STATISTICA, anno LXXV, n. 3, 2015

ON GENERALIZED UPPER(K)RECORD VALUES FROM WEIBULL DISTRIBUTION

Jerin Paul [1]
Department of Statistics, University of Kerala, Kariavattom, Trivandrum, India
Poruthiyudian Yageen Thomas
Department of Statistics, University of Kerala, Kariavattom, Trivandrum, India

1. Introduction

Record values and associated statistics are of great importance in several real life problems involving weather, economic studies, sports and so on. Chandler (1952) defined the record statistics as successive extremes occurring in a sequence of independent and identically distributed (*iid*) random variables. The prediction of a future record value is an interesting problem with many real life applications. For example the predicted value of the amount of next record level of water that a dam will capture from rain and hold or discharge is helpful for future planning purposes, predicted intensity of the next strongest earthquake is essential for disaster management planning, prediction of next level of new record in athletic events is helpful for subjecting the prospective athletes to rigorous training and practice and so on.

Usually in many events associated with athletics, temperature, wind velocity etc., only record breaking observations observed till then are made available to future reference. In such situations one is compelled to rely upon the available record data only to deal with inference problems of the parent distribution. A difficulty that one encounters in dealing with statistical inference problems based on record values is about their limited occurrence, as the expected values of interarrival times of records is infinite Glick (1978). However one may observe that generally the *kth* record values as introduced by Dziubdziela and Kopocinski (1976) occur more frequently than those of the classical records. Further the sequence of *kth* record values for $k > 1$ is free from the inclusion of outliers occurring in the data. Suppose $\{X_n\}$ is a sequence of *iid* random variables. Then for a positive integer $k \geq 1$, the sequence of *kth* upper record times $\{T_{U(n,k)}, n \geq 1\}$ is defined as Nevzorov (see, 2001,p. 82):

$$T_{U(1,k)} = k$$

and for n≥ 1,

$$T_{U(n+1,k)} = min\{j : j > T_{U(n,k)}, X_j > X_{T_{U(n,k)}-k+1:T_{U(n,k)}}\},$$

[1] Corresponding Author e-mail: jerinstat@gmail.com

where $X_{i:m}$ denotes the i-th order statistic in a sample of size m. Now if we write

$$X_{U(n,k)} = X_{T_{U(n,k)}-k+1:T_{U(n,k)}}, \text{ for n} = 1, 2, \ldots$$

then $\{X_{U(n,k)}\}$ is known as the sequence of the *kth* upper record values. In an analogous way, one may define the *kth* lower record times and *kth* lower record values. The *kth* member of the sequence of the classical record values is also called as *kth* record value. This contradicts with the *kth* record values as defined in Dziubdziela and Kopocinski (1976). Pointing out this conflict in the usage of *kth* record values of Dziubdziela and Kopocinski (1976), and as it generates the classical record values for $k = 1$, Minimol and Thomas (2013, 2014) and Paul and Thomas (2013) have called the *kth* record values as defined in Dziubdziela and Kopocinski (1976) as the Generalized(k)record values. Agreeing with the contention of Minimol and Thomas (2013, 2014) and Paul and Thomas (2013), we also call the *kth* record values of Dziubdziela and Kopocinski (1976) as generalized(k)record values all through this paper.

It is to be noted that, a lot of research is going on to detect outliers in a data so as to delete them for devising more reasonable statistical methods to the problem of interest. The integer parameter k involved in generalized upper(k)record value (GURV) can be chosen in such a manner that the record data generated discard away the specified number of outliers which are feared to be crept into the data. For example if some initial scrutiny of the data revels that there is a possibility of occurrence of only one outlier in terms of its largeness in the data and the if general interest is with upper record values, then it is enough to consider generalized upper(2)record values as the desirable record data that may be used for further analysis and storage of it for future purposes. Inventing more and more characterization results based on the distributional properties of the statistics arising from a distribution makes the model mathematically tractable for developing statistical methods to analyse the data arising from it. Arnold *et al.* (1998, pp. 43-44) have stated that all distributional properties of k-records (i.e. GURV's) arising from the cdf $F(x)$ can be studied from those of classical records arising from $F_{1:k}(x) = 1-(1-F(x))^k$. This statement cannot be taken unilaterally for every case. For example for the classical upper record values if it is proved that the statistics $g(X_{U(m)}, X_{U(n)})$ and $h(X_{U(n)})$ are independently distributed iff the parent cdf is $F(x)$ (see Dallas, 1982 for classical records from Weibull distribution (WD)) and again if it is proved that $g(X_{U(m,k)}, X_{U(n,k)})$ and $h(X_{U(n,k)})$ are independently distributed iff the parent cdf is $F(x)$ (see theorem 1 of this paper for GURV's from WD) then clearly for $k = 1$, the latter result implies the former. However if $X'_{U(m)}$ and $X'_{U(n)}$ are the classical records from $F_{1:k}(x)$, then one cannot claim that $g(X'_{U(m)}, X'_{U(n)})$ and $h(X'_{U(n)})$ are independently distributed as we know that it is impossible to have the same characterization result to $F(x)$ and $1-(1-F(x))^k$ simultaneously. Hence at least for generating characterization results in the general sense, one has to carry out the investigation based on generalized(k)record values rather than dealing with classical record values arising from $F_{1:k}(x)$. Again it is found that generalized(k)record values helps in estimating the location parameter of the parent distribution with more precision when compared with the estimate based on classical record values (For example see Remark 9 of this paper). The above considerations and the advantage observed in eliminating the outliers effect in the generalized(k)record value data the authors are motivated to make a study on GURV's arising from WD.

Several applications of GURV's can be found in the literature. For some recent works on characterization of parent distributions using recurrence relations on moments of generalized record values see, Minimol and Thomas (2013, 2014) and Pawlas and Szynal (1998, 1999) . Extensive applications of classical record values in inference problems are seen in the available liturature and for details one may refer, Arnold *et al.* (1998) and Gulati and Padgett (2003). However such works are not seen established for GURV's and hence detailed study on this problem also becomes much relevent.

Suppose $\{X_i, i \geq 1\}$ is a sequence of (*iid*) random variables with absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then the pdf of $X_{U(n,k)}$ and the joint pdf $f_{X_{U(m,k)},X_{U(n,k)}}(x,y)$ of $X_{U(m,k)}$ and $X_{U(n,k)}$ are given by (Arnold *et al.*, 1998)

$$f_{X_{U(n,k)}}(x) = \frac{k^n}{\Gamma(n)} \left[-\ln\{1 - F(x)\}\right]^{n-1} \left[1 - F(x)\right]^{k-1} f(x), -\infty < x < \infty. \tag{1}$$

and

$$f_{X_{U(m,k)},X_{U(n,k)}}(x,y) = k^n \frac{\left\{-\ln[\bar{F}(x)]\right\}^{m-1}}{\Gamma(m)} \frac{\left\{\ln[\bar{F}(x)] - \ln[\bar{F}(y)]\right\}^{n-m-1}}{\Gamma(n-m)} \frac{f(x)f(y)}{\bar{F}(x)} [\bar{F}(y)]^{k-1}, \tag{2}$$

for $x < y$, $1 \leq m < n; n \geq 2$, where $\bar{F}(x) = 1 - F(x)$.

A random variable Z is said to have a WD (see Weibull, 1951) if its probability density function is given by

$$g(z) = \frac{c}{\sigma}\left(\frac{z-\theta}{\sigma}\right)^{c-1} \exp\left\{-\left(\frac{z-\theta}{\sigma}\right)^c\right\},\ z > \theta,\ \sigma > 0,\ -\infty < \theta < \infty,\ c > 0, \tag{3}$$

where θ, σ and c are known as location, scale and shape parameters respectively. The cdf corresponding to the pdf given in (3) is

$$G(z) = 1 - \exp\left\{-\left(\frac{z-\theta}{\sigma}\right)^c\right\},\ z > \theta, \sigma > 0, -\infty < \theta < \infty,\ c > 0. \tag{4}$$

Also the pdf of the standard form of WD ($\theta = 0$ and $\sigma = 1$) is given by

$$f_0(y) = cy^{c-1} \exp\{-y^c\},\ y > 0, c > 0, \tag{5}$$

with corresponding cdf given by

$$F_0(y) = 1 - \exp\{-y^c\},\ y > 0,\ c > 0. \tag{6}$$

The WD has found applications in modelling and analysis wind speed data, rainfall data, flood data and so on. Also this model has been seen utilized in many health science problems, microscopic degradation studies, meteorological analysis and so on (for details see Johnson and Balakrishnan, 1994; Rinne, 2008).

Dallas (1982) has discussed some distributional and characterization results on classical record values arising from WD. Estimation of parameters of WD using classical record values are delt with in Balakrishnan and Chan (1993).

In this work we make a study on some interesting properties of GURV arising from WD and the application of those record values to deal with some inference problems of this distribution. In section 2 we derive the exact expressions for the means, variances and covariances of GURV's arising from WD. Further we have identified certain properties of GURV's which characterize the WD and those characterization theorems are given in section 3. In section 4 we derive some further interesting distributional aspects of statistics defined out of GURV arising from WD. Section 5 deals with the problem of estimation of the location and scale parameters of WD when the shape parameter involved in it is known. In this section we discuss the simultaneous estimation of all parameters of WD as well. In section 6 we consider the problem of prediction of next immediate future GURV that may occur from WD using Best Linear Unbiased Predictor (BLUP) based on initial GURV's arising from a WD. In the last section we have illustrated the estimation of the parameters and prediction of the next GURV based on the initial GURV's arising from WD by using a real life data set.

2. GURV Arising from Standard WD

Let $\{Y_{U(i,k)}\}$ be the sequence of GURV arising from standard WD defined in (5). Then by (1), the pdf of $Y_{U(n,k)}$ is given by

$$f_{Y_{U(n,k)}}(x) = \frac{k^n c}{\Gamma(n)} y^{cn-1} e^{-ky^c},\ y > 0,\ n = 1, 2, \ldots,\ k = 1, 2, \ldots,\ c > 0. \tag{7}$$

The joint pdf of $Y_{U(m,k)}$ and $Y_{U(n,k)}$ for $m < n$ based on (2) is given by

$$f_{Y_{U(m,k)},Y_{U(n,k)}}(x, y) = \frac{c^2 k^n}{\Gamma(m)\Gamma(n-m)} (y^c - x^c)^{n-m-1}\, x^{cm-1} y^{cn-1} e^{-ky^c}, \tag{8}$$

$0 < x < y < \infty$. Using (7) and (8), we have derived the exact expressions for the means, variances and covariances of GURV's arising from the standard WD and are given below:

$$E(Y^i_{U(n,k)}) = \frac{\Gamma(n + \frac{i}{c})}{k^{\frac{i}{c}}\Gamma(n)}, \tag{9}$$

$$Var(Y_{U(n,k)}) = \frac{1}{k^{\frac{2}{c}}\Gamma(n)} \left\{ \Gamma\left(n + \frac{2}{c}\right) - \frac{\left(\Gamma(n + \frac{1}{c})\right)^2}{\Gamma(n)} \right\}, \tag{10}$$

and

$$Cov(Y_{U(m,k)}, Y_{U(n,k)}) = \frac{\Gamma(m + \frac{1}{c})}{k^{\frac{2}{c}}\Gamma(m)} \left\{ \frac{\Gamma(n + \frac{2}{c})}{\Gamma(n + \frac{1}{c})} - \frac{\Gamma(n + \frac{1}{c})}{\Gamma(n)} \right\}. \tag{11}$$

3. Characterization of WD Using GURV

In this section we deal with some properties of GURV's which characterize a sub-family $\mathscr{F}$ of distributions belonging to the general family of WD defined by the pdf (3). We say that a random variable X has a distribution belonging to $\mathscr{F}$ if its pdf is given by

$$f(x) = \frac{c}{\sigma}\left(\frac{x}{\sigma}\right)^{c-1} e^{-(\frac{x}{\sigma})^c}, \quad x > 0, \sigma > 0. \tag{12}$$

The cdf corresponding to (12) is then defined by

$$F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^c}, \quad x > 0, \sigma > 0. \tag{13}$$

THEOREM 1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables each distriuted identically with a common continuous cdf $F(x)$ and pdf $f(x)$. Let $\bar{F}(0) = 1$ and $F(x) < 1$ for all $x > 0$. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then $F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^c}$ for all $x > 0$, $\sigma > 0$ and $c > 0$, if and only if $\frac{X_{U(m,k)}}{X_{U(n,k)}}$ and $X_{U(n,k)}$, (for, $m < n$) are independently distributed.*

PROOF. The joint probability density function of $X_{U(m,k)}$ and $X_{U(n,k)}$ for $1 \leq m < n$ is given by

$$f_{X_{U(m,k)},X_{U(n,k)}}(x,y) = k^n \frac{\left\{-\ln[\bar{F}(x)]\right\}^{m-1}}{\Gamma(m)} \frac{\left\{\ln[\bar{F}(x)] - \ln[\bar{F}(y)]\right\}^{n-m-1}}{\Gamma(n-m)} \frac{f(x)f(y)}{\bar{F}(x)} [\bar{F}(y)]^{k-1}, \tag{14}$$

where $0 \leq x < y < \infty$. Also the joint pdf can be written in the form

$$f_{X_{U(m,k)},X_{U(n,k)}}(x,y) = k^n \frac{\{R(x)\}^{m-1}}{\Gamma(m)} \frac{\{R(y) - R(x)\}^{n-m-1}}{\Gamma(n-m)} r(x) r(y) e^{-k\{R(y)\}}, \tag{15}$$

where $R(x) = -\ln(1 - F(x))$ and $r(x) = \frac{\mathrm{d}}{\mathrm{dx}}(R(x))$. If $F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^c}$ for all $x > 0$, $\sigma > 0$ and $c > 0$, then the joint pdf $f_{X_{U(m,k)},X_{U(n,k)}}(x,y)$ of $X_{U(m,k)}$ and $X_{U(n,k)}$ is

$$f_{X_{U(m,k)},X_{U(n,k)}}(x,y) = \frac{c^2 k^n}{\sigma^2 \Gamma(m)\Gamma(n-m)} \left(\frac{x}{\sigma}\right)^{cm-1} \left\{\left(\frac{y}{\sigma}\right)^c - \left(\frac{x}{\sigma}\right)^c\right\}^{n-m-1} \left(\frac{y}{\sigma}\right)^{c-1} e^{-k\left(\frac{y}{\sigma}\right)^c}, \tag{16}$$

where $1 \leq m < n$ and $c > 0$. If we make the transformation $T = \frac{X_{U(m,k)}}{X_{U(n,k)}}$ $Z = X_{U(n,k)}$, then the Jacobian of the transformation is $|J| = z$. Thus we can write the joint pdf $f_{T,Z}(t,z)$ of T and Z as

$$f_{T,Z}(t,z) = \frac{c^2 k^n}{\sigma \Gamma(m)\Gamma(n-m)} t^{cm-1} \{1 - t^c\}^{n-m-1} \left(\frac{z}{\sigma}\right)^{cn-1} e^{-k\left(\frac{z}{\sigma}\right)^c}, \tag{17}$$

where $0 < t < 1$, $z > 0$, $\sigma > 0$ and $c > 0$. The marginal pdf of T is given by

$$\begin{aligned} f_T(t) &= \frac{c^2 k^n}{\sigma \Gamma(m)\Gamma(n-m)} t^{cm-1} \{1 - t^c\}^{n-m-1} \int_0^\infty \left(\frac{z}{\sigma}\right)^{cn-1} e^{-k\left(\frac{z}{\sigma}\right)^c} \mathrm{d}z \\ &= \frac{c\Gamma(n)}{\Gamma(m)\Gamma(n-m)} t^{cm-1} \{1 - t^c\}^{n-m-1} \text{ for } 0 < t < 1, \ c > 0. \end{aligned} \tag{18}$$

Also, the pdf $f_Z(z)$ of Z is given by

$$f_Z(z) = \frac{ck^n}{\sigma \Gamma(n)} \left(\frac{z}{\sigma}\right)^{nc-1} e^{-k\left(\frac{z}{\sigma}\right)^c}. \tag{19}$$

From (17), (18) and (19), we obtain $f_{T,Z}(t,z) = f_T(t) f_Z(z)$. Hence T and Z are independently distributed.

Now we prove the sufficient part of the theorem. Let $T = \frac{X_{U(m,k)}}{X_{U(n,k)}}$ and $Z = X_{U(n,k)}$ be distributed independently. Then the Jacobian of the transformation is $|J| = z$. Now using (15) we can write the joint pdf $f_{T,Z}(t,z)$ of T and Z as

$$f_{T,Z}(t,z) = \frac{k^n}{\Gamma(m)\Gamma(n-m)}\left[R(zt)\right]^{m-1}\left[R(z) - R(zt)\right]^{n-m-1}\left[\bar{F}(z)\right]^k r(zt)\, r(z) z. \tag{20}$$

The pdf $f_Z(z)$ of Z is given by

$$f_Z(z) = \frac{k^n}{\Gamma(n)}\left[R(z)\right]^{n-1}\left[\bar{F}(z)\right]^k r(z). \tag{21}$$

Since T and Z are independent, we get the pdf $f_T(t)$ of T from (20) and (21) as

$$f_T(t) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)}\left[\frac{R(zt)}{R(z)}\right]^{m-1}\left[1 - \frac{R(zt)}{R(z)}\right]^{n-m-1}\frac{r(zt)}{R(z)}z.$$

The distribution function of T is then given by

$$F_T(t) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)}\int_0^t\left[\frac{R(zv)}{R(z)}\right]^{m-1}\left[1 - \frac{R(zv)}{R(z)}\right]^{n-m-1}\frac{\partial}{\partial v}\left[\frac{R(zv)}{R(z)}\right]\mathrm{d}v.$$

On putting $\frac{R(zv)}{R(z)} = u$, we obtain

$$F_T(t) = \frac{\Gamma(n)}{\Gamma(m)\Gamma(n-m)}\int_0^{\frac{R(zt)}{R(z)}} u^{m-1}(1-u)^{n-m-1}\,\mathrm{d}u.$$

The right side of above equation can be written as a Binomial sum and hence we have

$$F_T(t) = \sum_{i=m}^{n-1}\binom{n-1}{i}\left[\frac{R(zt)}{R(z)}\right]^i\left[1 - \frac{R(zt)}{R(z)}\right]^{n-i-1}. \tag{22}$$

The above result is true for every positive integer m such that $m < n$. On putting $m = n-1$, we have

$$F_T^*(t) \quad = \quad \left[\frac{R(zt)}{R(z)}\right]^{n-1},$$

where $F_T^*(t)$ is the distribution function of $\frac{X_{U(n-1,k)}}{X_{U(n,k)}}$. That is

$$\left[R(zt)\right]^{n-1} \quad = \quad F_T^*(t)\left[R(z)\right]^{n-1}, \tag{23}$$

Since $0 < z < \infty$, on putting $z = 1$, in the above equation we get

$$F_T^*(t) = \left[\lambda R(t)\right]^{n-1},$$

where $\lambda = \left[-\ln(1 - F(1))\right]^{-1}$ is a positive constant. Using this in the equation (23) we get

$$\left[R(zt)\right]^{n-1} = \left[\lambda R(t)\right]^{n-1}\left[R(z)\right]^{n-1} \text{ for } 0 < t < 1. \tag{24}$$

Observe that the above equation makes the right side of $F_T(t)$ given in (22) as well free of z for all $m < n$. Hence on multiplying both sides of the above equation by λ^{n-1} we get

$$\begin{aligned}[\lambda R\,(zt)]^{n-1} &= [\lambda R\,(t)]^{n-1}\,[\lambda R(z)]^{n-1}\\ \lambda R\,(zt) &= \lambda R\,(t)\,\lambda R(z).\end{aligned}$$

Subtituting $\lambda R\,(.)$ by $R_1\,(.)$, we get

$$R_1\,(zt) = R_1\,(t)\,R_1(z). \tag{25}$$

Clearly (25) is a well known Cauchy functional and hence from Aczel (1966), we observe that the only non-constant continuous solution of (25) is $R_1(x) = x^c$, for all $x > 0$ where c is a constant. Thus we have $\bar{F}(x) = e^{\frac{-x^c}{\lambda}}$. As $\bar{F}(x)$ is a survival function c cannot be negative . Hence we write $F(x) = 1 - e^{\frac{-x^c}{\lambda}}$, where $\lambda = \sigma^c$, $\sigma > 0$, $c > 0$. This completes the proof.

Corollary 2. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables each distriuted identically with a common continuous cdf $F(x)$ and pdf $f(x)$. Let $\bar{F}(0) = 1$ and $F(x) < 1$ for all $x > 0$. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then $F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^c}$ for all $x > 0$, $\sigma > 0$ and $c > 0$, if and only if $\frac{X_{U(m,k)}+X_{U(n,k)}}{X_{U(n,k)}}$ and $X_{U(n,k)}$, (for $m < n$) are independently distributed.*

Corollary 3. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables each distriuted identically with a common continuous cdf $F(x)$ and pdf $f(x)$. Let $\bar{F}(0) = 1$ and $F(x) < 1$ for all $x > 0$. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then $F(x) = 1 - e^{-\left(\frac{x}{\sigma}\right)^c}$ for all $x > 0$, $\sigma > 0$ and $c > 0$, if and only if $\frac{X_{U(m,k)}}{X_{U(m,k)}+X_{U(n,k)}}$ and $X_{U(n,k)}$, (for $m < n$) are independently distributed.*

The proof of the above corollaries are omitted as they are just similar to the proof of theorem 1.

Note 1. *If we have several data sets from a population, then for any fixed n and m, we may plot the points (T, Z) (where Tand Z are variables as defined in theorem 1) corresponding to each data set in a graph sheet. If the points in the graph corresponding to (T, Z) are seen scattered all over the region without any pattern, then as an implication of theorem 1, one may conclude that the WD as defined in* (12) *could be taken as a suitable distribution of the population random variable. Similar attempts may be made with the results in each of the Corollaries 2 to 3 to check the suitability of WD as a model to the population distribution.*

Though the result proved in theorem 1 for classical record values has been proved by Dallas (1982), there is importance to our results since the knowledge of the results of theorem 1 helps the practicing statisticians in applying the results in modelling problems when there is fear of possibility of contamination in the data by outliers with very large values. It is to be of interest to note that such outliers themselves become classical record values where as GURV's has the mechanism to eliminate those outliers. Further when outliers crept into the data it restrict the occurrence of the number of classical records and lead one to erroneous inferences where as we have scope to observe more number of GURV's and device inference procedures free from outliers in the original data.

4. Some Distributional Aspects of GURV Arising from Standard WD

In this section we derive some interesting properties of GURV's arising from standard WD. These results are also useful for dealing with some inferential aspects of the WD.

Proposition 4. *Suppose m and n are positive integers such that* $1 \leq m < n$ *and* $Y_{U(m,k)}$ *and* $Y_{U(n,k)}$ *are mth and nth GURV's arising from standard WD defined in (5). Then*

$$E\left\{\frac{Y_{U(m,k)}}{Y_{U(n,k)}}\right\} = \frac{E\left(Y_{U(m,k)}\right)}{E\left(Y_{U(n,k)}\right)}. \tag{26}$$

Proof. We have

$$E(Y_{U(m,k)}) \quad = \quad E\left(\left\{\frac{Y_{U(m,k)}}{Y_{U(n,k)}}\right\} Y_{U(n,k)}\right).$$

From the independence of $\frac{Y_{U(m,k)}}{Y_{U(n,k)}}$ and $Y_{U(n,k)}$ (see theorem 1) we write

$$E(Y_{U(m,k)}) \quad = \quad E\left\{\frac{Y_{U(m,k)}}{Y_{U(n,k)}}\right\} E\left(Y_{U(n,k)}\right).$$

The required result then easily follows.

As a consequence of the above proposition we then have

$$E\left\{\frac{Y_{U(m,k)}}{Y_{U(n,k)}}\right\} = \frac{\Gamma(n)\Gamma(m+\frac{1}{c})}{\Gamma(m)\Gamma(n+\frac{1}{c})}. \tag{27}$$

The folowing theorem describes a generalized independence property of a colleciton of statistics defined from GURV's arising from standard WD. To derive the results we require some basic results such as joint distribution of a collection of selected GURV's arising from an arbitrary distribution whose support set is restricted to the set of positive reals say $\mathbb{R}^+$. This result can be obtained by integrating out unnecessary terms from the joint distribution of GURV's and we state this result as a lemma without proof.

Lemma 5. *Let* $\{X_j,\ j \geq 1\}$ *be a sequence of iid random variables with pdf* $f(x)$ *and cdf* $F(x)$ *with support set* $\mathbb{R}^+$. *Let* $\{X_{U(n,k)}\}$ *be the sequence of GURV's generated from the sequence* $\{X_j, j \geq 1\}$. *Let* $r_1, r_2, \ldots, r_i$ *be positive integers such that* $1 \leq r_1 < r_2 < \ldots < r_i$ *and let* $X_{U(r_1,k)}, X_{U(r_2,k)}, \ldots, X_{U(r_i,k)}$ *be i selected GURV's from the sequence* $\{X_{U(n,k)}, n \geq 1\}$ *of GURV's of* $\{X_j, j \geq 1\}$. *Then the joint pdf of these selected GURV's is given by*

$$\begin{aligned} f_{X_{U(r_1,k)},\ldots,X_{U(r_i,k)}}(x_{r_1},\ldots,x_{r_i}) \quad &= \quad k^{r_i}\frac{f(x_{r_1})}{\bar{F}(x_{r_1})}\frac{\left[-\ln\bar{F}(x_{r_1})\right]^{r_1-1}}{\Gamma(r_1)}\prod_{h=2}^{i}\frac{f\left(x_{r_h}\right)}{\bar{F}\left(x_{r_h}\right)} \\ &\times \quad \frac{\left\{\left[-\ln\bar{F}\left(x_{r_h}\right)\right]-\left[-\ln\bar{F}\left(x_{r_{h-1}}\right)\right]\right\}^{r_h-r_{h-1}-1}}{\Gamma(r_h-r_{h-1})}\left[\bar{F}\left(x_{r_i}\right)\right]^k. \end{aligned} \tag{28}$$

Theorem 6. *Let* $r_1,\ r_2,\ \ldots,\ r_i$ *be positive integers such that* $1 \leq r_1 < r_2 < \ldots < r_i$ *and let* $Y_{U(r_1,k)},\ Y_{U(r_2,k)},\ \ldots,\ Y_{U(r_i,k)}$ *be the corresponding GURV's arising from (5). Then the random variables* $V_1 = \frac{Y_{U(r_1,k)}}{Y_{U(r_2,k)}},\ V_2 = \frac{Y_{U(r_2,k)}}{Y_{U(r_3,k)}},\ \ldots,\ V_{i-1} = \frac{Y_{U(r_{i-1},k)}}{Y_{U(r_i,k)}}$ *and* $V_i = Y_{U(r_i,k)}$, *are all statistically independent with the distribution of V_j is given by the pdf* $g_j(v_j) = \frac{c v_j^{c r_j - 1}(1-v_j^c)^{r_{j+1}-r_j-1}}{B(r_j, r_{j+1}-r_j)}$, $0 < v_j < 1$, $j = 1, 2, \ldots, i-1$ *and that of V_i is given by* $h(v_i) = \frac{c k^{r_i} e^{-k v_i^c} v_i^{c r_i - 1}}{\Gamma(r_i)}$, $0 < v_i < \infty$.

Proof. The joint distribution of $Y_{U(r_1,k)}, Y_{U(r_2,k)}, \ldots, Y_{U(r_i,k)}$ can be derived using (28) and is given by

$$\begin{aligned} f_{r_1,r_2,\ldots,r_i}(x_{r_1}, x_{r_2}, \ldots, x_{r_i}) &= \frac{k^{r_i} c^i}{\Gamma(r_1)\Gamma(r_2 - r_1)\ldots\Gamma(r_i - r_{i-1})} \\ &\times (x_{r_1}^c)^{r_1-1}\left(x_{r_2}^c - x_{r_1}^c\right)^{r_2-r_1-1}\ldots\left(x_{r_i}^c - x_{r_{i-1}}^c\right)^{r_i-r_{i-1}-1} \\ &\times (x_{r_1} x_{r_2}\ldots x_{r_i})^{c-1} e^{-k x_{r_i}^c},\ 0 < x_{r_1} < \ldots < x_{r_i} < \infty. \end{aligned} \tag{29}$$

If we put $v_1 = \frac{x_{r_1}}{x_{r_2}}, v_2 = \frac{x_{r_2}}{x_{r_3}}, \ldots, v_{i-1} = \frac{x_{r_{i-1}}}{x_{r_i}}$ and $v_i = x_{r_i}$ then we have

$$x_{r_i} = v_i, x_{r_{i-1}} = v_{i-1}v_i, \ldots, x_{r_2} = v_2 v_3 \ldots v_i, x_{r_1} = v_1 v_2 \ldots v_i. \tag{30}$$

The Jacobian of the transformation is given by

$$|J| = v_2 v_3^2 \ldots v_{i-1}^{i-2} v_i^{i-1}. \tag{31}$$

Using (30) in (29), multiplying it by $|J|$ and simplifying we get the required result.

Corollary 7. *Suppose $Y_{U(1,k)}, Y_{U(2,k)}, \ldots, Y_{U(n,k)}$ are the first n GURV's arising from standard WD. Then the random variables* $V_1 = \frac{Y_{U(1,k)}}{Y_{U(2,k)}}, V_2 = \frac{Y_{U(2,k)}}{Y_{U(3,k)}}, \ldots, V_{n-1} = \frac{Y_{U(n-1,k)}}{Y_{U(n,k)}}$ *and* $V_n = Y_{U(n,k)}$, *are all statistically independent with the pdf of V_j is given by* $g_j(v_j) = \frac{c v_j^{cj-1}}{B(j,1)}$, $0 < v_j < 1$, $j = 1, 2, \ldots, n-1$ *and that of V_n is given by* $h(v_n) = \frac{c k^n e^{-k v_n^c} v_n^{cn-1}}{\Gamma(n)}$, $0 < v_n < \infty$.

Proof. By putting $i = n$ and $r_1 = 1$, $r_2 = 2$, $\ldots$, $r_n = n$ in theorem 6, we get the required result.

The importance of theorem 6 is that it helps one to obtain the product moments of any order of GURV's arising from standard WD in a very simple manner. For example if p, q, r, s are positive integers such that $1 \le p < q < r < s$, then for the GURV's $Y_{U(p,k)}$, $Y_{U(q,k)}$, $Y_{U(r,k)}$, $Y_{U(s,k)}$ and positive integers a, b, d, e we have

$$E\left[Y_{U(p,k)}^a Y_{U(q,k)}^b Y_{U(r,k)}^d Y_{U(s,k)}^e\right] = E\left[\left\{\frac{Y_{U(p,k)}}{Y_{U(q,k)}}\right\}^a \left\{\frac{Y_{U(q,k)}}{Y_{U(r,k)}}\right\}^{a+b} \left\{\frac{Y_{U(r,k)}}{Y_{U(s,k)}}\right\}^{a+b+d} \{Y_{U(s,k)}\}^{a+b+d+e}\right].$$

If we put $V_1 = \frac{Y_{U(p,k)}}{Y_{U(q,k)}}$, $V_2 = \frac{Y_{U(q,k)}}{Y_{U(r,k)}}$, $V_3 = \frac{Y_{U(r,k)}}{Y_{U(s,k)}}$ and $V_4 = Y_{U(s,k)}$, then from theorem 6, we have

$$\begin{aligned} E\left[Y_{U(p,k)}^a Y_{U(q,k)}^b Y_{U(r,k)}^d Y_{U(s,k)}^e\right] &= E\left[\{V_1\}^a \{V_2\}^{a+b} \{V_3\}^{a+b+d} \{V_4\}^{a+b+d+e}\right] \\ &= E\left[\{V_1\}^a\right] E\left[\{V_2\}^{a+b}\right] E\left[\{V_3\}^{a+b+d}\right] E\left[\{V_4\}^{a+b+d+e}\right] \\ &= \frac{B(p + \frac{a}{c}, q-p)}{B(p, q-p)} \times \frac{B(q + \frac{a}{c} + \frac{b}{c}, r-q)}{B(q, r-q)} \\ &\times \frac{B(r + \frac{a}{c} + \frac{b}{c} + \frac{d}{c}, s-r)}{B(r, s-r)} \times \frac{\Gamma(s + \frac{a}{c} + \frac{b}{c} + \frac{d}{c} + \frac{e}{c})}{\Gamma(s)\, k^{\frac{a+b+d+e}{c}}} \\ &= \frac{\Gamma(p + \frac{a}{c})\Gamma(q + \frac{a+b}{c})\Gamma(r + \frac{a+b+d}{c})\Gamma(s + \frac{a+b+d+e}{c})}{\Gamma(p)\Gamma(q + \frac{a}{c})\Gamma(r + \frac{a+b}{c})\Gamma(s + \frac{a+b+d}{c})\, k^{\frac{a+b+d+e}{c}}}. \end{aligned} \tag{32}$$

5. Estimation of the Parameters of WD

In this section we consider the situation where the only experimental details available are the GURV's observed corresponding to sample of observations realized from WD. Hence the estimation methods proposed in the sub sections 5.1 and 5.2 utilize only the available GURV's.

5.1. Estimation of the Parameters of WD when the Shape parameter c is known

In this section we describe the problem of estimation of location parameter θ and scale parameter σ of WD defined by (3) based on the available GURV's provided the shape parameter c is known, by least-squares method. Let $X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}$ be the first n GURV's arising from WD defined in (3) with known c. Then clearly $Y_{U(i,k)}=\frac{X_{U(i,k)}-\theta}{\sigma}, i = 1, 2, \ldots, n$ are distributed as the first n GURV's arising from the standard WD defined in (5). From (9) and (11) we write $\alpha_i = E(Y_{U(i,k)}) = \frac{\Gamma(i+\frac{1}{c})}{k^{\frac{1}{c}}\Gamma(i)}$ and $\sigma_{i,j} = Cov\left(Y_{U(i,k)}, Y_{U(j,k)}\right) = \frac{\Gamma(i+\frac{1}{c})}{k^{\frac{2}{c}}\Gamma(i)}\left\{\frac{\Gamma(j+\frac{2}{c})}{\Gamma(j+\frac{1}{c})} - \frac{\Gamma(j+\frac{1}{c})}{\Gamma(j)}\right\}$.

Let $\mathbf{X} = [X_{U(1,k)}, X_{U(2,k)}, \ldots, X_{U(n,k)}]^T$ and $\mathbf{Y} = \left[Y_{U(1,k)}, Y_{U(2,k)}, \ldots, Y_{U(n,k)}\right]^T$. Then the mean vector $E(\mathbf{Y})$ and the dispersion matrix $D(\mathbf{Y})$ of $\mathbf{Y}$ are given by $E(\mathbf{Y}) = \boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$ and $D(\mathbf{Y}) = \boldsymbol{\Sigma} = ((\sigma_{i,j}))$. Now we can write

$$E(\mathbf{X}) = \mathbf{1}\theta + \boldsymbol{\alpha}\sigma\,, \tag{33}$$

where $\mathbf{1}$ is a column vector of n ones and

$$D(\mathbf{X}) = \sigma^2\,\boldsymbol{\Sigma}. \tag{34}$$

Clearly (33) and (34) form the well-known generalized Gauss-Markoff setup. Then for

$$\boldsymbol{\Delta} = \left(\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}\right)\left(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}\right) - \left(\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}\right)^2,$$

we obtain the BLUEs θ^* and σ^* of θ and σ respectively as (see Arnold *et al.*, 1998)

$$\theta^* = \left[\frac{\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\alpha}\mathbf{1}^T - \mathbf{1}\boldsymbol{\alpha}^T\right)\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\Delta}}\right]\mathbf{X} = \sum_{j=1}^{n} a_{j,n}X_{U(j,k)} = \mathbf{a}^T\mathbf{X} \tag{35}$$

and

$$\sigma^* = \left[\frac{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\left(\mathbf{1}\boldsymbol{\alpha}^T - \boldsymbol{\alpha}\mathbf{1}^T\right)\boldsymbol{\Sigma}^{-1}}{\boldsymbol{\Delta}}\right]\mathbf{X} = \sum_{j=1}^{n} b_{j,n}X_{U(j,k)} = \mathbf{b}^T\mathbf{X}, \tag{36}$$

where $\mathbf{a} = (a_{1,n}, a_{2,n}, a_{3,n}, \ldots, a_{n,n})^T$ and $\mathbf{b} = (b_{1,n}, b_{2,n}, b_{3,n}, \ldots, b_{n,n})^T$ are the vectors of constants. Further

$$\begin{aligned} &Var(\theta^*) = \sigma^2\left\{\frac{\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\alpha}}{\boldsymbol{\Delta}}\right\},\ Var(\sigma^*) = \sigma^2\left\{\frac{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\boldsymbol{\Delta}}\right\} \\ &\text{and } Cov(\theta^*, \sigma^*) = \sigma^2\left\{\frac{-\boldsymbol{\alpha}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\boldsymbol{\Delta}}\right\} \end{aligned} \tag{37}$$

Now based on the expectations, variances and covariances of GURV's arising from WD we state and prove the following theorem, which helps one to obtain the coefficients of GURV's involved in the estimators of θ and σ of WD from the corresponding coefficients of the classical upper record values involved in the estimates of the above parameters.

Theorem 8. *Let $\{X_i,\ i \geq 1\}$ be a sequence of iid random variables each distributed identically as a WD with pdf $f(x)$ given in* (3)*. Let $\{X_{U(n,k)}\}$ be the sequence of GURV's generated from the sequence $\{X_i\}$. Then the coefficient vectors $\boldsymbol{a}^T$ and $\boldsymbol{b}^T$ of the vector of first n GURV's involved in the of Best Linear Unbiased Estimates θ^* and σ^* of the location parameter (θ) and the scale parameter (σ) respectively are given by*

$$\boldsymbol{a}^T = (\boldsymbol{a}^*)^T \qquad \text{and} \qquad \boldsymbol{b}^T = k^{\frac{1}{c}} (\boldsymbol{b}^*)^T , \tag{38}$$

where $(\boldsymbol{a}^*)^T$ and $(\boldsymbol{b}^*)^T$ are the coefficient vectors of the vector of first n classical upper record values involved in the Best Linear Unbiased Estimates θ_0^* and σ_0^* of the location and scale parameters of the WD respectively. Further

$$Var(\theta^*) = k^{\frac{-2}{c}} Var(\theta_0^*) \qquad Var(\sigma^*) = Var(\sigma_0^*) \text{ and } \qquad Cov(\theta^*, \sigma^*) = k^{\frac{-1}{c}} Cov(\theta_0^*, \sigma_0^*). \tag{39}$$

Proof. Let $\mathbf{a}^T$ be the coefficient vector of BLUE of location parameter (θ) based on first n GURV's. Then from (35) we have

$$\mathbf{a}^T = \left[\frac{\boldsymbol{\alpha}^T \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\alpha} \mathbf{1}^T - 1 \boldsymbol{\alpha}^T \right) \boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\alpha}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\alpha}) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \right) - (\boldsymbol{\alpha}^T \boldsymbol{\Sigma}^{-1} \mathbf{1})^2} \right].$$

From (9), (10) and (11) we can write

$$\boldsymbol{\alpha} = k^{\frac{-1}{c}} \boldsymbol{\alpha}_0 \qquad \text{and} \qquad \boldsymbol{\Sigma}^{-1} = k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1},$$

where $\boldsymbol{\alpha}_0$ and $\boldsymbol{\Sigma}_0^{-1}$ are the mean vector and inverse of the dispersion matrix of the vector of first n classical upper record values arising from the standard WD. So the coefficient vector of BLUE of location parameter based on first n GURV's can be simplified as

$$\begin{aligned} \mathbf{a}^T &= \left[\frac{k^{\frac{-1}{c}} \boldsymbol{\alpha}_0^T k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1} \left(k^{\frac{-1}{c}} \boldsymbol{\alpha}_0 \mathbf{1}^T - 1 k^{\frac{-1}{c}} \boldsymbol{\alpha}_0^T \right) k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1}}{\left(k^{\frac{-1}{c}} \boldsymbol{\alpha}_0^T k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1} k^{\frac{-1}{c}} \boldsymbol{\alpha}_0 \right) \left(\mathbf{1}^T k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1} \mathbf{1} \right) - \left(k^{\frac{-1}{c}} \boldsymbol{\alpha}_0^T k^{\frac{2}{c}} \boldsymbol{\Sigma}_0^{-1} \mathbf{1} \right)^2} \right] \\ &= \left[\frac{\boldsymbol{\alpha}_0^T \boldsymbol{\Sigma}_0^{-1} \left(\boldsymbol{\alpha}_0 \mathbf{1}^T - 1 \boldsymbol{\alpha}_0^T \right) \boldsymbol{\Sigma}_0^{-1}}{\left(\boldsymbol{\alpha}_0^T \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\alpha}_0 \right) \left(\mathbf{1}^T \boldsymbol{\Sigma}_0^{-1} \mathbf{1} \right) - \left(\boldsymbol{\alpha}_0^T \boldsymbol{\Sigma}_0^{-1} \mathbf{1} \right)^2} \right] = (\mathbf{a}^*)^T . \end{aligned}$$

The proof of $\mathbf{b}^T = k^{\frac{1}{c}} (\mathbf{b}^*)^T$, $Var(\theta^*) = k^{\frac{-2}{c}} Var(\theta_0^*)$, $Var(\sigma^*) = Var(\sigma_0^*)$ and $Cov(\theta^*, \sigma^*) = k^{\frac{-1}{c}} Cov(\theta_0^*, \sigma_0^*)$, follows easily, if we proceed as in the case of proving $\mathbf{a}^T = (\mathbf{a}^*)^T$.

Remark 9. *From the results of the theorem 8 we conclude that the estimate θ^* of θ is higher in precision than the estimate θ_0^*, for any $k \geq 2$ and $c > 0$. Further the precision of θ^* for fixed k in estimating θ decreases as c increases. However σ is estimated with equal precision by σ^* and σ_0^* for fixed values of c and k. Hence we recommend the experimenters to keep (or store) generalized upper(k)record values of the sequence instead of the classical record values observed in the data for future purpose and for estimating the parameters of the Weibull model.*

Balakrishnan and Chan (1993) have tabulated the component values of the vectors $\mathbf{a}^*$, $\mathbf{b}^*$, $Var(\theta_0^*)$ and $Var(\sigma_0^*)$ for different values of n. We can use those tabulated values and the results of theorem 8, to obtain the coefficient vectors $\mathbf{a}^T$ and $\mathbf{b}^T$ of θ^* and σ^* respectively and their variances without any direct computation for determining them.

5.2. Estimation of the parameters of WD when all parameters are unknown

It is unrealistic to assume always that the shape parameter c of WD is known. However when c is unknown we may recommend first an estimator of c and then use the BLUE's θ^* and σ^* for estimating θ and σ. In this case we make use of a systematic approach utilizing the concept of method of moments estimation and Best Linear Unbiased Estimation (BLUE) based on GURV's to estimate the parameters. If $X_{U(1,k)}$, $X_{U(2,k)}$, $\ldots$, $X_{U(n,k)}$ are the first n available GURV's arising from (3), then it is clear to note that $\theta \leq X_{U(1,k)} < X_{U(2,k)} < \ldots, < X_{U(n,k)} < \infty$. As $X_{U(1,k)}$ is the closest observation point to θ we may take $X_{U(1,k)} = \theta_0$ as an estimate of θ. Now from (27), we can write

$$E\left\{\frac{X_{U(i,k)} - \theta}{X_{U(i+1,k)} - \theta}\right\} = \frac{\Gamma(i+1)\Gamma(i+\frac{1}{c})}{\Gamma(i)\Gamma(i+1+\frac{1}{c})} = \frac{i}{i+\frac{1}{c}}$$

as $\frac{X_{U(i,k)}-\theta}{X_{U(i+1,k)}-\theta}$ is distributed identically as the ratio, $\frac{Y_{U(i,k)}}{Y_{U(i+1,k)}}$ of GURV's arising from standard WD defined by the pdf $f_0(y)$ which is given in (5). Now using θ_0 for θ and keeping in mind the approach adopted in method of moments estimation we write the following

$$\frac{X_{U(i,k)} - \theta_0}{X_{U(i+1,k)} - \theta_0} = \frac{i}{i+\frac{1}{c}}, \quad i = 1, 2, \ldots, n-1.$$

Then an estimator $\hat{c}_i$ of c based on $X_{U(i,k)}$ and $X_{U(i+1,k)}$ is given by

$$\hat{c}_i = \frac{X_{U(i,k)} - \theta_0}{i(X_{U(i+1,k)} - X_{U(i,k)})}, \quad i = 1, 2, \ldots, n-1. \tag{40}$$

Now there are $n-1$ estimates $\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{n-1}$ for c so that, we may take the mean of these estimates as an estimate $\hat{c}$ of c and is given by

$$\hat{c} = \frac{\sum_{i=1}^{n-1} \hat{c}_i}{n-1}. \tag{41}$$

Using $\hat{c}$ as a known value of c, now use the results of Section 5.1 to estimate ultimately θ and σ by BLUEs.

To illustrate the closeness of the estimated value of c with its true value, we have used *Mathematica* software and simulated 100 independent observations from WD with scale parameter $\sigma = 2.0$ and location parameter $\mu = 1.0$ and for some assumed values of c, collected GURV's from them and repeated it for 1000 runs. In each case we have estimated the shape parameter c by $\hat{c}$ as given in (41), taken the average $\hat{\hat{c}}$ from 1000 times and it is given below.

TABLE 1
Estimates of c for various values of k based on GURV's of a simulation study with 1000 runs, for $\mu = 1.0$, $\sigma = 2.0$, $c = 2.5(0.5)3.5$ and $k = 2(1)4$.

	Values of c					
k	True	Estimated ($\hat{\hat{c}}$)	True	Estimated ($\hat{\hat{c}}$)	True	Estimated ($\hat{\hat{c}}$)
2	2.5	2.50034	3.0	3.01361	3.5	3.53560
3	2.5	2.51804	3.0	3.02162	3.5	3.52452
4	2.5	2.50124	3.0	3.01464	3.5	3.50258

It seems that in each case the estimated value $\hat{\hat{c}}$ is very close to the true value of the parameter c.

6. Prediction for future records using BLUP

Prediction of future records are of natural interest in many contexts. Statistical prediction is the problem of inferring about the occurance of future value of a phinomena based on current and past values recorded in the phenomena. In the available literature one can observe that considerable works have been carried out on both parametric and nonparametric predictions. For more details on prediction one may refer to Gulati and Padgett (2003).

Now we discuss the problem of predicting the future GURV using the available GURV's of WD for known values of c. Suppose $X_{U(1,k)}$, $X_{U(2,k)}, \ldots$, $X_{U(n,k)}$ are the n available GURV's. Then our interest here is on predicting the next GURV $X_{U(n+1,k)}$. The best linear unbiased predicted value of the next record can be obtained (see Arnold *et al.*, 1998) as

$$X^*_{U(n+1,k)} = \theta^* + \alpha_{n+1}\sigma^* + \mathbf{w}^T\Sigma^{-1}(\mathbf{X} - \theta^*\,\mathbf{1} - \sigma^*\boldsymbol{\alpha}), \tag{42}$$

where $\mathbf{X}$ is the vector of the n observed GURV's from WD as defined in (3), $\boldsymbol{\alpha}$ and $\boldsymbol{\Sigma}$ are the vector of means and dispersion matrix of the vector of n GURV's arising from standard WD, $\mathbf{1}$ is a vector of n one's, α_{n+1} is the expected value of $(n+1)th$ GURV arising from the standard WD and $\mathbf{w}^T = (\sigma_{1,n+1}, \ \sigma_{2,n+1}, \ ..., \ \sigma_{n,n+1})$ is the vector of the covariance between the $(n+1)th$ GURV with the initial GURV's arising from standard WD and θ^* and σ^* are the BLUEs of θ and σ of (3) based on the first n GURV's respectively. The equation (42) can also be written as

$$X^*_{U(n+1,k)} = \sum_{i=1}^{n} f_{i,n}X_{U(i,k)} = \mathbf{f}^T\mathbf{X}, \tag{43}$$

where $\mathbf{f}^T$ is the coefficient vector of X and is given by $\mathbf{f}^T = (f_{1,n}, f_{2,n}, \ldots, f_{n,n})$.
In the following theorem we establish a connection between Best Linear Unbiased Prediction (BLUP) based on classical record values and that based on GURV's in predicting $(n+1)th$ GURV respectively.

Theorem 10. *Let $X_{U(1,k)}$, $X_{U(2,k)}$, $\ldots X_{U(n,k)}$ be the first n available GURV's and let $X_{U(1,1)}$, $X_{U(2,1)}, \ldots X_{U(n,1)}$ be the first n classical record values. Then the coefficients $f_{i,n}$ of*

$X_{U(i,k)}$, $i = 1, 2, \ldots, n$ involved in the BLUP of immediate next GURV $X_{U(n+1,k)}$ as given in $X^*_{U(n+1,k)} = \sum_{i=1}^{n} f_{i,n} X_{U(i,k)}$ are identically same as the coefficents of the classical record values $X_{U(i,1)}$, $i = 1, 2, \ldots, n$ involved in the BLUP of the next classical record value $X_{U(n+1,1)}$. That is

$$f^T = (f^*)^T, \tag{44}$$

where $(f^*)^T$ is the coefficient vector in the BLUP of $X_{U(n+1,1)}$ based on the first n classical upper record values.

Proof. The best linear unbiased predicted value of the next GURV can be obtained by the equation

$$X^*_{U(n+1,k)} = \theta^* + \alpha_{n+1}\sigma^* + \mathbf{w}^T\Sigma^{-1}(\mathbf{X} - \theta^*\,\mathbf{1} - \sigma^*\boldsymbol{\alpha}).$$

If $\mathbf{a}$ and $\mathbf{b}$ are the vectors as defined in (35) and (36), then we have

$$\begin{aligned} X^*_{U(n+1,k)} &= \mathbf{a}^T\mathbf{X} + \alpha_{n+1}\mathbf{b}^T\mathbf{X} + \mathbf{w}^T\Sigma^{-1}(\mathbf{X} - (\mathbf{a}^T\mathbf{X})\,\mathbf{1} - (\mathbf{b}^T\mathbf{X})\boldsymbol{\alpha}) \\ &= \left[\mathbf{a}^T + \alpha_{n+1}\mathbf{b}^T + \mathbf{w}^T\Sigma^{-1}(\mathbf{1} - \mathbf{a}^T\,\mathbf{1} - \mathbf{b}^T\boldsymbol{\alpha})\right]\mathbf{X} \qquad (45) \\ &= \mathbf{f}^T\mathbf{X}, \qquad (46) \end{aligned}$$

where $\mathbf{f}^T$ is defined as in (43). Let $\mathbf{a}^*$ and $\mathbf{b}^*$ are the coefficient vectors of the vector of first n classical record values in the BLUE's of θ and σ arising from (3). Further using the notaions used in theorem 8 if we replace the quantities involved in the coefficient vector of X involved in (45) corresponding to those involved in the BLUP $X^*_{U(n+1,1)}$ based on first n classical record values then we have $a = a^*$ and $b = k^{\frac{1}{c}b^*}$. Clearly

$$\begin{aligned} \mathbf{f}^T &= \mathbf{a}^T + \alpha_{n+1}\mathbf{b}^T + \mathbf{w}^T\boldsymbol{\Sigma}^{-1}(1 - \mathbf{a}^T - \mathbf{b}^T\boldsymbol{\alpha}) \\ &= (\mathbf{a}^*)^T + k^{\frac{-1}{c}}\alpha^*_{n+1}\left(k^{\frac{1}{c}}\mathbf{b}^*\right)^T + k^{\frac{-2}{c}}\mathbf{w}_0^T k^{\frac{2}{c}}\boldsymbol{\Sigma}_0^{-1}\left(1 - (\mathbf{a}^*)^T\,\mathbf{1} - \left(k^{\frac{1}{c}}\mathbf{b}^*\right)^T k^{\frac{-1}{c}}\boldsymbol{\alpha_0}\right), \end{aligned}$$

where α^*_{n+1} is the expected value of $(n+1)th$ classical record value arising from standard WD and $\mathbf{w}_0^T = (\sigma_{0(1,n+1)},\ \sigma_{0(2,n+1)},\ \ldots,\ \sigma_{0(n,n+1)})$ is the vector of the covariance between the $(n+1)th$ classical record value with the initial classical record values arising from standard WD. Hence we have

$$\begin{aligned} \mathbf{f}^T &= (\mathbf{a}^*)^T + \alpha^*_{n+1}(\mathbf{b}^*)^T + \mathbf{w}_0^T\Sigma_0^{-1}\left\{\mathbf{1} - (\mathbf{a}^*)^T\ \mathbf{1} - (\mathbf{b}^*)^T\boldsymbol{\alpha}_0\right\} \\ &= (\mathbf{f}^*)^T. \qquad (47) \end{aligned}$$

This completes the proof.

The coefficients of the available first n classical upper record values or those of first n GURVs in predicting the appropriate future records are not seen determined specifically for WD. Hence we have determined the coefficients of $X_{U(i,k)}$ for $i = 1, 2, \ldots, n$, $n = 2(1)5$, $c = 2.5(0.5)5$ in the BLUP of $X^*_{U(n+1,k)}$ and are presented in Table 2. Due to the invariance nature of these coefficients for different choices of k as seen in theorem 10, the coefficients in Table 2 can be used as such for prediction of a future classical upper record value or for the prediction of a future GURV for any $k \geq 2$ as well.

TABLE 2

Coefficients $f_{i,n}$ of $X_{U(i,k)}$ involved in the BLUP $X^*_{U(n+1,k)} = \sum_{i=1}^{n} f_{i,n}(X_{U(i,k)})$ for n=2(1)5, c=2.5(0.5)5

		Coefficients $f_{i,n}$ of $X_{U(i,k)}$ in $X^*_{U(n+1,k)} = \sum_{i=1}^{n} f_{i,n}(X_{U(i,k)})$				
n	c	$f_{1,n}$	$f_{2,n}$	$f_{3,n}$	$f_{4,n}$	$f_{5,n}$
2	2.5	-0.70000	1.70000			
	3	-0.66667	1.66667			
	3.5	-0.64286	1.64286			
	4	-0.62500	1.62500			
	4.5	-0.61111	1.61111			
	5	-0.60000	1.60000			
3	2.5	-0.30000	-0.07143	1.37143		
	3	-0.27778	-0.08333	1.36111		
	3.5	-0.26191	-0.09259	1.35450		
	4	-0.25000	-0.10000	1.35000		
	4.5	-0.24074	-0.10606	1.34680		
	5	-0.23333	-0.11111	1.34444		
4	2.5	-0.17927	-0.04268	-0.03049	1.25244	
	3	-0.16260	-0.04878	-0.03659	1.24797	
	3.5	-0.15081	-0.05332	-0.04147	1.24560	
	4	-0.14205	-0.05682	-0.04545	1.24432	
	4.5	-0.13528	-0.05960	-0.04876	1.24364	
	5	-0.12990	-0.06186	-0.05155	1.24330	
5	2.5	-0.12358	-0.02942	-0.02102	-0.01659	1.19062
	3	-0.11028	-0.03308	-0.02481	-0.02030	1.18847
	3.5	-0.10096	-0.03569	-0.02776	-0.02332	1.18773
	4	-0.09409	-0.03763	-0.03011	-0.02581	1.18763
	4.5	-0.08882	-0.03913	-0.03202	-0.02789	1.18785
	5	-0.08467	-0.04032	-0.03360	-0.02964	1.18822

Clearly the above table serves as an aid for practicing statisticians in predicting the immediate next GURV $X_{U(n+1,k)}$ for $n = 2(1)6$ and given values of $c = 2.5(0.5)5$. It is to be noted that the values in the table remains the same whatever is k for $k = 1, 2, \ldots$

7. Illustration of the Results by a Real Life Data

Roberts (1979,p. 4) has given the monthly and annual maxima of the one-hour mean concentration of Sulfur dioxide (SO_2) from Long Beach, California, for the years 1956-1974. Chan (1993) indicated that the WD is a reasonable model for this data set Chan (also see 1998). From these monthly maxima of the hourly concentration of SO_2, we obtain five generalized(2)record values and are given as

$$X_{U(1,2)} = 31.0,\ X_{U(2,2)} = 44.0,\ X_{U(3,2)} = 47.0,\ X_{U(4,2)} = 51.0,\ X_{U(5,2)} = 55.0.$$

These generalized(2)record values are used to estimate the parameters of WD. The shape parameter c can be estimated by the equation (41) and we estimated its value as $\hat{c}= 1.1875$. Now we take the estimated value of c as its known value and use the results of this paper (given in section 5.1) to estimate the location and scale parameters by BLUE based on GURV's for $k = 2$ and the estimated values are

$$\begin{aligned}\theta^* &= 1.28351 * X_{U(1,2)} + 0.04099 * X_{U(2,2)} + 0.022249 * X_{U(3,2)} + 0.014249 * X_{U(4,2)} \\ &\quad -0.36099 * X_{U(5,2)} = 23.51, \qquad (48)\\ \sigma^* &= -0.60108 * X_{U(1,2)} - 0.01919 * X_{U(2,2)} - 0.010419 * X_{U(3,2)} - 0.006673 * X_{U(4,2)} \\ &\quad +0.63736 * X_{U(5,2)} = 14.747, \qquad (49)\end{aligned}$$

with $Var(\theta^*) = 0.278518\sigma^2$, $Var(\sigma^*) = 0.203228\sigma^2$ and $Cov(\theta^*, \sigma^*) = -0.130432\ \sigma^2$. Now we consider the BLUP given in section 6 and the formula given in (43) to predict the 6th generalized(2)upper record value and is obtained as

$$\begin{aligned}X^*_{U(6,2)} &= -0.21617 * X_{U(1,2)} - 0.00690 * X_{U(2,2)} - 0.00375 * X_{U(3,2)} - 0.00239 * X_{U(4,2)} \\ &\quad +1.22922 * X_{U(5,2)} = 60.3036.\end{aligned}$$

Clearly the above predicted value of the 6th GURV eliminates one possible outlier in the data and gives a clue to workout the extent to which the society has to divise ways and means to keep the SO_2 level under control in the Californian Beach.

Acknowledgements

The authors express their gratefulness to the learned referee for some suggestions which lead to the improvement in the present version of the paper. The first author is much grateful to the University Grants Commission, Government of India for the financial support received for this research work in the form of MAN-JRF. The second author acknowledges for the financial support received from Kerala State Council for Science, Technology & Environment in the form of Emeritus Scientist Fellowship.

References

J. Aczel (1966). *Lectures on Functional Equations and Their Applications*. Academic Press, New York.

B. C. Arnold, N. Balakrishnan, H. N. Nagaraja (1998). *Records*. John Wiley and Sons, New York.

N. Balakrishnan, P. S. Chan (1993). *Record values from Rayleigh and Weibull distributions and associated inference*. National Institute of Standards and Technology Journal of Research, Special Publication, 866, pp. 41–51.

P. S. Chan (1993). *A statistical study of log-gamma distribution*. Ph.D. thesis, McMaster University, Canada.

P. S. Chan (1998). *Interval estimation of location and scale parameters based on record values*. Statistics & probability letters, 37, no. 1, pp. 49–58.

K. N. Chandler (1952). *The distribution and frequency of record values*. Journal of the Royal Statistical Society. Series B, 14, pp. 220–228.

A. C. Dallas (1982). *Some results on record values from the exponential and Weibull law*. Acta Mathematica Hungarica, 40, no. 3, pp. 307–311.

W. Dziubdziela, B. Kopocinski (1976). *Limiting properties of the kth record values*. Zastos. Mat., 15, pp. 187–190.

N. Glick (1978). *Breaking records and breaking boards*. American Mathematical Monthly, 85, pp. 2–26.

S. Gulati, W. J. Padgett (2003). *Parametric and Nonparametric Inference from Record Breaking Data*. 172. Springer-Verlag, New York.

K. S. Johnson, N. L., N. Balakrishnan (1994). *Continuous univariate distributions, Vol. 1*. John Wiley & Sons, New York.

S. Minimol, P. Y. Thomas (2013). *On some properties of Makeham distribution using generalized record values and its characterizations*. Brazilian Journal of Probability and Statistics, 27, no. 4, pp. 487–501.

S. Minimol, P. Y. Thomas (2014). *On characterization of Gompertz distribution by generalized record values*. Journal of Statistical Theory and Applications, 13, pp. 38–45.

V. B. Nevzorov (2001). *Records: Mathematical Theory. Translation of Mathematical Monographs, vol. 194*. American Mathematical Society, Providence, RI, USA.

J. Paul, P. Y. Thomas (2013). *On a property of generalized record values arising from exponential distribution*. Indian Association for Productivity, Quality and Reliability Transactions, 38, pp. 19–27.

P. Pawlas, D. Szynal (1998). *Relations for single and product moment of k-th record values from exponential and Gumbel distribution*. J. Appl. Statist. Sci, 7, pp. 53–62.

P. Pawlas, D. Szynal (1999). *Recurrence relations for single and product moment of k-th record values from Pareto, generalized Pareto and Burr distributions*. Communications in Statistics-Theory Methods, 28, pp. 1699–1709.

H. Rinne (2008). *The Weibull Distribution : A Handbook*. CRC Press, Taylor & Francis Group, LLC.

E. M. Roberts (1979). *Review of statistics of extreme values with application to air quality data, part ii: applications*. Journal of the Air Pollution Control Association, 29, no. 7, pp. 733–740.

W. Weibull (1951). *A statistical distribution function of wide applicability*. J. Appl. Mech.-Trans. ASME, 18, pp. 293–297.

SUMMARY

In this paper we study the generalized upper(k)record values arising from Weibull distribution. Expressions for the moments and product moments of those generalized upper(k)record values are derived. Some properties of generalized upper(k)record values which characterize the Weibull distribution have been established. Also some distributional properties of generalized upper(k)record values arising from Weibull distribution are considered and used for suggesting an estimator for the shape parameter of Weibull distribution. The location and scale parameters are estimated using the Best Linear Unbiased Estimation procedure. Prediction of a future record using Best Linear Unbiased Predictor has been studied. A real life data is used to illustrate the results generated in this work.

Keywords: Best Linear Unbiased Estimation; Best Linear Unbiased Predictor; Characterization; Generalized upper(k)record values; Weibull Distribution.