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CHARACTERIZATIONS OF A FAMILY OF BIVARIATE PARETO DISTRIBUTIONS

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1. Introduction

In the literature, Pareto distributions have been extensively employed for modeling and analysis of statistical data under different contexts. Orginally, the distribution was first proposed as a model to explain the allocation of wealth among individuals. Later, various forms of the Pareto distribution have been formulated for modeling and analysis of data from engineering, environment, geology, hydrology etc. These diverse applications of the Pareto distributions lead researchers to develop different kinds of bivariate(multivariate)Pareto distributions. Accordingly, Mardia (1962) introduced two types of bivariate(multivariate) Pareto models which are referred as bivariate Pareto distributions of first kind and second kind respectively. Since then there has been a lot of works in the form, alternative derivation of bivariate Pareto models, their extensions, inference, characterizations and applications to a variety of fields. Various types of bivariate (multivariate) Pareto distributions discussed and studied in literature include those of Lindley and Singpurwalla (1986), Arnold (1985), Arnold (1990), Sankaran and Nair (1993), Hutchinson and Lai (1990), Langseth (2002), Balakrishnan and Lai (2009) and Sankaran and Kundu (2014).

The models discussed above are individual in nature and are appropriate for a particular data set that meet the specified requirements. However, when there is little information about the data generating process, it is desirable to start with a family of distributions and then choose a member of the family that agrees with the patterns in the data. Motivated by this fact, we introduced a family of bivariate Pareto distributions arising from a generalization of the univariate dullness property (Talwalker (1980)) that characterized the univariate Pareto law.(see Sankaran *et al.* (2014))

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The major aim of the present paper is to develop characterizations of the above family of Pareto distributions. One of the problems that is usually addressed while examining the characteristic properties of the bivariate distribution is to investigate how far the characterizations of the corresponding univariate version can be extended to the bivariate forms. Among the characterizations of the univariate Pareto I law, one with important practical applications is the dullness property. The bivariate version of the dullness property is employed to characterize the family of Pareto distributions.

A second concept that has applications in economics is income gap ratio which is used for developing indices of affluence (Sen (1988)). The bivariate generalization of the concept is proposed and characterizations using this concept are derived. Another function of interest that has applications in reliability also is the bivariate generalized failure rate. Characterizations of the family of distributions using the generalized failure rate are also discussed. As the bivariate versions of these functions are not unique, different versions of these concepts lead to various types of bivariate Pareto distributions which are members of the family.

The rest of the paper is organized as follows. In Section 2, we present a family of bivariate Pareto distributions characterized through a generalized version of the univariate dullness property. In Section 3, we introduce bivariate versions of dullness property and present characterizations using these versions. The bivariate version of income gap ratio and related concepts in economics are discussed in Section 4. The forms of these functions for various bivariate Pareto distributions are given. Section 5 discusses bivariate generalized failure rate. Characterizations using the generalized failure rate are also developed. Finally, Section 6 provides brief conclusions of the study.

2. Bivariate Pareto family

Let (X,Y) be a non-negative random vector having absolutely continuous survival function $\bar{F}(x,y) = P(X > x, Y > y)$. Assume that Z is a non-negative random variable with continuous and strictly decreasing survival function $\bar{G}(z)$ and cumulative hazard function $H(z)$ defined by $H(z) = -\log \bar{G}(z)$. Then the family of distributions specified by the survival function

$$\bar{F}(x,y) = [g(x,y)]^{-1},\ x,y > 1 \tag{1}$$

is a bivariate Pareto family if and only if there exist a function $g(x,y)$ satisfying

$$H(\log\, g(x,y)) = H(a \log x) + H(b \log y) \tag{2}$$

Notice that $g(x,y)$ is a function of (x,y) in $R_2^+ = \{(x,y)|x,y > 0\}$ satisfying the following properties

(a) $g(1,y) = y^b, g(x,1) = x^a$,

(b) $g(\infty,y) = \infty, g(x,\infty) = \infty$,

(c) since $H(.)$ is increasing and continuous, $g(x,y)$ is also increasing and continuous in x and y and

(d) it is assumed that $g(x, y)$ satisfies the inequality $\frac{2}{g(x,y)} \frac{\partial g}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial^2 g}{\partial x \partial y} \geq 0$.

Further, (2) holds if and only if

$$P(Z > \log g(x, y) | Z > a \log x) = P(Z > b \log y). \tag{3}$$

The proof of this is given in Sankaran *et al.* (2014). Notice that the marginal distributions of X and Y are Pareto I distributions with survival functions $\bar{F}_X(x) = x^{-a}$, $x > 1$ and $\bar{F}_Y(y) = y^{-b}$, $y > 1$. The property (3) is referred as extended dullness property. The family (1) includes various known and unknown bivariate distributions. The members of the family can be derived by choosing appropriate univariate distribution of Z. Table 1 provides some members of family according to different choices of the distribution of Z. It may be noted that the properties of $\bar{F}(x, y)$ can be inferred from the corresponding properties of $\bar{G}(z)$. The members of the family listed in Table 1, include bivariate Pareto with independent marginals, Mardia's(1962) Type 1 model and bivariate Burr distribution. For more properties one could refer to Sankaran *et al.* (2014).

3. Bivariate dullness property

We first discuss the univariate dullness property. Let Z_1 be a non-negative random variable representing the income in a population. Then the distribution of Z_1 is said to have dullness property if

$$P(Z_1 > xy | Z_1 > x) = P(Z_1 > y) \tag{4}$$

for all $x, y \geq 1$. This means that the conditional probability that true income Z_1 is at least y times the reported value x is the same as the unconditional probability that Z_1 has at least income y. In otherwords, the distribution of error in an income is independent of the reported value. It is proved that the property (4) holds if and only if the distribution of Z_1 is Pareto I (Talwalker (1980)). There have been many works on models that accounts for underreporting, characterizations and related concepts, like those of Wong (1982), Xekalaki (1983), Korwar (1985), Artikis *et al.* (1994), Pham-Gia and Turkkan (1997) and Nadarajah (2009). Sometimes it is more convenient to work with an equivalent form of dullness property, as the property (4) is not helpful in practice to verify whether a given data set follows Pareto distribution or not. In this context, we have the following result.

Theorem 1. *The random variable Z_1 satisfies dullness property* (4) *if and only if*

$$m(x) = E(Z_1 | Z_1 > x) = \mu x \tag{5}$$

for all $x > 1$, where $\mu = E(Z_1) < \infty$.

Proof. Assume that (4) holds. Then we can write

$$S(x, y) = S(x)S(y) \;\; for \;\; all \;\; x, y > 1 \tag{6}$$

TABLE 1
Some members of bivariate Pareto family

Type	Survival function of the distribution of Z	Survival function of the bivariate Pareto distribution
1	$\bar{G}_1(z) = \exp(-\lambda z), z > 0$	$\bar{F}_1(x, y) = x^{-a} y^{-b};\ x > y > 1; a, b > 0$
2	$\bar{G}_2(z) = \exp[-\theta(e^{\alpha z} - 1)]; z \geq 0; \alpha, \theta > 0$	$\bar{F}_2(x, y) = (x^{a\alpha} + y^{b\alpha} - 1)^{\frac{-1}{\alpha}}; x, y > 1, \alpha, a > 0$
3	$\bar{G}_4(z) = (1 + \beta z)^{-\alpha}$	$\bar{F}_4(x, y) = x^{-a-c\log y} y^{-b}, x, y > 1, a, b > 0; 0 \leq c \leq 1$
4	$\bar{G}_5(z) = 2(1 + e^{\frac{z}{\sigma}})^{-1},\ z > 0, \sigma > 0$	$\bar{F}_5(x, y) = [\frac{1}{2}(x^{\alpha} + y^{\beta} + x^{\alpha} y^{\beta} - 1)]^{-\sigma}; \alpha = \frac{a}{\sigma} > 0, \sigma > 0, \beta = \frac{b}{\sigma} > 0$
5	$\bar{G}_6(z) = (1 + z^c)^{-k},\ z > 0;\ c,\ k > 0$	$\bar{F}_6(x, y) = \exp[-(a\log x)^c - (b\log y)^c - (ab\log x \log y)^c]^{\frac{1}{c}}$
6	$\bar{G}_7(z) = (2e^z - 1)^{-\sigma}, z > 0; \sigma > 0$	$\bar{F}_7(x, y) = (1 + 2x^a y^b - x^a - y^b)^{-1}$
7	$\bar{G}_8(z) = e^{-(\lambda z)^{\alpha}}\ \alpha, \lambda > 0, z > 0$	$\bar{F}_8(x, y) = \exp[\frac{-1}{\lambda}\{(\lambda a \log x)^{\alpha} + (\lambda b \log y)^{\alpha}\}^{\frac{1}{\alpha}}]$
8	$\bar{G}_9(z) = \frac{p}{e^{\lambda z} - q},\ z > 0; \lambda > 0,\ 0 < p < 1,\ q = 1 - p$	$\bar{F}_9(x, y) = (q + p^{-1}(x^{a\lambda} - q)(y^{b\lambda} - q))^{\frac{-1}{\lambda}}$
9	$\bar{G}_{10}(z) = (1 + \frac{e^{\lambda z} - 1}{\alpha})^{-1},\ \alpha, \lambda > 0$	$\bar{F}_{10}(x, y) = (1 + \alpha^{-1}(\alpha + x^{a\lambda} - 1)(\alpha + y^{b\lambda} - 1) - \alpha)^{\frac{-1}{\lambda}}$

where $S(x) = P(Z_1 > x)$. Integrating (6) from 1 to ∞, we obtain

$$\int_1^\infty S(xy)\,dy = S(x)\int_1^\infty S(y)\,dy$$

or

$$\int_x^\infty S(t)\,dt = x\,S(x)(\mu-1) \tag{7}$$

Since left side of (7) is $E(Z_1 - x|Z_1 > x)$, we get

$$m(x) - x = x(\mu - 1)$$

which leads to (5).
Conversely, when (5) hold, we obtain (7) and hence

$$\frac{S(x)}{\int_x^\infty S(t)\,dt} = \frac{1}{x(\mu-1)} \tag{8}$$

Integrating (8), we obtain

$$\int_x^\infty S(t)\,dt = cx^{-\frac{1}{\mu-1}} \tag{9}$$

where c is the integrating constant. Differentiating (9) with respect to x, we get

$$S(x) = \frac{c}{\mu-1}x^{-\frac{\mu}{\mu-1}} \tag{10}$$

Since $S(1)$ =1, we obtain $c = \mu - 1$ and thus

$$S(x) = x^{-\frac{\mu}{\mu-1}}$$

which means that the distribution of Z_1 is Pareto I. From Talwalker (1980) it follows that the Pareto distribution satisfies (4), which completes the proof. □

Now we propose bivariate versions of (5) and examine whether they characterize members of the bivariate Pareto family. A natural extension of (5) with respect to the random vector (X, Y) is the vector $(m_1(x, y), m_2(x, y))$ where

$$m_1(x, y) = E(X|X > x, Y > y) \tag{11}$$

and

$$m_2(x, y) = E(Y|X > x, Y > y). \tag{12}$$

We first observe that the joint survival function $\bar{F}(x, y)$ of (X, Y) can be determined from (11) and (12).

The joint survival function $\bar{F}(x,y)$ is obtained as

$$\bar{F}(x,y) = \exp\left[-\int_1^x \frac{\frac{\partial m_1(t,1)}{\partial t}}{m_1(t,1)-t}dt - \int_1^y \frac{\frac{\partial m_2(x,t)}{\partial t}}{m_2(x,t)-t}dt\right] \tag{13}$$

$$= \exp\left[-\int_1^y \frac{\frac{\partial m_2(1,t)}{\partial t}}{m_2(1,t)-t}dt - \int_1^x \frac{\frac{\partial m_1(t,y)}{\partial t}}{m_2(t,y)-t}dt\right] \tag{14}$$

The proof follows from Nair and Nair (1989) by using the relationship between $(m_1(x_1,x_2), m_2(x_1,x_2))$ and bivariate mean residual life functions.

Definition 2. *The distribution of the random vector (X,Y) is said to have bivariate dullness property one (BDP-1) if and only if for $x,y>1$*

$$P(X > xt|X > x, Y > y) = P(X > t|Y > y), \;\; t > 1$$

and

$$P(Y > ys|X > x, Y > y) = P(Y > s|X > x), \;\; s > 1.$$

Definition 3. *The distribution of (X,Y) satisfies bivariate dullness property two (BDP-2) if and only if for $x,y>1$*

$$P(X > xt, Y > ys|X > x, Y > y) = P(X > t, Y > s). \tag{15}$$

Now we have the following theorems characterizing bivariate Pareto distributions belonging to the family by two versions of the bivariate dullness property.

Theorem 4. *Let (X,Y) be a bivariate random vector as described in Section 2, with $\mu_X = E(X) < \infty$ and $\mu_Y = E(Y) < \infty$. Denote $\mu_X(y) = E(X|Y > y)$ and $\mu_Y(x) = E(Y|X > x)$. Then the following statements are equivalent.*

(a) $\bar{F}_4(x,y) = x^{-(a+c\log y)}y^{-b}; \quad x,y>1, \quad a,b>1, \quad 0<c\le a\,b$

(b) (X,Y) *satisfies BDP-1.*

(c) $m_1(x,y) = x\mu_X(y)$ *and* $m_2(x,y) = y\mu_Y(x)$

Proof. To prove $(a) \Rightarrow (b)$, we have

$$P[X > xt|X > x, Y > y] = \frac{(xt)^{-(a+c\log y)}y^{-b}}{x^{-(a+c\log y)}y^{-b}} = t^{-(a+c\log y)} = P(X > t|Y > y)$$

The second part is proved similarly.
To establish $(b) \Rightarrow (c)$, we note that the first statement in (b) is same as

$$\frac{\bar{F}(xt,y)}{\bar{F}(x,y)} = \frac{\bar{F}(t,y)}{P(Y>y)} = \bar{F}_X(t|Y>y) \tag{16}$$

where $\bar{F}_X(t|Y>y)$ is the conditional survival function of X given $Y>y$. Integrating (16) with respect to t in $(1,\infty)$, we obtain

$$E(X|X > x, Y > y) = xE(X|Y > y)$$

or

$$m_1(x, y) = x\mu_X(y).$$

The expression for $m_2(x, y)$ is proved similarly.
Now we establish $(c) \Rightarrow (a)$. From (c), we have

$$\frac{\partial m_1(t,1)}{\partial t} = \frac{\partial}{\partial t}(t\mu_X(y)) = \mu_X,$$
$$\frac{\partial m_2(1,t)}{\partial t} = \mu_Y$$
$$m_1(t, y) - t = t\mu_X(y) - t$$

and

$$m_2(x, t) - t = t\mu_Y(x) - t.$$

Substituting the above expressions in (13) and (14) and equating the resulting expressions, we get, after some simplifications,

$$\frac{\mu_X}{\mu_X - 1}\log x + \frac{\mu_Y(x)}{\mu_Y(x) - 1}\log y = \frac{\mu_Y}{\mu_Y - 1}\log y + \frac{\mu_X(y)}{\mu_X(y) - 1}\log x$$

which leads to the functional equation

$$\left(\frac{\mu_X}{\mu_X - 1} - \frac{\mu_X(y)}{\mu_X(y) - 1}\right)\log x = \left(\frac{\mu_Y}{\mu_Y - 1} - \frac{\mu_Y(x)}{\mu_Y(x) - 1}\right)\log y \qquad (17)$$

To solve (17), we rewrite it as

$$\frac{\log x}{\frac{\mu_Y}{\mu_Y-1} - \frac{\mu_Y(x)}{\mu_Y(x)-1}} = \frac{\log y}{\frac{\mu_X}{\mu_X-1} - \frac{\mu_X(y)}{\mu_X(y)-1}} \qquad (18)$$

The right(left) side of (18) is a function of $y(x)$ alone and therefore the equality of the two sides hold good for all $x, y > 1$ if and only if each side must be a constant say $\frac{1}{c}$. Hence

$$\frac{\mu_Y(x)}{\mu_Y(x) - 1} = \frac{\mu_Y}{\mu_Y - 1} - c\log x$$

and

$$\frac{\mu_X(y)}{\mu_X(y) - 1} = \frac{\mu_X}{\mu_X - 1} - c\log y \qquad (19)$$

Using (19) in (13) or (14), we get

$$\bar{F}_4(x, y) = x^{-\frac{\mu_X}{\mu_X-1}} y^{-\frac{\mu_Y}{\mu_Y-1}+c\log x}$$

Taking $\mu_X = \frac{a}{a-1}$ and $\mu_Y = \frac{b}{b-1}$, $a, b > 1$, we obtain

$$\bar{F}_4(x, y) = x^{-a}y^{-(b+c\log x)}$$

since $x^{-c\log y} = y^{-c\log x}$, we have (a) and the theorem is completely proved. The parameter values $a, b > 1$ is required for the existence of the means. □

Setting $c = 0$ in $\bar{F}_4(x, y)$ and working similarly, we have the following result.

THEOREM 5. *For the random vector (X, Y) in Theorem 4, the following statements are equivalent*

(a) $\bar{F}_1(x, y) = x^{-a}y^{-b}; \quad x, y > 1, \quad a, b > 1$

(b) (X, Y) *satisfies BDP-2.*

(c) $(m_1(x, y), m_2(x, y)) = (x\mu_X, y\mu_Y)$

REMARK 6. *It may be noticed that* $BDP-1$ *is stronger than* $BDP-2$.

REMARK 7. *The properties* $BDP-1$ *and* $BDP-2$ *can be interpreted in income analysis as follows. Let* X *and* Y *be the incomes from two different sources of a unit in a population. Assume that the incomes of* X *and* Y *are at least* x *and* y *respectively. The average under-reporting error is proportional to the amount by which the income exceeds the tax exemption level. The under-reporting error in* $X(Y)$ *is a linear function of the reported income if and only if the incomes* (X, Y) *follow bivariate Pareto law. In the case of* $BDP-1$, *the proportionality is independent of* x *and* y, *while in* $BDP-2$, *it is independent of* x *in the case of* X *and independent of* y *in the case of* Y.

REMARK 8. *Hanagal (1996) has considered another version of the dullness property defined as*

$$P(X > xt, Y > yt) = P(X > x, Y > y)P(X > t, Y > t)$$

which characterized a Marshall-Olkin type bivariate Pareto model. Note that this distribution does not belongs to the family (1).

THEOREM 9. *Let* (X, Y) *be a non-negative exchangeable random vector with absolutely continuous survival function and* $\mu = E(X) = E(Y) < \infty$. *Then*

$$(m_1(x, y), m_2(x, y)) = (\mu x + (\mu - 1)p(y), \mu y + (\mu - 1)p(x)) \tag{20}$$

for some non-negative function $p(.)$ *with* $p(1) = 0$ *iff the survival function of* (X, Y) *is*

$$\bar{F}_{10}(x, y) = \left(\frac{x + y - cxy - 1}{1 - c}\right)^{-\frac{\mu}{\mu - 1}}; \; x, y > 1 \tag{21}$$

where c *is a real constant different from unity.*

PROOF. Assume that (X, Y) has the distribution (21). Then

$$\begin{aligned} m_1(x, y) &= x + \frac{1}{\bar{F}(x, y)} \int_x^\infty \bar{F}(t, y)dt \\ &= x + \frac{x + y - cxy - 1}{1 - cy}(\mu - 1) \\ &= \mu x + \frac{y - 1}{1 - cy}(\mu - 1) \end{aligned}$$

which is of the form (20) with $p(y) = \frac{y-1}{1-cy}$ and $p(1) = 0$. The proof for $m_2(x, y)$ is similar.
Conversely, if the relation (20) holds, from (14),

$$\bar{F}(x, y) = \exp\left[-\int_1^x \frac{\mu}{(\mu-1)t}dt - \int_1^y \frac{\mu}{(\mu-1)t + (\mu-1)p(x)}dt\right]$$
$$= \left(\frac{x(y+p(x))}{1+p(x)}\right)^{-\frac{\mu}{\mu-1}}. \tag{22}$$

From (14) and (20), we obtain

$$\bar{F}(x, y) = \left(\frac{y(x+p(y))}{1+p(y)}\right)^{-\frac{\mu}{\mu-1}}. \tag{23}$$

Equating (22) and (23) and simplifying,

$$\frac{xp(x)}{1+p(x)-x} = \frac{yp(y)}{1+p(y)-y}. \tag{24}$$

Since (24) holds for all $x, y > 1$, one should have

$$\frac{xp(x)}{1+p(x)-x} = \frac{1}{c},$$

a constant independent of x and y. Solving the above, we get

$$p(x) = \frac{x-1}{1-cx}.$$

Substituting $p(x)$ in (22), we have (21). This completes the proof. □

Remark 10. *The distribution specified by* (21) *is a bivariate distribution with Pareto I marginals. It contains some members of our family. When* $c = 0$

$$m_1(x, y) = \mu x + (\mu-1)(y-1)$$

and

$$m_2(x, y) = \mu y + (\mu-1)(x-1)$$

characterized the bivariate Pareto distribution

$$\bar{F}_3(x, y) = (x+y-1)^{-a}; x, y > 1,$$

the well known Mardia's(1962) type I bivariate Pareto model. Similarly when $c = -1$, *we have*

$$(m_1(x, y), m_2(x, y)) = \left(\mu x + (\mu-1)\frac{y-1}{y+1}, \mu y + (\mu-1)\frac{x-1}{x+1}\right)$$

characterized the bivariate Pareto distribution

$$\bar{F}_{11}(x,y) = [\frac{1}{2}(x+y+xy-1)]^{-a}, x,y>1, a>1$$

which is a special case of $\bar{F}_5(x,y)$ *in Table 1 when* $\alpha=\beta=1$ *so that* $\sigma=a$. *Finally* $c=\frac{1}{q},\ q>0$ *gives*

$$m_1(x,y) = \mu x + \frac{(\mu-1)q(y-1)}{q-y}$$

and

$$m_2(x,y) = \mu y + \frac{(\mu-1)q(x-1)}{q-x}$$

that characterizes

$$\bar{F}_{12}(x,y) = (q+p^{-1}(x-q)(y-q))^{-a}$$

a special case of $\bar{F}_9(x,y)$ *obtained by taking* $\lambda a=\lambda b=1$.

Remark 11. *It is easy to see that all the bivariate distributions discussed in Remark 10, including our models* $\bar{F}_3(x,y)$, $\bar{F}_5(x,y)$ *and* $\bar{F}_9(x,y)$ *do not satisfy the dullness properties* $BDP-1$ *and* $BDP-2$.
The extent to which they depart from $BDP-1$ *is accounted for by the terms* $\mu p(x)$ *and* $\mu p(y)$.

Remark 12. *A closely related function to* $(m_1(x,y), m_2(x,y))$ *used extensively in reliability analysis is the bivariate mean residual life function (see Nair and Nair, 1988) defined as*

$$(E(X-x|X>x,Y>y), E(Y-y|X>x,Y>y)) = (m_1(x,y)-x, m_2(x,y)-y) \tag{25}$$

It follows that Theorems 4 through 9 provide useful characterizations of the concerned distributions by the form of the bivariate mean residual life function that can be easily deduced from the relationship (25).

4. Bivariate income gap ratio

In the context of applications in economics, in the univariate case, two functions that are closely related to $m(x)$ are the income gap ratio and the left proportional residual income. For a continuous non-negative random variable Z which represents the income of a population, those with income exceeding x are deemed to be affluent or rich. We call $Z=x$ to be the affluence line. Then $\bar{G}(z)=P(Z>z)$ represents the proportion of rich in the population. The proportion of rich, their average income and the measures of income inequality are important indices discussed in connection with income analysis and also for comparison between the rich and poor. Of these, Sen (1988) defines the income gap ratio among the affluent as

$$i(x) = 1 - \frac{x}{E(X|X>x)} \tag{26}$$

TABLE 2
Bivariate income gap ratios

Distribution	$(i_1(x,y), i_2(x,y))$
$\bar{F}_1(x,y)$	$\left(\frac{\mu_X-1}{\mu_X}, \frac{\mu_Y-1}{\mu_Y}\right)$
$\bar{F}_2(x,y)$	$\left(\frac{\mu_X(y)-1}{\mu_X(y)}, \frac{\mu_Y(x)-1}{\mu_Y(x)}\right)$
$\bar{F}_3(x,y)$	$\left(\frac{(\mu-1)(x+y-2)}{\mu x+(\mu-1)(y-1)}, \frac{(\mu-1)(x+y-2)}{\mu x+(\mu-1)(y-1)}\right)$
$\bar{F}_{11}(x,y)$	$\left(\frac{(\mu-1)(x+y+xy-1)}{\mu x(y+1)+(\mu-1)(y-1)}, \frac{(\mu-1)(x+y+xy-1)}{\mu x(y+1)+(\mu-1)(y-1)}\right)$
$\bar{F}_{12}(x,y)$	$\left(\frac{(\mu-1)(x(q-y)+q(y-1))}{\mu x(q-y)+q(\mu-1)(y-1)}, \frac{(\mu-1)(y(q-x)+q(x-1))}{\mu x(q-y)+q(\mu-1)(y-1)}\right)$

The measure $i(x)$ is used in defining indices of affluence in Sen (1988). On the other hand, Belzunce *et al.* (1998) defined the mean left proportional residual income($MLPRI$) as

$$l(x) = E\left(\frac{X}{x}|X > x\right) = 1 - \frac{1}{i(x)}. \tag{27}$$

We propose bivariate generalization of these concepts. For a non-negative random vector (X, Y), the bivariate income gap ratio is defined by the vector

$$\begin{aligned}(i_1(x,y), i_2(x,y)) &= (1 - \frac{x}{E(X|X > x, Y > y)}, 1 - \frac{y}{E(Y|X > x, Y > y)}) \\ &= (1 - \frac{x}{m_1(x,y)}, 1 - \frac{y}{m_2(x,y)}). \end{aligned} \tag{28}$$

Equation (28) shows that there is one-to-one relationship between $(i_1(x,y), i_2(x,y))$ and $(m_1(x,y), m_2(x,y))$, so that each determine other and the corresponding distribution uniquely. The functional forms of $(i_1(x,y), i_2(x,y))$ characterizing some members of our family are given in Table 2.
The bivariate generalization of $MLPRI$ is proposed as the vector

$$\begin{aligned}(l_1(x,y), l_2(x,y)) &= \left(E\left(\frac{X}{x}|X > x, Y > y\right), E\left(\frac{Y}{y}|X > x, Y > y\right)\right) \\ &= (\frac{m_1(x,y)}{x}, \frac{m_2(x,y)}{y}) \end{aligned} \tag{29}$$

The calculation of $(l_1(x,y), l_2(x,y))$ is easily facilitated from those of $(m_1(x,y), m_2(x,y))$. Thus the characterizations established in Section 3 using $(m_1(x,y), m_2(x,y))$ can be translated in terms of $(l_1(x,y), l_2(x,y))$.

5. Bivariate generalized failure rate

For a non-negative random variable Z, the generalized failure rate is given by

$$r(x) = -x\frac{dlog\bar{G}(x)}{dx} \tag{30}$$

Lariviere and Porteus (2001) and Lariviere (2006) discussed the properties of $r(x)$ and its applications in operations management. The well known income model derived by Singh and Maddala (1976) is based on a relationship between $r(x)$ and $\bar{G}(x)$ as

$$r(x) = \alpha x^{\beta}(\bar{G}(x))^{\gamma}, \alpha, \beta, \gamma > 0.$$

For a non-negative random vector (X, Y), the bivariate generalized failure rate vector is defined by

$$(r_1(x,y), r_2(x,y)) = \left(-x\frac{\partial \log \bar{F}(x,y)}{\partial x}, -y\frac{\partial \log \bar{F}(x,y)}{\partial y}\right). \tag{31}$$

There exists an identity connecting $(r_1(x,y), r_2(x,y))$ and $(m_1(x,y), m_2(x,y))$. Differentiating

$$m_1(x,y) = x + \frac{1}{\bar{F}(x,y)}\int_x^{\infty} \bar{F}(t,y)\, dt$$

with respect to x and rearranging terms,

$$\bar{F}(x,y)\frac{\partial m_1(x,y)}{\partial x} = (m_1(x,y) + x)\,\frac{\partial \bar{F}(x,y)}{\partial x}.$$

This gives

$$r_1(x,y) = \frac{x\frac{\partial m_1(x,y)}{\partial x}}{m_1(x,y) - x} \tag{32}$$

and similarly

$$r_2(x,y) = \frac{y\frac{\partial m_2(x,y)}{\partial y}}{m_2(x,y) - y}. \tag{33}$$

A redeeming feature of $(r_1(x,y), r_2(x,y))$ is that it allows simple analytically tractable expression for various distributions in the bivariate Pareto family, while the other functions can be expressed in terms of special functions only for many members. See Table 3 for expressions of $(r_1(x,y), r_2(x,y))$. It may be noted that the characterizations developed in Section 3 can be transformed in terms of $(r_1(x,y), r_2(x,y))$.

From Theorems 4 and 5 and the above deliberations, the following result is apparent.

Theorem 13. *The following statements are equivalent:*

a. (X, Y) *possesses* $BDP-1$ *(*$BDP-2$*)*

b. $(i_1(x,y), i_2(x,y))$ *is constant(locally constant)*

c. $(l_1(x,y), l_2(x,y))$ *is constant(locally constant)*

d. $(r_1(x,y), r_2(x,y))$ *is constant(locally constant)*

e. (X, Y) *is distributed as* $\bar{F}_1(x,y)$*(*$\bar{F}_4(x,y)$*)*

(Local constancy of a vector means that the first component is independent of x *and the second component is independent of* y*).*

TABLE 3
Bivariate generalized failure rates

Distribution	$(r_1(x,y), r_2(x,y))$
$\bar{F}_1(x,y)$	$\left(a = \frac{\mu_X}{\mu_X-1}, b = \frac{\mu_Y}{\mu_Y-1}\right)$
$\bar{F}_2(x,y)$	$\left(\frac{ax^{a\alpha}}{x^{a\alpha}+y^{b\alpha}-1}, \frac{by^{b\alpha}}{x^{a\alpha}+y^{b\alpha}-1}\right)$
$\bar{F}_3(x,y)$	$\left(\frac{ax}{x+y-1}, \frac{by}{x+y-1}\right)$
$\bar{F}_4(x,y)$	$\left(\frac{\mu_X(y)}{\mu_X(y)-1}, \frac{\mu_Y(x)}{\mu_Y(x)-1}\right)$
$\bar{F}_5(x,y)$	$\left(\frac{\sigma\alpha x^\alpha(1+y^\beta)}{(x^\alpha+y^\beta+x^\alpha y^\beta-1)}, \frac{\sigma\alpha y^\beta(1+x^\alpha)}{(x^\alpha+y^\beta+x^\alpha y^\beta-1)}\right)$
$\bar{F}_6(x,y)$	$(ca^c(\log x)^{c-1}(1-(b\log y)^c), cb^c(\log y)^{c-1}(1-(a\log x)^c))$
$\bar{F}_7(x,y)$	$(1+2x^a y^b - x^a - y^b)^{-1}\left(ax^a(2y^b-1), by^b(2x^a-1)\right)$
$\bar{F}_8(x,y)$	$\lambda^{\alpha-1}\left\{\lambda^\alpha a^\alpha\,(\log x)^\alpha + \lambda^\alpha b^\alpha(\log y)^\alpha\right\}^{1-\frac{1}{\alpha}}\,(a^\alpha(\log x)^{\alpha-1}, b^\alpha(\log y)^{\alpha-1})$
$\bar{F}_9(x,y)$	$\left[pq + (x^{\lambda a}-q)(y^{\lambda b}-q)\right]^{-1} ap\left(x^{\lambda a}(y^{\lambda b}-q), y^{\lambda b}(x^{\lambda a}-q)\right)$
$\bar{F}_{11}(x,y)$	$a(x+y+xy-1)^{-1}\,(x(1+y), y(1+x))$
$\bar{F}_{12}(x,y)$	$ap\,[pq+(x-q)(y-q)]^{-1}\,(x(y-q), y(x-q))$

6. Conclusion

In this paper, we have developed characterizations of a family of bivariate Pareto distributions. The well known dullness property was extended to the bivariate set up and characterizations of bivariate Pareto distributions using this property were derived. The measures of income inequality such as income gap ratio and mean left proportional residual income were proposed and studied in the bivariate case. The generalized failure rate was extended to the bivariate set up and characterizations using this concept were derived. The properties of the family of bivariate Pareto distribtions using copula theory are yet to be studied. The work in this direction will be reported elsewhere.

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Summary

In the present paper, we study properties of a family of bivariate Pareto distributions. The well known dullness property of the univariate Pareto model is extended to the bivariate setup. Two measures of income inequality viz. income gap ratio and mean left proportional residual income are defined in the bivariate case. We also introduce bivariate generalized failure rate useful in reliability analysis. Characterizations, using the above concepts, for various members of the family of bivariate Pareto distributions are derived.

Keywords: Bivariate Pareto distributions; characterization; dullness property; income gap ratio; generalized failure rate.