ON THE COVARIANCE OF RESIDUAL LIVES

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1. INTRODUCTION

The concept of residual life is centuries old and has been used extensively in various disciplines. Characteristics of residual life such as mean (Watson and Wells, 1961) median (Schittlein and Morrison, 1981) percentiles (Arnold and Brochett, 1983) second moment and variance (Gupta and Gupta, 1983) and partial means (Nair, 1987) coefficient of variation (Gupta and Kirmani, 2000) find a predominant role in modeling and analysis of life time data and in describing various notions of aging of equipments and devices. Mean residual life occurs naturally in other areas like optimal disposal of assets, renewal theory, branching process, dynamic programming, social sciences and in setting rates and benefits of life insurance. For details we refer to Guess and Proschan (1988).

The definition of mean residual life extends to higher dimensions in a natural way. In the bivariate case it can be conceived in vector form

$$(r_1(x_1,x_2), r_2(x_1,x_2))$$

where

$$r_i(x_1,x_2) = \mathrm{E}(X_i - x_i \,|\, X_1 > x_1, X_2 > x_2) \tag{1}$$

and (X_1, X_2) is a non-negative random vector representing the life times of components in a two-component system. It is well known that (1) determines the distribution of (X_1, X_2) uniquely. Many characterizations of life distributions based on the functional form of (1) are available in literature that helps to identify the distributions of life lengths. Some of the papers in this direction are Kotz and Shanbhag (1980), Zahedi (1985), Galambos and Kotz (1978), Nair and Nair (1988), Sankaran and Nair (1993).

An important aspect to be considered while modeling bivariate data on life times is the dependency structure between them, which can be measured in terms of the covariance. Since covariance between life times can also be studied in terms of their residual lives, a discussion of covariance of residual lives becomes relevant. It appears that much attention has not been devoted to study the properties and to investigate the role of covariance structure of the residual lives in determining the model. In this paper we define the product moment and covariance residual life of a two component system and obtain some of their properties and applications in characterization of life distributions.

In Section 2 we define product moment and covariance of residual lives in the bivariate case and establish some properties. The relationship between Basu's (1971) bivariate failure rate and the product moment of residual lives is analysed in Section 3. Finally in Section 4 we prove that the proportionality between product moment to the product of components of vector valued mean residual life defined in (1) is a characteristic property of the bivariate Lomax and beta laws.

2. DEFINITION AND PROPERTIES

Let (X_1, X_2) be a random vector that takes values in the positive octant $R_2^+ = \{(x_1, x_2) | x_1 > 0, x_2 > 0\}$ of the two dimensional space with absolutely continuous survival function

$$R(x_1, x_2) = P[X_1 > x_1, X_2 > x_2]$$

and density function $f(x_1, x_2)$ with $\mathrm{E}(X_1 X_2) < \infty$. Then the product moment residual life function (PMRL) of (X_1, X_2) is defined as

$$\mathrm{M}(x_1, x_2) = \mathrm{E}[(X_1 - x_1)(X_2 - x_2) | X_1 > x_1, X_2 > x_2] \quad (2)$$

The covariance residual life function (CVRL) is

$$\mathrm{C}(x_1, x_2) = \mathrm{M}(x_1, x_2) - r_1(x_1, x_2) r_2(x_1, x_2) \quad (3)$$

where $r_1(x_1, x_2)$ and $r_2(x_1, x_2)$ are as defined in (1). The function $\mathrm{M}(x_1, x_2)$ and $\mathrm{C}(x_1, x_2)$ satisfy the following properties.

Proposition 2.1

For a random vector (X_1, X_2) that takes value in the positive octant $R_2^+ = \{(x_1, x_2) | x_1 > 0, x_2 > 0\}$ of the two dimensional space with $\mathrm{E}(X_1 X_2) < \infty$

$$\mathrm{C}(0,0) = \mathrm{Cov}(X_1, X_2) \quad (4)$$

Proposition 2.2

For a random vector (X_1, X_2) that takes value in the positive octant $R_2^+ = \{(x_1, x_2) | x_1 > 0, x_2 > 0\}$ of the two dimensional space with $\mathrm{E}(X_1 X_2) < \infty$

the following partial differential equation of second order connects the survival function and PMRL

$$M(x_1,x_2)\frac{\partial^2 R(x_1,x_2)}{\partial x_1 \partial x_2}+\frac{\partial M(x_1,x_2)}{\partial x_1}\frac{\partial R(x_1,x_2)}{\partial x_2}+$$

$$\frac{\partial M(x_1,x_2)}{\partial x_2}\frac{\partial R(x_1,x_2)}{\partial x_1}+\left(\frac{\partial^2 M(x_1,x_2)}{\partial x_1 \partial x_2}-1\right)R(x_1,x_2)=0$$

Proof

From equation (2), it follows that

$$M(x_1,x_2)=[R(x_1,x_2)]^{-1}\int_{x_1}^{\infty}\int_{x_2}^{\infty}R(t_1,t_2)dt_1dt_2 \tag{5}$$

Differentiating with respect to x_2

$$M(x_1,x_2)\frac{\partial R(x_1,x_2)}{\partial x_2}+R(x_1,x_2)\frac{\partial M(x_1,x_2)}{\partial x_2}=-\int_{x_1}^{\infty}R(t_1,x_2)dt_1 \tag{6}$$

and further differentiation of (6) with respect to x_1 yields

$$M(x_1,x_2)\frac{\partial^2 R(x_1,x_2)}{\partial x_1 \partial x_2}+\frac{\partial M(x_1,x_2)}{\partial x_1}\frac{\partial R(x_1,x_2)}{\partial x_2}+$$

$$\frac{\partial M(x_1,x_2)}{\partial x_2}\frac{\partial R(x_1,x_2)}{\partial x_1}+\left(\frac{\partial^2 M(x_1,x_2)}{\partial x_1 \partial x_2}-1\right)R(x_1,x_2)=0\,.$$

Proposition 2.3

If $(h_1(x_1,x_2),h_2(x_1,x_2))$ is the bivariate failure rate (Johnson and Kotz, 1975) with

$$h_i(x_1,x_2)=-\frac{\partial \log R(x_1,x_2)}{\partial x_i},\ i=1,2,$$

then

$$\frac{\partial M(x_1,x_2)}{\partial x_1}=h_1(x_1,x_2)M(x_1,x_2)-r_2(x_1,x_2) \tag{7}$$

$$\frac{\partial M(x_1,x_2)}{\partial x_2}=h_2(x_1,x_2)M(x_1,x_2)-r_1(x_1,x_2) \tag{8}$$

Proof

From (1) it follows that

$$r_i(x_1,x_2)=\mathrm{E}(X_i-x_i\,|X_1>x_1,X_2>x_2)$$

so that

$$r_1(x_1,x_2)=\mathrm{E}(X_1-x_1\,|X_1>x_1,X_2>x_2)$$

$$=\ [\mathrm{R}(x_1,x_2)]^{-1}\int_{x_1}^{\infty}\ \int_{x_2}^{\infty}(X_1-t_1)f(t_1,t_2)dt_1dt_2$$

or

$$[\mathrm{R}(x_1,x_2)]r_1(x_1,x_2)=\int_{x_1}^{\infty}\mathrm{R}(t_1,x_2)dt_1 \tag{9}$$

Similarly proceeding for $r_2(x_1,x_2)$ we can see that

$$\mathrm{R}(x_1,x_2)r_2(x_1,x_2)=\int_{x_2}^{\infty}\mathrm{R}(x_1,t_2)dt_2 \tag{10}$$

Further from (5) we have

$$\mathrm{M}(x_1,x_2)[R(x_1,x_2)]^{-1}\frac{\partial\mathrm{R}(x_1,x_2)}{\partial x_2}+\frac{\partial\mathrm{M}(x_1,x_2)}{\partial x_2}=-[\mathrm{R}\ \ (x_1,x_2)]^{-1}\int_{x_1}^{\infty}\mathrm{R}(t_1,x_2)dt_1$$

and

$$\mathrm{M}(x_1,x_2)[R(x_1,x_2)]^{-1}\frac{\partial\mathrm{R}(x_1,x_2)}{\partial x_1}+\frac{\partial\mathrm{M}(x_1,x_2)}{\partial x_1}=-[\mathrm{R}\ \ (x_1,x_2)]^{-1}\int_{x_2}^{\infty}\mathrm{R}(x_1,t_2)dt_1\ .$$

Using (9) and (10) in the above expressions (7) and (8) follow.

Corollary 2.1

If $(h_1(x_1,x_2),h_2(x_1,x_2))$ is the bivariate failure rate (Johnson and Kotz, 1975) with

$$h_i(x_1,x_2)=\ -\frac{\partial\log\mathrm{R}(x_1,x_2)}{\partial x_i},i=1,2$$

then

$$r_1(x_1,x_2)\frac{\partial\mathrm{M}(x_1,x_2)}{\partial x_1}=\left(1+\frac{\partial r_1(x_1,x_2)}{\partial x_1}\right)\mathrm{M}(x_1,x_2)-r_1(x_1,x_2)r_2(x_1,x_2) \tag{11}$$

$$r_2(x_1,x_2)\frac{\partial\mathrm{M}(x_1,x_2)}{\partial x_2}=\left(1+\frac{\partial r_2(x_1,x_2)}{\partial x_2}\right)\mathrm{M}(x_1,x_2)-r_1(x_1,x_2)r_2(x_1,x_2) \tag{12}$$

Proof

The proof follows from Proposition (4) and observing that differentiation of (8) and (9) with respect to x_1 and x_2 respectively yields

$$h_i(x_1,x_2) = [r_2(x_1,x_2)]^{-1}\left(1+\frac{\partial lr_i(x_1,x_2)}{\partial x_i}\right), i=1,2.$$

Note

Equations (11) and (12) enable the calculation of $\mathrm{M}(x_1,x_2)$ and hence $C(x_1,x_2)$ in terms of the mean residual life (MRL). Thus unlike the usual covariance which cannot be determined from the mean values alone, CVRL can be evaluated from the knowledge of the MRL. This fact leaves scope for characterizing life distributions by the functional relationship between $\mathrm{M}(x_1,x_2)$ and $(r_1(x_1,x_2), r_2(x_1,x_2))$ Further MRL can give indication of the covariance structure of residual lives.

Proposition 2.4

A necessary condition for a function $\mathrm{M}(x_1,x_2)$ to be a PMRL is

$$r_1^2(x_1,x_2)\,\frac{\partial}{\partial x_1}\left(\frac{\mathrm{M}(x_1,x_2)}{r_1(x_1,x_2)}\right) = r_2^2(x_1,x_2)\,\frac{\partial}{\partial x_2}\left(\frac{\mathrm{M}(x_1,x_2)}{r_2(x_1,x_2)}\right)$$

Proof

Equating expressions for $r_1(x_1,x_2)r_2(x_1,x_2)$ from (11) and (12) we get

$$r_1(x_1,x_2)\frac{\partial \mathrm{M}(x_1,x_2)}{\partial x_1} - \mathrm{M}(x_1,x_2)\frac{\partial r_1(x_1,x_2)}{\partial x_1} = r_2(x_1,x_2)\frac{\partial \mathrm{M}(x_1,x_2)}{\partial x_2} - \mathrm{M}(x_1,x_2)\frac{\partial r_2(x_1,x_2)}{\partial x_2}$$

from which the necessary condition follow.

Proposition 2.5

PMRL is increasing in x_1, x_2 whenever $\mathrm{M}(x_1,x_2) < \min\left(\frac{\partial r_1(x_1,x_2)}{\partial x_1}, \frac{\partial r_2(x_1,x_2)}{\partial x_2}\right)$ and decreasing whenever $\mathrm{M}(x_1,x_2) > \max\left(\frac{\partial r_1(x_1,x_2)}{\partial x_1}, \frac{\partial r_2(x_1,x_2)}{\partial x_2}\right)$.

Proof

$M(x_1,x_2)$ is increasing in both x_1 and x_2 whenever $\frac{\partial M(x_1,x_2)}{\partial x_1}>0$ and $\frac{\partial M(x_1,x_2)}{\partial x_2}>0$. From (11) and (12) this is true if simultaneously the inequalities $M(x_1,x_2)>\frac{r_1(x_1,x_2)r_2(x_1,x_2)}{1+\frac{\partial r_1(x_1,x_2)}{\partial x_1}}$ and $M(x_1,x_2)>\frac{r_1(x_1,x_2)r_2(x_1,x_2)}{1+\frac{\partial r_2(x_1,x_2)}{\partial x_2}}$ hold. This leads to $M(x_1,x_2)<\min(\frac{\partial r_1(x_1,x_2)}{\partial x_1},\frac{\partial r_2(x_1,x_2)}{\partial x_2})$. The proof of the second part is similar.

Proposition 2.6

The CMRL $C(x_1,x_2)=0$ for all (x_1, x_2) in R_2^+ if and only if X_1 and X_2 are independently distributed.

Proof

When X_1 and X_2 are independent,

$r_1(x_1,x_2)=r_1(x_1,0)$ and $r_2(x_1,x_2)=r_2(0,x_2)$.

Also $M(x_1,x_2)=r_1(x_1,x_2)\ r_2(x_1,x_2)$ so that $C(x_1,x_2)=0$. Conversely $C(x_1,x_2)=0$ implies

$$M(x_1,x_2)=r_1(x_1,x_2)\ r_2(x_1,x_2)$$

or

$$\int_{x_1}^{\infty}\int_{x_2}^{\infty}R(t_1,t_2)dt_1dt_2=R(x_1,x_2)r_1(x_1,x_2)r_2(x_1,x_2)$$

or

$$-\int_{x_2}^{\infty}R(x,t_2)dt_2=r_1(x_1,x_2)r_2(x_1,x_2)\frac{\partial R(x_1,x_2)}{\partial x_1}$$
$$+r_2(x_1,x_2)R(x_1,x_2)\frac{\partial r_1(x_1,x_2)}{\partial x_1}+r_1(x_1,x_2)R(x_1,x_2)\frac{\partial r_2(x_1,x_2)}{\partial x_2}$$

Dividing by $R(x_1,x_2)$ and using the relation between $h(x_1,x_2)$ and $r_1(x_1,x_2)$

$$-r_2(x_1,x_2) = -r_2(x_1,x_2)\left(1+\frac{\partial r_2(x_1\ ,x_2)}{\partial x_1}\right) + r_2(x_1,x_2)\frac{\partial r_1(x_1,x_2)}{\partial x_1}$$
$$+ r_2(x_1,x_2)\frac{\partial r_2(x_1,x_2)}{\partial x_2}$$

and hence $r_2(x_1,x_2)\dfrac{\partial r_2(x_1,x_2)}{\partial x_2} = 0$ implying $r_2(x_1,x_2)$ is a function of x_2 alone and hence $r_2(x_1,x_2) = r_2(0,x_2)$. Similarly $r_1(x_1,x_2) = r_1(x_1,0)$ and hence X_1 and X_2 are independent (Nair and Nair, 1989).

Corollary 2.2

$\mathrm{M}(x_1,x_2) = r_1(x_1,0)r_2(0,x_2)$ for all $x_1,x_2 > 0$ if and only if X_1 and X_2 are independent.

The proof follows from Proposition 2.6.

Remark 1

Normally, zero covariance does not imply independence. But in the case of residual lives the property holds. This is useful in tests of independence.

Remark 2

CVRL need not determine a distribution uniquely. This statement will follow from our discussions in the next section and Theorem 3.2.

3. RELATIONSHIP WITH BASU'S FAILURE RATE

Basu (1971) defined the failure rate of a continuous random vector in R_2^+ as

$$h(x_1,x_2) = f(x_1,x_2)/\mathrm{R}(x_1,x_2) \tag{13}$$

Of particular interest is the case when (X_1,X_2) has constant Basu failure rate. The following theorem identifies the consequences of the constancy of the failure on the PMRL.

Theorem 3.1

For a continuous bivariate random vector in R_2^+ with exponential marginals, $h(x_1,x_2) = k$ if and only if $\mathrm{M}(x_1,x_2) = k^{-1}$

Proof

The first part of the Theorem is evident from the equivalence of the relationship

$$\frac{\partial^2 R(x_1,x_2)}{\partial x_1 \partial x_2} = kR(x_1,x_2) \text{ and } kR(x_1,x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} R(t_1,t_2)dt_1dt_2 .$$

The second part follows from Basu (1971).

Theorem 3.2

The only absolutely continuous distributions in R_2^+ for which $M(x_1,x_2)$ is constant are mixture of exponential distributions.

$$f(x_1,x_2) = \lambda \int_0^{\infty}\int_0^{\infty} \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}\mu(\lambda_1,\lambda_2), \quad x_1,x_2 > 0 \tag{14}$$

where μ is a measure on the set $A = \{\lambda_1\lambda_2 = \lambda \mid \lambda_1 > 0, \lambda_2 > 0\}$.

The proof of the theorem follows directly from the definition of PMRL and Puri and Rubin (1974).

In Theorem 3.2, $h(x_1,x_2)\mathrm{M}(x_1,x_2)^{-1} = 1$ and thus PMRL is the reciprocal of failure rate.

Let us consider a more general relationship

$$h(x_1,x_2)\mathrm{M}(x_1,x_2) = c \tag{15}$$

for some constant *c>0.*

Equation (15) is equivalent to the fourth order partial differential equation

$$u\frac{\partial^4 u}{\partial x_1^2 \partial x_2^2} = k\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 \tag{16}$$

where

$$u(x_1,x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty} R(t_1,t_2)dt_1t_2. \tag{17}$$

Being a fourth order equation (16) has four independent solutions. We are interested in only those solutions that can generate a probability distribution. One solution is $u(x_1,x_2) = a_0 + a_1x_1 + a_2x_2$, which cannot produce a probability distribution. The second solution is (14), that corresponds to *c*=1.When *c* is of the form $\beta(\beta+1)/(\beta-1)(\beta-2) > 1$

$$f(x_1,x_2) = \lambda \int_0^{\infty}\int_0^{\infty} (1 + d_1x_1 + d_2x_2)^{-\beta-2}\mu(\lambda_1,\lambda_2), \; x_1, x_2 > 0$$

where μ is a measure on $A=\{d_1 d_2=\lambda | d_1>0, d_2>0\}$ provides $u(x_1,x_2)$ that satisfies (16) for $\beta>0$. A fourth solution is

$$f(x_1,x_2)=\lambda\int_0^{a_1}\int_0^{a_2}\left(1-\frac{x_1}{a_1}-\frac{x_2}{a_2}\right)^{d-2}\mu(\lambda_1,\lambda_2)\ ,\ x_1,x_2>0$$

where μ is a measure on $A=\{a_1 a_2=\lambda | a_1,a_2>0\}$, which corresponds to $c=d(d-1)/(d+1)(d+2)<1, d>2$ and the support of (X_1,X_2) is $(0,a_1)\times\left(0,a_2\left(1-\frac{x_1}{a_1}\right)\right)$. It can be verified that linear combinations of the above four solutions with constant or variable coefficients do not satisfy the conditions for a probability distribution in R_2^+ for which (15) is true.

If X is a continuous non-negative random variable with survival function $R(x)=P(X>x)$, the distribution with density function.

$$g(x)=R(x)/E(X),\ x>0 \tag{18}$$

is called the equilibrium distribution corresponding to X. For a discussion on the properties and applications of equilibrium distribution we refer to Gupta (1979). In analogy with (18), if we define the bivariate equilibrium distribution as

$$g(x_1,x_2)=R(x_1,x_2)/E(X_1,X_2),$$

then the Basu's failure rate $k(x_1,x_2)$ of $g(x_1,x_2)$ is seen to be related to $M(X_1,X_2)$ as

$$k(x_1,x_2)=[M(X_1,X_2)]^{-1} \tag{19}$$

For a similar result in the univariate case and the applications of equilibrium distributions in reliability modeling we refer to Gupta (1979) and Nair and Hitha (1989). As pointed out in those papers the result in (19) is useful in mutual characterization of the original and equilibrium distributions via the PMRL. Also from the above discussions.

$$k(x_1,x_2)=h(x_1,x_2)$$

if and only if the distribution of (X_1,X_2) is as in (14). Thus we have

Theorem 3.3

The only bivariate continuous distribution in R_2^+ in which the equilibrium distribution is identical with the original distribution has density (14).

The equilibrium distribution belongs to the class of bivariate weighted distribution by virtue of the representation.

$$g(x_1,x_2)=[h(x_1,x_2)]^{-1}f(x_1,x_2)/\mathrm{E}[h(x_1,x_2)]^{-1}$$

provided the expectation exists finitely. In general PMRL can be expressed as a truncated mean of the reciprocal failure rate, $\mathrm{M}(x_1,x_2)=\mathrm{E}[h(x_1,x_2)^{-1}|X_1>x_1,X_2>x_2]$.

4. SOME CHARACTERIZATIONS

The relationship between PMRL and the mean residual life was derived in Section 2. This relationship involves first order partial differential equations in $\mathrm{M}(x_1,x_2)$ and the components $r_1(x_1,x_2)$ and $r_2(x_1,x_2)$ of mean residual life that provide characterization of some useful life distributions. For other characterization of these distributions by reliability concepts we refer to Sankaran and Nair (1993).

Theorem 4.1

A random vector in (X_1,X_2) with $\mathrm{E}(X_1X_2)<\infty$ in the support of R_2^+ satisfies the condition.

$$\mathrm{M}(x_1,x_2)=kr_1(x_1,x_2)r_2(x_1,x_2) \tag{20}$$

if and only if the survival function of (X_1,X_2) is bivariate Lomax with survival function.

$$\mathrm{R}(x_1,x_2)=(1+a_1x_1+a_2x_2)^{-d},\ a_1,a_2,\ d>0,\ x_1,x_2>0 \tag{21}$$

for $k>1$ and bivariate beta survival function

$$\mathrm{R}(x_1,x_2)=(1-p_1x_1-p_2x_2)^c, 0<x_1<p_1^{-1}, 0<x_2<\frac{1-p_1x_1}{p_2}, p_1,p_2,c>0 \tag{22}$$

for $k<1$

For $k=1$, the variables are independently distributed.

Proof

Assuming the relationship (20), equations (10) and (9) reduce to

$$kr_1(x_1,x_2)\frac{\partial r_2(x_1,x_2)}{\partial x_1}=(k-1)r_2(x_1,x_2) \tag{23}$$

and

$$k r_2(x_1,x_2)\frac{\partial r_1(x_1,x_2)}{\partial x_2}=(k-1)r_1(x_1,x_2) \tag{24}$$

Hence

$$\frac{k}{k-1}\ \frac{\partial r_2(x_1,x_2)}{\partial x_1}=\ \frac{k-1}{k}\left(\frac{\partial r_1(x_1,x_2)}{\partial x_2}\right)^{-1} \tag{25}$$

holds for all x_1, $x_2 > 0$ and $k \neq 1$. This happens if and only if both sides of (25) is a constant, say *c*. Solving the resulting differential equations

$$r_2(x_1,x_2)=\frac{(k-1)c}{k}x_1+A_1(x_2)$$

and

$$r_1(x_1,x_2)=\frac{(k-1)}{kc}x_2+A_2(x_1)$$

We substitute these values of $r_1(x_1,x_2)$ and $r_2(x_1,x_2)$ in (23) and (24). Since one side of the resulting equations is linear in x_1 (x_2) the other must also be linear in x_1 (x_2). Thus the final solution of (23) and (24) are

$$r_1(x_1,x_2)=\frac{k-1}{k}x_1+\frac{k-1}{kc}x_2+A_2(0)$$

$$r_2(x_1,x_2)=\frac{(k-1)c}{k}x_1+\frac{k-1}{kc}x_2+cA_2(0)$$

Substituting these expressions in the inversion formula (Nair and Nair,(1989))

$$R(x_1,x_2)=\frac{r_1(0,0)r_1(x_1,0)}{r_2(x_1,0)r_2(x_1,x_2)}\exp\left[-\int_0^{x_1}\frac{dt_1}{r_1(t_1,0)}-\int_0^{x_2}\frac{dt_2}{r_2(x_1,t_2)}\right]$$

we get

$$R(x_1,x_2)=\left(1+\frac{k-1}{kA_2}x_1+\frac{k-1}{kcA_2}x_2\right)^{-\left(1+\frac{k}{k-1}\right)}$$

which is of the form stated in the Theorem. Conversely for the distribution specified by (21)

$$r_1(x_1,x_2)=(d-1)^{-1}a_1^{-1}(1+a_1x_1+a_2x_2)$$

$$r_2(x_1,x_2)=(d-1)^{-1}a_2^{-1}(1+a_1x_1+a_2x_2)$$

and $M(x_1,x_2)=(d-1)^{-1}(d-2)^{-1}a_1^{-1}a_2^{-1}(1+a_1x_1+a_2x_2)^2$

so that $M(x_1,x_2)=kr_1(x_1,x_2)r_2(x_1,x_2)$ with $k=\dfrac{d-1}{d-2}>1$.

The proof for k<1 is exactly similar and is therefore omitted. Proposition 2.6 of Section 2 is the case when k=1 and this establishes Theorem 4.1

Corollary 4.1

The covariance of residual life $C(x_1,x_2)=Ar_1(x_1,x_2)r_2(x_1,x_2)$ if and only if of the distribution of (X_1,X_2) is either bivariate Lomax for A>0 or bivariate beta for A<0 with survival functions stated in (21) and (22).

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RIASSUNTO

La covarianza delle vite residue

Varie proprietà della vita residua quali la media, la mediana, i percentili, la varianza sono state discusse nella letteratura sull'analisi della sopravvivenza. Tuttavia uno studio dettagliato sulla covarianza tra le vite residue in un sistema a due componenti non sembra essere stato affrontato. Nel presente lavoro vengono discusse diverse proprietà del momento dei prodotti e della covarianza delle vite residue. Si studiano poi le relazioni che il momento dei prodotti ha con la vita residua media e il tasso di insuccesso.

SUMMARY

On the covariance of residual lives

Various properties of residual life such as mean, median, percentiles, variance etc have been discussed in literature on reliability and survival analysis. However a detailed study on covariance between residual lives in a two component system does not seem to have been undertaken. The present paper discusses various properties of product moment and covariance of residual lives. Relationships the product moment has with mean residual life and failure rate are studied and some characterizations are established.