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ON HYBRID CENSORED INVERSE LOMAX DISTRIBUTION: APPLICATION TO THE SURVIVAL DATA

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1. Introduction

In survival analysis, the study of the lifetime of any random phenomenon is an extensive work to explain the characteristics of existing phenomenon. Several life time models namely exponential, gamma, Weibull, etc. are introduced in market to illustrate the real pattern of failure data in medical as well as in engineering sciences. The generalized version of these models are also advocated and well justified for the different situations of the failure rate behavior, see Ebrahimi (1990). The Lomax distribution is one of these and frequently used in economics, geography, econometrics and medical fields, see; Chandrasekar *et al.* (2002), Kleiber and Kotz (2003), Kleiber (2004).

The considered distribution belongs to inverted family of distributions and found to be very flexible to analyze the situation where the non-monotonicity of the failure rate has been realized, see Singh *et al.* (2012). If a random variable Y has Lomax distribution, then $X = \frac{1}{Y}$ has an Inverse Lomax distribution (ILD). It has been used to obtain the Lorenz ordering relationship among ordered statistics; Kleiber (2004). Besides this, it has also lots of applications in stochastic modeling, economics and actuarial sciences, see Kleiber and Kotz (2003). Kleiber (2004) have implemented this model on geophysical data, particularly on the sizes of land fires in California state of US. Rahman *et al.* (2013) have discussed the estimation and prediction problems for the inverse Lomax distribution via Bayesian approach. Yadav *et al.* (2016) have used this distribution for reliability estimation based on Type-II censored observations. But no one has paid attention about

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the consideration of hybrid censored observation from ILD in both classical and Bayesian approach. Therefore, the authors have considered this problem under investigation. The probability density function and cumulative distribution function of ILD are given by the following equations.

$$f(x, \alpha, \beta) = \frac{\alpha\beta}{x^2}\left(1 + \frac{\beta}{x}\right)^{-(1+\alpha)} \quad ; \; x \geq 0, \alpha, \beta > 0 \tag{1}$$

where, α is the shape parameter and β is the scale parameter of the distribution.

$$F(x) = \left(1 + \frac{\beta}{x}\right)^{-\alpha} \quad ; x \geq 0, \alpha, \beta > 0 \tag{2}$$

respectively.

The reliability and hazard functions, denoted as $R(t)$ and $H(t)$ of the ILD for specified values of t are given in following equations,

$$R(t) = 1 - \left(1 + \frac{\beta}{t}\right)^{-\alpha} \quad ; t > 0 \tag{3}$$

and

$$H(t) = \frac{\alpha\beta(1 + \frac{\beta}{t})^{-(1+\alpha)}}{t^2\left(1 - (1 + \frac{\beta}{t})^{-\alpha}\right)} \quad ; t > 0 \tag{4}$$

In industrial Statistics, censored observations are preferred due to cost, time or some other constraints. Hence, the variety of censoring schemes are used to obtain the censored observations. In practice most commonly used censoring schemes are Type-I and Type-II censoring schemes. In any life time experiments, experimenters are unable to investigate each experimental units. Therefore, generally experiments are terminated at prefixed time T or after getting predetermined number of failure R. In Type-I censoring scheme the termination time T is fixed and number of failure R is random and vice versa for Type-II censoring scheme respectively.

Hybrid censoring scheme is the mixture of Type-I and Type-II censoring schemes and it can be described as follows; Suppose n identical units are put on test and test is terminated when a pre-chosen number R out of n items are failed, or when a prefixed time T on the test has been obtained. Such censoring scheme is called as hybrid censoring scheme and it was introduced by Epstein (1954). This censoring scheme is quite useful in reliability acceptance plan, see MIL-STD-781-C (1977). Many authors have used this censoring scheme for their different purpose; see Draper and Guttman (1987), Chen and Bhattacharya (1988), Ebrahimi (1990) and the references cited therein. Hybrid censoring scheme is also comprises in two parts due to specification of censoring parameters R and T. In Type-I hybrid censoring scheme; suppose n units put on test, then experiment is terminated at the random time $T^* = Min(X_{R:n}, T)$, where R and T are prefixed numbers and $x_{R:n}$ is the time of R^{th} failure in a sample of size n. Type-I hybrid censoring scheme has its own limitations as conventional Type-I censoring. The main demerits of Type-I hybrid censoring scheme, the number of observed failures is at least one and there may be very few failures at the termination time of the experiment. Thus, it has the adverse effect over the efficiency of the estimators. Therefore, Childs *et al.* (2003) introduced a new type of censoring scheme which is an alternative to the Type-I hybrid censoring scheme, called as Type-II hybrid censoring scheme. In this censoring scheme, we terminate the experiments at the random time $T^* = Max(X_{R:n}, T)$. The advantage of this scheme is that at least R failures are observed at the end of the experiment. Fairbanks *et al.* (1982) have obtained the exact distribution of the maximum likelihood estimator (MLE) of the mean and interval estimates in one-parameter exponential distribution based on a hybrid type II censored data. Banerjee and Kundu (2008) considered the statistical inference of the two-parameter Weibull distribution based on Type-II hybrid censored samples. Several works are available in the literature, see for references Gupta and Kundu (1998), Kundu and Pradhan (2009), Dube *et al.* (2011) etc.

The organization of the paper is as follows; In Section 1, we described the problem and hybrid censoring scheme. The maximum likelihood estimation is discussed in Section 2. Bayesian estimation procedure is carried out in Section 3. The numerical illustration of the considered methodology for the ILD are provided by considering two data set in Section 4. Finally, the conclusion of the paper is given in Section 5.

2. Maximum Likelihood Estimation

Case 1: Under type-I hybrid censored data (HCD-I)

In this section, we assume that the data are Type-I hybrid censored, then we have the one of the following types observations;

$$Data(\underline{X}) = \begin{cases} x_{1:n} < x_{2:n} < \cdots < x_{r:n} \; when \; x_{r:n} \leq T \\ x_{1:n} < x_{2:n} < \cdots < x_{d:n} \; when \; x_{r:n} > T \end{cases} \tag{5}$$

where, d denotes the number of failure observed before time T. Thus, the likelihood function for given p.d.f (1) under Type-I HCD is written as;

$$L(\alpha, \beta|\underline{x}) = \begin{cases} \dfrac{n!}{(n-r)!}\alpha^r \beta^r \left[1 - \left(1 + \dfrac{\beta}{x_{r:n}}\right)^{\alpha}\right]^{(n-r)} \prod\limits_{i=1}^{r} x_i^{-2} \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \qquad\qquad\qquad\qquad\qquad\qquad when \; x_{r:n} \leq T \\ \\ \dfrac{n!}{(n-d)!}\alpha^d \beta^d \left[1 - \left(1 + \dfrac{\beta}{x_{d:n}}\right)^{\alpha}\right]^{(n-d)} \prod\limits_{i=1}^{d} x_i^{-2} \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \qquad\qquad\qquad\qquad\qquad\qquad when \; x_{r:n} > T \end{cases} \tag{6}$$

therefore, the combined likelihood function is given as;

$$L(\alpha, \beta|\underline{x}) = \frac{n!}{(n-m)!}\alpha^m \beta^m \left[1 - \left(1 + \frac{\beta}{R}\right)^{\alpha}\right]^{(n-m)} \prod_{i=1}^{m} x_i^{-2} \left(1 + \frac{\beta}{x_i}\right)^{-(1+\alpha)} \tag{7}$$

where;

$$m = \begin{cases} r & when \;\; x_{r:n} \leq T \\ d & when \;\; x_{r:n} > T \end{cases} \tag{8}$$

and

$$R = \begin{cases} x_{r:n} & when \;\; x_{r:n} \leq T \\ x_{d:n} & when \;\; x_{r:n} > T \end{cases} \tag{9}$$

Hence, the log-likelihood function is written as;

$$L = \ln L(\alpha, \beta|\underline{x}) = Const. + m\ln\alpha + m\ln\beta - \\ 2\sum_{i=1}^{m}\ln x_i - (1+\alpha)\sum_{i=1}^{m}\ln\left(1+\frac{\beta}{x_i}\right) + (n-m)\ln\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right] \tag{10}$$

now for MLE's of the parameters, we differentiate the log-likelihood function w.r.t to the parameter and equate to zero. Then, we have two likelihood equations which are obtained in implicit form. Therefore, N-R method is used to secure MLE's. Hence,

$$\frac{\partial L}{\partial \alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m}\ln\left(1+\frac{\beta}{x_i}\right) - \frac{(n-m)\left(1+\frac{\beta}{R}\right)^{\alpha}}{1-\left(1+\frac{\beta}{R}\right)^{\alpha}}\ln\left(1+\frac{\beta}{R}\right) = 0 \tag{11}$$

and

$$\frac{\partial L}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m}\frac{1+\alpha}{x_i+\beta} - \frac{(n-m)\alpha\left(1+\frac{\beta}{R}\right)^{\alpha-1}}{R\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]} = 0 \tag{12}$$

Let $\hat{\alpha}$ and $\hat{\beta}$ are the MLE's of the parameters, then the MLE's of the reliability and hazard functions are given by using the invariance properties of MLE, which are obtained as;

$$\hat{R}(t) = 1 - \left(1+\frac{\hat{\beta}}{t}\right)^{-\hat{\alpha}} \quad and \quad \hat{H}(t) = \frac{\hat{\alpha}\hat{\beta}(1+\frac{\hat{\beta}}{t})^{-(1+\hat{\alpha})}}{t^2\left[1-(1+\frac{\hat{\beta}}{t})^{-\hat{\alpha}}\right]}$$

The exact distribution of the maximum likelihood estimators are not available, thus, we derived the 95% asymptotic confidence interval based on fisher information matrix. The Fisher information matrix can be obtained by using equation (10). Thus we have

$$I(\hat{\alpha}, \hat{\lambda}) = \begin{pmatrix} -\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 L}{\partial \beta \partial \alpha} & -\frac{\partial^2 L}{\partial \beta^2} \end{pmatrix}_{(\hat{\alpha}, \hat{\beta})}$$

where, the derivatives are given as,

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{m}{\alpha^2} + \frac{(n-m)\left(1+\frac{\beta}{R}\right)^{\alpha}\left[\ln\left(1+\frac{\beta}{R}\right)\right]^2}{\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]}\left\{1+\frac{\left(1+\frac{\beta}{R}\right)^{\alpha}}{\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]}\right\}$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \frac{\partial^2 L}{\partial \beta \partial \alpha} = -\sum_{i=1}^{m}\frac{1}{(x_i+\beta)} + \frac{(n-m)\alpha\left(1+\frac{\beta}{R}\right)^{\alpha-1}}{R\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]}\left\{1+\ln\left(1+\frac{\beta}{R}\right)^{\alpha}+\frac{\left(1+\frac{\beta}{R}\right)^{\alpha}\ln\left(1+\frac{\beta}{R}\right)^{\alpha}}{\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]^2}\right\} \tag{13}$$

$$\frac{\partial^2 L}{\partial \beta^2} = -\frac{m}{\beta^2} + \sum_{i=1}^{m}\frac{1+\alpha}{(x_i+\beta)^2} + \frac{(n-m)\alpha\left(1+\frac{\beta}{R}\right)^{\alpha-2}}{R^2\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]}\left\{(\alpha-1)-\frac{\alpha\left(1+\frac{\beta}{R}\right)^{\alpha}}{\left[1-\alpha\left(1+\frac{\beta}{R}\right)^{\alpha}\right]}\right\}$$

All the derivatives are evaluated at the point $(\hat{\alpha}, \hat{\beta})$. The above matrix can be inverted to obtain the estimate of the asymptotic variance-covariance matrix of the MLEs and diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\beta})$ provides asymptotic variance of α and β respectively. Then by using large sample theory a two sided $100(1-\delta)\%$ approximate confidence interval for α and β are constructed as;

$$[\hat{\alpha}_L, \hat{\alpha}_U] = \hat{\alpha} \mp Z_{\delta/2}\sqrt{var(\hat{\alpha})}, \quad [\hat{\beta}_L, \hat{\beta}_U] = \hat{\beta} \mp Z_{\delta/2}\sqrt{var(\hat{\beta})}$$

respectively.

Case 2: Under Type-II hybrid censored data (HCD-II)

If the data is Type-II hybrid censored, then we have one of the following types observations;

$$Data(\underline{\mathrm{X}}) = \begin{cases} x_{1:n} < x_{2:n} < \cdots < x_{r:n} \; when \; x_{r:n} \geq T \\ x_{1:n} < x_{2:n} < \cdots < x_{d:n} \; when \; x_{r:n} < T \\ x_{1:n} < x_{2:n} < \cdots < x_{n:n} \; complete \; data \end{cases} \tag{14}$$

where, $r \leq d \leq n$ denotes the number of failure observed before time T. Thus, the likelihood function under Type-II HCD is given as;

$$L(\alpha,\beta|\underline{x}) = \begin{cases} \dfrac{n!}{(n-r)!}\alpha^r\beta^r\left[1-\left(1+\dfrac{\beta}{x_{r:n}}\right)^{\alpha}\right]^{(n-r)}\prod\limits_{i=1}^{r} x_i^{-2}\left(1+\dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \qquad\qquad when\ x_{r:n}\geq T \\ \dfrac{n!}{(n-d)!}\alpha^d\beta^d\left[1-\left(1+\dfrac{\beta}{x_{d:n}}\right)^{\alpha}\right]^{(n-d)}\prod\limits_{i=1}^{d} x_i^{-2}\left(1+\dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \qquad\qquad when\ x_{r:n}< T \\ \alpha^n\beta^n\prod\limits_{i=1}^{n} x_i^{-2}\left(1+\dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \quad whenCompletedataobserved \end{cases} \tag{15}$$

therefore, the combined likelihood function is given as;

$$L(\alpha,\beta|\underline{x}) = \frac{n!}{(n-m^*)!}\alpha^{m^*}\beta^{m^*}\left[1-\left(1+\frac{\beta}{R}\right)^{\alpha}\right]^{(n-m^*)}\prod_{i=1}^{m^*} x_i^{-2}\left(1+\frac{\beta}{x_i}\right)^{-(1+\alpha)} \tag{16}$$

where;

$$m^* = \begin{cases} r \ \ when \ \ x_{r:n}\geq T \\ d \ \ when \ \ x_{r:n}< T \\ n \ \ when \ \ x_{r:n}<\cdots<x_{d:n}<T \end{cases} \tag{17}$$

All the other mathematical expressions can be obtained by replacing m by m^* in the expression of HCD-I.

3. Bayesian Estimation

In this section, we considered Bayes procedure to obtain the point and interval estimates of the parameters α, β in presence of hybrid censored data. The Bayes estimators are derived under Jeffrey's non-informative priors with squared error loss function. The considered priors are improper but they leads the proper posterior. Thus, the joint prior is given as;

$$\pi_1(\alpha,\beta) \propto \frac{1}{\alpha\beta} \qquad ;\alpha,\ \beta>0 \tag{18}$$

Then by using (7), (16) and (18) the joint posterior under considered cases (HCD-I, HCD-II) are given by;

$$p(\alpha,\beta|\underline{x}) \propto \begin{cases} \alpha^m\beta^m\left[1-\left(1+\dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m)}\prod\limits_{i=1}^{m} x_i\left(1+\dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \alpha^{m^*}\beta^{m^*}\left[1-\left(1+\dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m^*)}\prod\limits_{i=1}^{m^*} x_i\left(1+\dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \end{cases} \tag{19}$$

Now, after simplification the Bayes estimators of α, β, reliability function $R(t)$ and Hazard function $H(t)$ under SELF are simply obtained by taking the mean of the posterior distribution. From above, we noticed that the posterior distributions are not in explicit form. Therefore, it seems to be tedious to calculate the posterior expectations analytically. Therefore, Markov Chain Monte Carlo (MCMC) method has implemented for obtaining the approximate solution of the posterior expectations.

3.1. Markov Chain Monte Carlo Method

Here, we used Metropolis Hastings algorithm to extract the sample from posterior distribution to obtain the approximate Bayes estimates of the parameters and reliability characteristics. Further, we have also constructed 95% highest posterior density (HPD) credible intervals of the parameters on the basis of generated posterior sample. For more detail about MCMC, see Geman and Geman (1984), Upadhyay *et al.* (2001), Smith and Roberts (1993). Thus for implementation of the metropolis Hastings algorithm see Hastings (1970), the full conditional posterior densities for α and β under HCD-I and HCD-II are written as;

$$p_1(\alpha|\underline{x}, \beta) \propto \begin{cases} \alpha^n \left[1 - \left(1 + \dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m)} \prod\limits_{i=1}^{m} x_i \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \beta^m \left[1 - \left(1 + \dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m)} \prod\limits_{i=1}^{m} x_i \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \end{cases} \tag{20}$$

and

$$p_2(\alpha|\underline{x}, \beta) \propto \begin{cases} \alpha^n \left[1 - \left(1 + \dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m^*)} \prod\limits_{i=1}^{m^*} x_i \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \\ \beta^{m^*} \left[1 - \left(1 + \dfrac{\beta}{R}\right)^{\alpha}\right]^{(n-m^*)} \prod\limits_{i=1}^{m^*} x_i \left(1 + \dfrac{\beta}{x_i}\right)^{-(1+\alpha)} \end{cases} \tag{21}$$

respectively.

The following steps are used to draw the posterior samples from their respective full conditional density;

- Set the initial values of α and β say (α_0, β_0)
- Set l=1
- Generate posterior sample for α and β from (20) and (21) respectively.
- Repeat step 2, for all $l = 1, 2, 3, \cdots, N$ and obtained (α_1, β_1), (α_2, β_2), $\cdots$, (α_N, β_N)

  After obtaining the posterior samples the Bayes estimate of the parameters, reliability function and hazard function under SELF are the mean of the corresponding posterior samples. Therefore, we have,

$$\hat{\alpha}_B \approx E(\alpha|\underline{x}) = \frac{1}{N}\sum_{l=1}^{N}\alpha_l, \quad \hat{\beta}_B \approx E(\beta|\underline{x}) = \frac{1}{N}\sum_{l=1}^{N}\beta_l$$

and

$$\hat{R}(t) = \frac{1}{N}\sum_{l=1}^{N}\left[1 - \left(1+\frac{\beta_l}{t}\right)^{-\alpha_l}\right], \quad \hat{H}(t) = \frac{1}{N}\sum_{l=1}^{N}\frac{\alpha_l\beta_l(1+\frac{\beta_l}{t})^{-(1+\alpha_l)}}{t^2\left(1-(1+\frac{\beta_l}{t})^{-\alpha_l}\right)}$$

- To construct 95% HPD credible intervals for α and β based on MCMC samples, order $\alpha_1, \alpha_2, ..., \alpha_N$ as $\alpha_1 < \alpha_2 < ... < \alpha_N$ and $\beta_1, \beta_2, \cdots, \beta_N$ as $\beta_1 < \beta_2 < ... < \beta_N$. Then $100(1-\delta)\%$ credible intervals of α and β are

  $$(\alpha_1, \alpha_{[N(1-\delta)+1]}), \cdots, (\alpha_{[N\delta]}, \alpha_N)$$

  and

  $$(\beta_1, \beta_{[N(1-\delta)+1]}), \cdots, (\beta_{[N\delta]}, \beta_N)$$

  Here $[x]$ denotes the greatest integer less than or equal to x. Then, the HPD credible interval is that interval which has the shortest length, for more details see Chen and Shao (1999) .

4. Numerical Illustration

In this section, we illustrate the proposed estimation procedure based on one survival data and one simulated data.

Bladder Cancer Data-I: It represents the remission times (in months) of a 128 bladder cancer patients. This data set was initially used by Lee and Wang (2003). The data is as follows:

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

The fitting of the above data set has checked for Inverse Lomax model over inverted exponential distribution (IED), generalized Inverted exponential distribution (GIED) and Inverse Weibull distribution (IWD); see Table 1. Table 1 lists the MLEs of the model parameters and the following statistics; Akaike information criterion (AIC), Bayesian information criterion (BIC) and log-likelihood (-Log L) values. These results show that the all the fitted life time models are provide good fitting but among these Inverse Lomax distribution has the lowest AIC, BIC and LogL values, and so it can be chosen as the best model. The empirical cumulative distribution function plot and Q-Q plot are also given to show the appropriateness of the considered data set for the Inverse Lomax distribution; see Figure 1

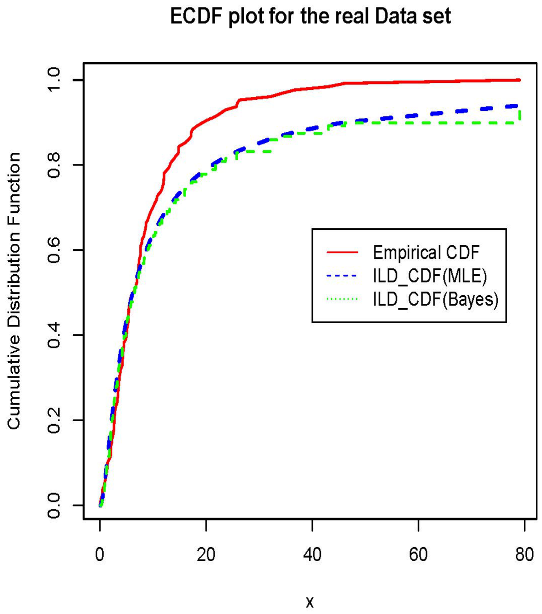

Figure 1 – Empirical cumulative distribution function plot for the data set-I.

and Figure 5. Q-Q plot shows that the considered data has non-monotone hazard rate pattern which suits for the ILD. For the above data set, the maximum likelihood estimates and Bayes estimates of the parameters are obtained using different choices of censoring parameters R and T, see Table 2. The reliability and hazard estimates are also evaluated for $t = 9$. The 95% asymptotic confidence and highest posterior density intervals are also constructed for the same combination of censoring parameters R and T.

Simulated Data-II: For the above considered model, a simulated data of size 100 is also taken for illustrative purpose of the study. The data is generated from $ILD(2.5, 3)$ using inverse cdf transformation method.

0.68, 1.08, 1.17, 1.18, 1.2, 1.38, 1.5, 1.66, 1.75, 1.9, 1.94, 2.03, 2.12, 2.16,

TABLE 1
Values of different adaptive measures of model discrimination

Models	ML Estimates	AIC	BIC	-LogL
$IED(\beta)$	[2.4847]	922.7646	925.6166	460.3823
$GIED(\alpha, \beta)$	[0.7463, 1.9945]	918.4050	924.1090	457.2024
$IWD(\alpha, \beta)$	[2.4311, 0.7521]	892.0015	897.7056	444.0008
$ILD(\alpha, \beta)$	[2.4621, 2.0023]	853.3514	859.0560	424.6757

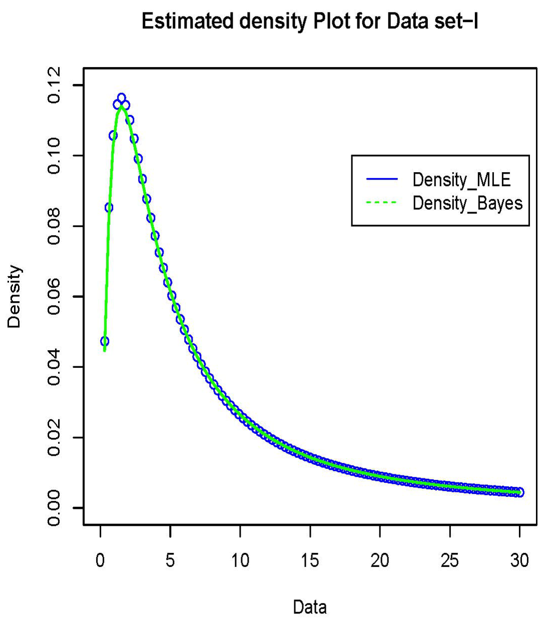

Figure 2 – Estimated density function plot for the data set-I.

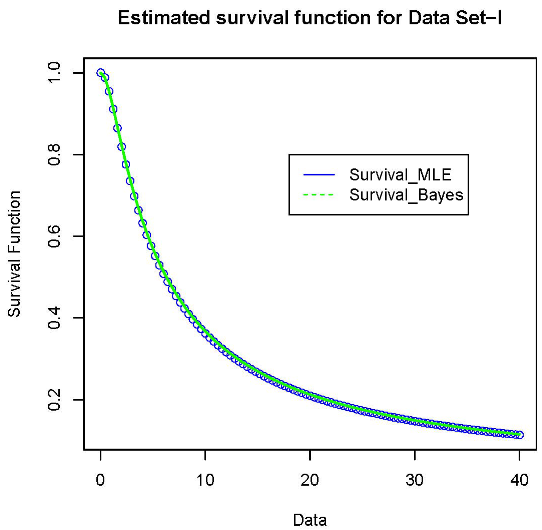

Figure 3 – Estimated survival function plot for the data set-I.

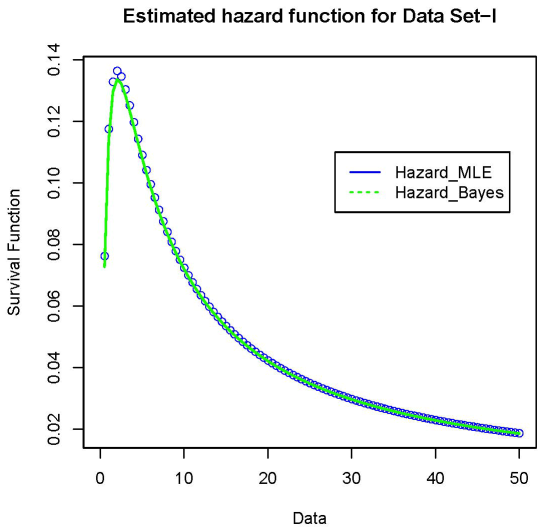

Figure 4 – Estimated hazard function plot for the data set-I.

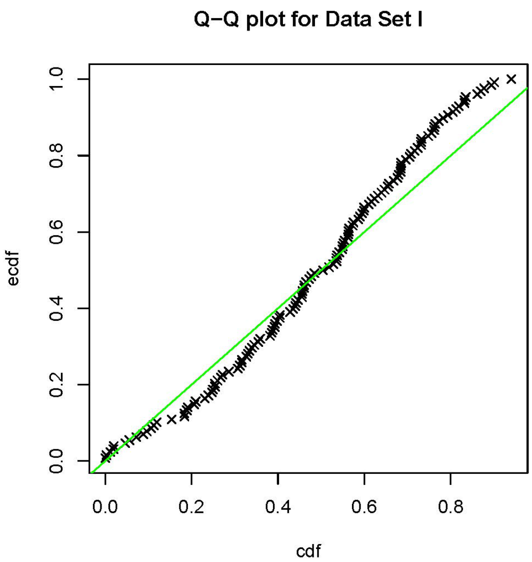

Figure 5 – QQ plot for the data set-I.

2.44, 2.58, 2.62, 2.64, 2.94, 3.03, 3.17, 3.21, 3.26, 3.39, 3.52, 3.53, 3.7, 3.76, 3.8, 3.86, 3.91, 3.96, 4.25, 4.33, 5.06, 5.07, 5.2, 5.49, 5.94, 6.77, 6.78, 6.82, 7.94, 8.05, 8.32, 8.69, 8.82, 8.85, 8.97, 9.14, 9.2, 9.7, 9.97, 10.1, 10.25, 10.27, 10.44, 10.59, 10.65, 10.75, 10.84, 11.36, 11.53, 11.55, 12.36, 13.27, 13.98, 14.01, 14.53, 16.42, 16.44, 17.07, 18.07, 18.11, 19.98, 22.21, 22.66, 23.24, 23.31, 23.8, 26.66, 29.7, 33.62, 35, 36.15, 41.75, 45.93, 50.88, 52.53, 61.31, 67.02, 67.8, 90.39, 97.82, 105.69, 114.02, 144.98, 150.84, 160.65, 239.36

Also for the simulated data, the maximum likelihood and Bayes estimates of the parameters are evaluated. The reliability and hazard estimates in both setup are also presented for $t = 20$ in Table 3. In table 4, the 95% asymptotic and HPD interval estimates for the parameters are provided.

5. Conclusion

In this paper, we proposed the estimation procedure for Inverse Lomax distribution under Type-I and Type-II hybrid censoring schemes. The mathematical expression for the maximum likelihood estimators and Bayes estimators are derived. The applicability of the considered model in real life has been illustrated based on bladder cancer data and simulated data and it was observed that the considered model is a good competitor of IED, GIED and ILD. Thus it can be used as a good alternative to these models whenever the situation of non-monotonicity of hazard rate is realized.

TABLE 2
Estimates of the parameters, reliability and hazard functions for different choices of censoring parameters R and T under Type-I hybrid and Type-II hybrid.

R	T	$\hat{\alpha}$	$\hat{\beta}$	$\hat{R}(t)$	$\hat{H}(t)$	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{R}(t)_B$	$\hat{H}(t)_B$
Type-I hybrid									
50	2	1.2180	9.0766	0.5724	4.1128	1.3694	6.9674	0.5362	4.5663
	4	1.7253	3.8813	0.4613	5.4636	1.7870	3.6601	0.4535	5.5543
	10	1.7759	3.6597	0.4544	5.5470	1.8281	3.4811	0.4474	5.6253
80	3	1.5837	4.6601	0.4836	5.1930	1.6556	4.2976	0.4722	5.3260
	6	1.9934	2.9189	0.4288	5.8536	2.0352	2.8265	0.4241	5.9018
	10	2.1522	2.5358	0.4139	6.0297	2.1887	2.4649	0.4092	6.0743
100	5	1.7836	3.6282	0.4534	5.5591	1.8310	3.4693	0.4470	5.6300
	7	1.9830	2.9504	0.4301	5.8389	2.0282	2.8440	0.4248	5.8942
	12	2.2635	2.3167	0.4046	6.1378	2.3003	2.2583	0.4006	6.1754
	15	2.3114	2.2328	0.4009	6.1807	2.3469	2.1748	0.3963	6.2215
120	14	2.3304	2.2011	0.3994	6.1971	2.3592	2.1535	0.3952	6.2332
	17	2.3702	2.1372	0.3965	6.2306	2.4048	2.0862	0.3924	6.2673
	25	2.4271	2.0516	0.3925	6.2763	2.4614	2.0043	0.3885	6.3115
Type-II hybrid									
50	2	1.7759	3.6597	0.4544	5.5470	1.8275	3.4960	0.4484	5.6163
	5	1.7836	3.6282	0.4534	5.5591	1.8302	3.4693	0.4469	5.6304
	7	1.9830	2.9504	0.4301	5.8389	2.0217	2.8533	0.4249	5.8910
80	5	2.1522	2.5358	0.4139	6.0297	2.1924	2.4567	0.4089	6.0778
	10	2.2164	2.4047	0.4084	6.0938	2.2534	2.3398	0.4040	6.1353
	12	2.2635	2.3167	0.4046	6.1378	2.3020	2.2511	0.3999	6.1810
100	10	2.3114	2.2328	0.4009	6.1807	2.3489	2.1699	0.3960	6.2244
	15	2.3683	2.1402	0.3966	6.2291	2.4060	2.0831	0.3922	6.2692
	20	2.4090	2.0781	0.3937	6.2621	2.4453	2.0283	0.3898	6.2977
120	20	2.4297	2.0478	0.3923	6.2783	2.4695	1.9935	0.3882	6.3164
	30	2.4457	2.0250	0.3912	6.2906	2.4811	1.9745	0.3869	6.3289
	45	2.4567	2.0097	0.3905	6.2989	2.4948	1.9587	0.3865	6.3355

TABLE 3
Classical and Bayes interval estimates of the parameters

		Type-I							
		Classical Interval				Bayes Interval			
R	T	$\hat{\alpha}_L$	$\hat{\alpha}_U$	$\hat{\lambda}_L$	$\hat{\lambda}_U$	$\hat{\alpha}^L$	$\hat{\alpha}^U$	$\hat{\lambda}^L$	$\hat{\lambda}^U$
50	2	0.4301	2.0060	0.0000	22.8656	1.1280	1.6330	4.3054	9.7186
	4	0.8520	2.5985	0.7695	6.9932	1.4989	2.0598	2.8040	4.5846
	10	0.9047	2.6471	0.9207	6.3986	1.5764	2.1207	2.6924	4.2232
80	3	0.7304	2.4370	0.2989	9.0213	1.3926	1.9374	3.1429	5.5161
	6	1.0290	2.9577	0.8952	4.9427	1.7151	2.3334	2.2605	3.4608
	10	1.1263	3.1781	0.8720	4.1997	1.8500	2.5049	1.9427	2.9574
100	5	0.9109	2.6563	0.9306	6.3258	1.5530	2.1038	2.6906	4.2327
	7	1.1918	3.3353	0.8412	3.7923	1.9812	2.6458	1.8096	2.6862
	12	1.2162	3.4066	0.8192	3.6465	2.0102	2.7051	1.7290	2.5852
	15	1.1918	3.3353	0.8412	3.7923	1.9760	2.6357	1.8153	2.6724
120	14	1.2272	3.4336	0.8134	3.5887	2.0127	2.7088	1.7288	2.5692
	17	1.2486	3.4917	0.7983	3.4762	2.0705	2.7898	1.6911	2.4969
	25	1.2788	3.5753	0.7763	3.3269	2.1120	2.8482	1.6135	2.3927
		Under Type-II Hybrid							
50	2	0.9047	2.6471	0.9207	6.3986	1.5551	2.1058	2.6563	4.2442
	5	0.9109	2.6563	0.9306	6.3258	1.5663	2.1208	2.7391	4.2382
	7	1.0352	2.9307	0.9421	4.9588	1.7259	2.3129	2.3133	3.4500
80	5	1.1263	3.1781	0.8720	4.1997	1.8900	2.5372	1.9638	2.9411
	10	1.1918	3.3353	0.8412	3.7923	1.9678	2.6414	1.8326	2.7161
	12	1.2162	3.4066	0.8192	3.6465	2.0060	2.7040	1.7552	2.5730
100	10	1.2471	3.4895	0.7979	3.4825	2.0377	2.7439	1.6784	2.4799
	15	1.2690	3.5489	0.7828	3.3734	2.0875	2.8121	1.6685	2.4463
	20	1.2803	3.5792	0.7754	3.3201	2.1099	2.8298	1.6295	2.3655
120	20	1.2887	3.6028	0.7691	3.2809	2.1241	2.8594	1.6181	2.3907
	30	1.2846	3.6173	0.7656	3.2741	2.1336	2.8465	1.6016	2.3229
	45	1.2946	3.6187	0.7653	3.2541	2.1191	2.8482	1.5809	2.3204

TABLE 4
Estimates of the parameters, reliability and hazard functions for the simulated data set under different variations of the censoring parameters.

R	T	$\hat{\alpha}$	$\hat{\beta}$	$\hat{R}(t)$	$\hat{H}(t)$	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{R}(t)_B$	$\hat{H}(t)_B$
Under Type-I hybrid									
25	3	7.4044	0.7222	0.2310	17.1819	7.3807	0.7347	0.2301	17.1858
	5	7.9984	0.6573	0.2279	17.2448	7.9436	0.6741	0.2282	17.2325
40	7	4.3176	1.4320	0.2581	16.5827	4.3592	1.4221	0.2572	16.5949
	6	5.6319	1.0125	0.2428	16.9261	5.6273	1.0211	0.2416	16.9353
55	9	4.0476	1.5626	0.2625	16.4815	4.0754	1.5547	0.2618	16.4910
	11	4.3525	1.4183	0.2578	16.5912	4.3863	1.4113	0.2570	16.6022
70	12	5.4655	1.0565	0.2452	16.8800	5.4919	1.0547	0.2433	16.9023
	17	4.9036	1.2132	0.2508	16.7528	4.9575	1.2060	0.2496	16.7683
85	23	4.9061	1.2124	0.2508	16.7535	4.9452	1.2098	0.2499	16.7631
	35	4.8896	1.2176	0.2510	16.7495	4.9226	1.2126	0.2493	16.7690
95	68	4.8681	1.2245	0.2512	16.7441	4.9025	1.2201	0.2498	16.7602
	105	4.8373	1.2344	0.2515	16.7363	4.8837	1.2267	0.2501	16.7546
Under Type-II hybrid									
25	3	7.9984	0.6573	0.2279	17.2448	7.9688	0.6660	0.2283	17.2349
	5	10.8873	0.4562	0.2177	17.4476	10.8557	0.4590	0.2177	17.4446
40	6	4.3176	1.4320	0.2581	16.5827	4.3094	1.4303	0.2569	16.5951
	7	4.8500	1.2280	0.2510	16.7452	4.8623	1.2259	0.2505	16.7495
55	9	4.3525	1.4183	0.2578	16.5912	4.3527	1.4164	0.2570	16.5992
	11	5.2219	1.1194	0.2475	16.8283	5.2263	1.1178	0.2466	16.8383
70	12	4.9036	1.2132	0.2508	16.7528	4.9148	1.2098	0.2500	16.7617
	17	5.0409	1.1711	0.2494	16.7864	5.0479	1.1679	0.2484	16.7974
85	23	4.9250	1.2064	0.2506	16.7583	4.9339	1.2031	0.2497	16.7684
	50	4.8444	1.2321	0.2514	16.7381	4.8519	1.2307	0.2508	16.7441
95	110	4.8500	1.2303	0.2514	16.7395	4.8369	1.2285	0.2498	16.7569
	150	4.8626	1.2263	0.2513	16.7427	4.8734	1.2236	0.2506	16.7495

TABLE 5
Interval estimates of the parameters for the simulated data set for different variation of R and T.

Under Type-I Hybrid									
		Classical Interval				Bayes Interval			
R	T	$\hat{\alpha}_L$	$\hat{\alpha}_U$	$\hat{\lambda}_L$	$\hat{\lambda}_U$	$\hat{\alpha}^L$	$\hat{\alpha}^U$	$\hat{\lambda}^L$	$\hat{\lambda}^U$
25	3	0.0000	24.4781	0.0000	2.6653	5.7423	9.3947	0.5053	0.9865
	5	0.0000	26.1804	0.0000	2.3660	6.1495	9.7578	0.4545	0.8837
40	7	0.0000	14.4885	0.0000	2.8936	3.4069	5.3571	1.1927	1.6575
	6	0.0000	9.9483	0.0000	3.7350	4.3994	6.9624	0.7707	1.3220
55	9	0.0000	8.9152	0.0000	3.8921	3.2880	4.9862	1.3466	1.7867
	11	0.0000	9.7375	0.0000	3.5588	3.4967	5.4001	1.1680	1.6333
70	12	0.0000	13.1952	0.0000	2.8107	4.2795	6.8271	0.8052	1.3352
	17	0.0000	11.3073	0.0000	3.1009	3.8830	6.1839	0.9251	1.4641
85	23	0.0000	11.2828	0.0000	3.0883	3.9083	6.0879	0.9530	1.4795
	35	0.0000	11.2166	0.0000	3.0933	3.8485	6.1167	0.9611	1.4919
95	68	0.0000	11.1428	0.0000	3.1045	3.8277	6.0605	0.9576	1.4885
	105	0.0000	11.0495	0.0000	3.1247	3.7760	6.0282	0.9684	1.4960
Under Type-II Hybrid									
25	3	0.0000	26.1804	0.0000	2.3660	6.9636	9.0631	0.5115	0.8006
	5	0.0000	41.3595	0.0000	1.8591	9.9542	11.6752	0.3512	0.5564
40	6	0.0000	9.9483	0.0000	3.7350	3.4208	5.0818	1.3246	1.5276
	7	0.0000	11.6176	0.0000	3.2955	3.9994	5.8275	1.1066	1.3495
55	9	0.0000	9.7375	0.0000	3.5588	3.5131	5.1404	1.3096	1.5215
	11	0.0000	12.4162	0.0000	2.9434	4.3086	6.3320	0.9838	1.2554
70	12	0.0000	11.3073	0.0000	3.1009	3.9797	5.8753	1.0842	1.3385
	17	0.0000	11.7357	0.0000	3.0161	4.1779	6.0565	1.0434	1.3003
85	23	0.0000	11.3240	0.0000	3.0704	4.0253	5.8928	1.0769	1.3321
	50	0.0000	11.0747	0.0000	3.1212	3.9572	5.7761	1.1092	1.3643
95	110	0.0000	11.0873	0.0000	3.1162	3.9794	5.7546	1.1029	1.3469
	150	0.0000	11.1249	0.0000	3.1077	3.9280	5.7750	1.1029	1.3475

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Summary

In this paper, we proposed the estimation procedures to estimate the unknown parameters, reliability and hazard functions of Inverse Lomax distribution. The mathematical expressions for maximum likelihood and Bayes estimators are derived in presence of hybrid censoring scheme. In most of the cases, it has been seen that maximum likelihood and Bayes estimators of the parameters are not appear in explicit form. Hence, Newton-Raphson (N-R) method has been used to draw the maximum likelihood estimates of the parameters. The Bayes estimators are obtained under Jeffrey's non-informative prior for both shape α and scale λ using Markov Chain Monte Carlo (MCMC) technique. Further, we have also constructed the 95% asymptotic confidence interval based on maximum likelihood estimates (MLEs) and highest posterior density (HPD) credible intervals based on MCMC samples. Finally, two data sets have been used to demonstrate the proposed methodology.

Keywords: Parameters estimation; hybrid censoring; MCMC method