not even allow events to be framed into classificatory schemes from which frequencies could be drawn. The outcome of the voyage remained a unique, uncertain event as there were too many factors that could determine it. And yet in any case Lloyd became rich, as did the skilled sixteenth-century doctors who from a small number of symptoms were able to diagnose a disease and, where possible, cure it. Even diagnosis became a bet, just as when presented with presumed evidence as opposed to certain proof, the judge's verdict of guilt or acquittal becomes a bet.

A response to the uncertainty of the single event can always and only be given with the value of a bet in which the stake is measured in such a way as to express the degree of confidence that each person, individually, has that the event will occur. Therefore, it is not only the player who gambles; so does the judge, the doctor, the "experts" called upon to give their enlightened opinions. Yet unlike the gambler, who is indifferent to any consistent degree of rationality, or the researcher who makes use of frequency distributions or of distribution models, none of the others have equally solid systems of reference against which to measure their uncertainties.

The distinction between the two categories of probability *probability-1* and *probability-2* nonetheless remains ambiguous, although useful in classificatory terms specifically for dealing with the different categories of uncertainty.

In this process, Pascals Wager is doubly interesting: 1) as it confirms to us that from its very beginnings, probability theory has run on twin tracks: one being the combinatorial system within which the event is placed, the other the degree of confidence that an external party attributes to the occurrence of the event; and 2) as it instructs us in the use of probability theory to estimate the expediency of a bet based on what is at stake (Pascal, 1670).

When nothing is at stake, we cannot speak of a bet. When it is external parties who have to estimate the probability of an uncertain event occurring, their degree of confidence in the proposition (forecast) must stand the test of the stake (which they risk losing) and the prize (which they may win). Everything is trusted to their skill, but at the same time is also trusted to their propensity to risk.

Subjective probability. Endless debates, some of major epistemological interest, have sprung up around subjective probability, which is the expression of the degree of conviction which an individual has that a given event will occur.

The purest proponent of subjectivism was Bruno De Finetti, who dedicated his whole life to championing his idea, which met with little success in advanced scientific research, although it had a major impact on economics and business science, particularly in decision theory. De Finetti's subjectivist position is extreme and rejects any other conception of probability, which it peremptorily consigns to other logical categories.

In his 1931 essay *"Sul significato soggettivo della probabilità"* (On the subjective meaning of probability), De Finetti imagines an individual forced to keep a bank of bets on an event, E, and specifically on a race in which n individuals participate. The bank undertakes to accept the bets setting the odds for each competitor and the events E_1 $E_2 \ldots$ $E_i \ldots$ E_n represent victory for competitor $1, 2, \ldots i \ldots n$ respectively. The bank determines the stake $p_i S$ for anyone wishing to bet on E_i, and if competitor i wins, he receives the sum S; otherwise he loses hhis stake. The quantity pi changes for each competitor according to the degree of confidence that the bank has that E_i will occur. For De Finetti p_i is the "probability" of the event as estimated by the bank and in his interpretation is a subjective probability.

In order to make his theory work, De Finetti had to focus exclusively on the bank and ignore the probability estimates made by the opposite party, who was also a gambler and who can only participate or not participate in the bet. He writes in his essay: "There is an essential difference between the case of one occasional and well defined betting, and the case of an individual who would systematically and endlessly be driven to betting [the bank]" Here the bank acted as a mediator for the variability spread among the population of gamblers for whom it opened its betting shop and became the figurative representation of the nineteenth-century concept of mathematical expectation and fair play, to which De Finetti explicitly refers. In fact, he points out: "When one intends always to refer to the same specific subject [the bank], or when all the individuals one want to consider have the same opinion, the indication of the subject might be only implied and one might simply speak of probability of event E". Finally, he concludes: "In fact, it may appear that in the action of establishing the betting conditions we are rather influenced by love and fear of risk, or by similar totally alien circumstances, more than by that degree of belief which corresponds to the more or less intuitive notion of probability that we have set out to measure. It would evidently be true if it were the case of one single and well defined bet; it is not so any more if we put ourselves in the assumed situation: of an individual who has to keep a betting shop for certain events, accepting at the same conditions any bet in one sense or the other. We will see that he has to follow certain restrictions, which are the theorems of probability theory. Otherwise, he will be at fault with coherence and he will be for sure at loss, provided the antagonist knows how to exploit his errors. We will call coherent an individual who does not make such a mistake, that is one who values probabilities so as not to put the competitors in a situation where they can win with certainty. The theory of probability is then nothing but the mathematical theory teaching one to be coherent". (De Finetti, 1931).

In this long quotation there is very little that is subjectivist. De Finetti's subjective probability is nothing other than an expected value, a mathematical expectation: statistically we would say that it is an arithmetic mean. Yet the arithmetic mean is a theoretical model which does not conform to observed reality or even to perceptions: it is the model of invariance, the one that makes us all equal, as if we were not different. What is subjective about a theoretical "cage" in which each gamblers, to be coherent, must have his opinion converge upon the average of the opinions? There is no answer: subjective does not mean arbitrary. Each of our "subjective" opinions is always formed through the coherent and not contradictory synthesis of our experience and knowledge.

De Finetti's "coherent gambler" is a nineteenth-century man: he is not so different from Walras' "economic man" (1883) or Quetelet's "average man" (1837), and simply exists in the imagination of one who, in keeping with the times, was afraid to consider variability, reassured by the regularities that an average could offer.

The whole of the nineteenth century sought to lead the social and human sciences to the precepts of a deterministic description that aimed to cancel out the variability of phenomena, which was interpreted as a disturbance factor rather than a symptom of evolution and innovation. Adolphe Quetelet (1794-1874) offered his research into people's anthropometric characteristics and social behaviour, arriving at a description of what would subsequently be termed Quetelet's average man.

Quetelet based his arguments exclusively on the theory of instrumental errors in measurement and on the interpretation of the variability modelled by the Gaussian function, which contained a logical fallacy: it attributed to observed phenomena the same causal premised which in completely different circumstances had determined that same curve. Yet the theory of the average man met with great success, and only the scientific achievements of Mendelian genetics enabled the variability of anthropometric characteristics to be reinterpreted according to a combinatorial, probabilistic logic, which once again found its theoretical reference model in the Gaussian function, thus returning to the games of chance and to the toss of a coin described by Newton's binomial, towards which as is well known the Gaussian converges. (Scardovi, 1978).

Not so different is the deterministic principle that inspired "economic man", ascribed to Walras (1883) as an extremely concise summary of his economic theory of general equilibrium.

The concept of "economic man" represented a synthesis of the so-called rational behaviours which unequivocally determine the decisions of a virtual subject who is obliged to choose his actions in a system strictly governed by rigorously deterministic economic models. He is a man without imagination from who any free will has been removed. The systematic refutations of the concept on the part of the markets have diminished the fascination of this little man, the perfect hedonist, who sixty years later was replaced by the so-called "decision maker" (Wald, 1942-47), rational, modern and with one great virtue: he no longer reasons in terms of certainties but probabilities and expediencies. The rational decision maker recognises in individual variability the basic elements of his decision-making process, yet translates them into risk propensity, profit and loss functions, and combines this kind of personal equation of the decision maker (De Finett's bank) with a subjective measurement of the uncertainty of events in order to arrive at the optimal decision, that is to say coherent in terms of the average of expectations.

Moving beyond these considerations of an epistemological nature concerning the rational gambler, subjective probability *á la* Savage and *á la* De Finetti has been extensively applied in many areas of economics, medicine, quality control, business management and so on. And an extensive body of literature concerning the methodology of statistics and probability theory has grown up around numerical estimates of probability made by "experts". In the same way, mathematicians and economists developed the Theory of games (Neumann, Morgenstern,1944) which is one of the most interesting outcome of modern economic science.

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Summary

The article sets out a number of considerations on the distinction between variability and uncertainty over the centuries. Games of chance have always been useful random experiments which through combinatorial calculation have opened the way to probability theory and to the interpretation of modern science through statistical laws. The article also looks briefly at the stormy nineteenth-century debate concerning the definitions of probability which went over the same grounds sometimes without any historical awareness as the debate which arose at the very beginnings of probability theory, when the great probability theorists were open to every possible meaning of the term.

Keywords: History of probability; Definitions of Probability; P. S. Laplace; B. De Finetti.