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SOME PROPERTIES OF GAMMA GENERATED DISTRIBUTIONS

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1. Introduction

The classical univariate distributions, like normal, exponential, Weibull, gamma, beta, etc., have been studied in great details by many authors (cf. Kotz et al., 1994, 1995). Over the past decade, extensive studies have been carried out to develop generalized families of distributions based on the classical ones, which exhibit greater flexibility in modeling real-life data. Mudholkar and Srivastava (1993) introduced the exponentiated Weibull distribution to analyze bathtub failure rate data. Gupta et al. (1998) proposed a generalization of the standard exponential distribution. Thereafter, many authors introduced and studied various classes of univariate distributions, see, for example, Gupta and Kundu (2001), Eugene et al. (2002), Nadarajah (2005), Pal et al. (2006), Lee et al. (2007), Zografos (2008), Sarhan and Zaindin (2009), Aryal, and Tsokos (2011), Cordeira (2013), Pal and Tiensuwan (2014).

Zografos and Balakrishnan (2009) introduced a family of univariate distributions generated by the standard Gamma distribution. Based on a baseline continuous distribution $G(x)$ with survival function $\overline{G}(x)$, and density function $g(x)$, they defined the cumulative distribution function (cdf) of the gamma generated distribution as

$$F(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-log\overline{G}(x)} t^{\alpha-1} e^{-t} dt, \qquad -\infty < x < \infty, \alpha > 0. \tag{1}$$

The density function of the distribution is, therefore, given by

$$f(x) = \frac{1}{\Gamma(\alpha)} \{-log\overline{G}(x)\}^{\alpha-1} g(x), \qquad -\infty < x < \infty, \alpha > 0. \tag{2}$$

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Zografos and Balakrishnan (2009) obtained the general expressions for the moments and Shannon entropy of the distribution, and gave the specific forms under some standard baseline distributions. Not much study has been carried out on the distribution thereafter.

In this paper, we carry out further investigation on the distribution for general baseline distribution $G(.)$. The paper is organized as follows. In Section 2 we study some properties of the distribution. In Section 3, we obtain distributions of the order statistics. In Section 4, the analytical forms of the properties for some specific baseline distributions are obtained. Finally in Section 5, some concluding remarks on the study have been made.

2. Properties of gamma generated distributions

The class of gamma generated distributions is flexible in the sense that, depending on the value of α in (2), it can represent distributions of different shapes. Below are some gamma generated density curves for varied values of α.

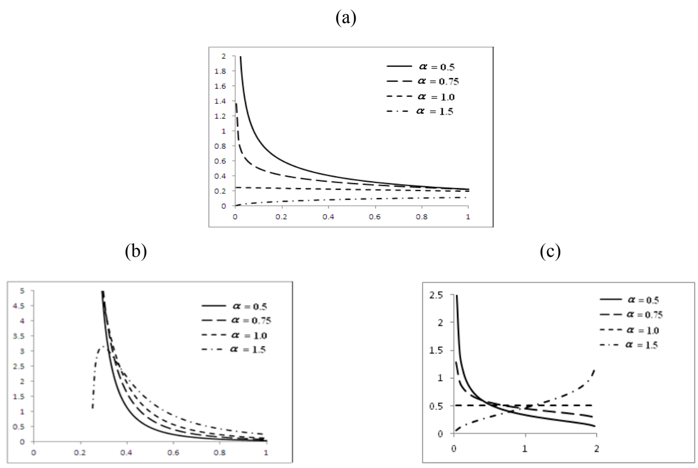

Figure 1 – The density curves of gamma generated (a) exponential distribution with cdf $G(x) = 1 - e^{-0.25x}$, $x > 0$; (b) Pareto distribution with cdf $G(x) = 1 - (\frac{0.25}{x})^2$, $x \geq 0.25$, and (c) uniform distribution with cdf $G(x) = \frac{x}{2}$, $0 \geq x \geq 2$, for $\alpha = 0.5$, 0.75, 1, 1.5.

In this section we study some interesting properties of gamma generated distributions.

2.1. Survival Function and Hazard Rate

The survival function of the distribution (1) is

$$\overline{F}(x) = 1 - F(x) = \frac{1}{\Gamma(\alpha)} \int_{-log\overline{G}(x)}^{\infty} t^{\alpha-1} e^{-t} dt,$$

and its hazard rate is given by

$$h_F(x) = \frac{f(x)}{\overline{F}(x)} = \frac{\{-log\overline{G}(x)\}^{\alpha-1} g(x)}{\int_{-log\overline{G}(x)}^{\infty} t^{\alpha-1} e^{-t} dt}, \quad -\infty < x < \infty, \quad \alpha > 0, \tag{3}$$

LEMMA 1. *If $G(.)$ be the baseline distribution with hazard rate $h_G(x)$ and the generating distribution be Gamma(α) with hazard rate $h_{gamma}(x;\alpha)$, then the hazard rate of the gamma generated distribution $F(x)$ is the product of $h_G(x)$ and $h_{gamma}(-ln\overline{G}(x);\alpha)$.*

PROOF. We can write

$$\begin{aligned} h_F(x) &= \frac{\overline{G}(x)\{-log\overline{G}(x)\}^{\alpha-1} g(x)/\overline{G}(x)}{\int_{-log\overline{G}(x)}^{\infty} t^{\alpha-1} e^{-t} dt} \\ &= h_G(x) \frac{e^{-\{ln\overline{G}(x)\}}\{-log\overline{G}(x)\}^{\alpha-1}}{\int_{-log\overline{G}(x)}^{\infty} t^{\alpha-1} e^{-t} dt} \\ &= h_G(x) \times h_{gamma}(-ln\overline{G}(x);\alpha). \end{aligned}$$

LEMMA 2. *For fixed x, $h_F(x)$ is a decreasing function of α.*

PROOF. From Lemma 1 we have that $h_F(x) = h_G(x) \times h_{gamma}(-ln\overline{G}(x);\alpha)$, which is dependent on α only through $h_{gamma}(-ln\overline{G}(x);\alpha)$.
For the distribution Gamma(α), the hazard rate can be written as

$$h_{gamma}(x;\alpha) = \left[\int_0^{\infty} e^{-z}(1 + \frac{z}{x})^{\alpha-1} dz\right]^{-1},$$

which decreases as α increases for fixed x.

Hence the lemma.

LEMMA 3. *If the baseline distribution $G(.)$ has the increasing failure rate (IFR) (decreasing failure rate (DFR)) property, then for $\alpha > 1$ ($\alpha < 1$) the gamma generated distribution (2) also has the IFR (DFR) property.*

PROOF. The Gamma(α) distribution has the IFR (DFR) property according as $\alpha >$ (<)1. Further, $-ln\overline{G}(x)$ is a non-decreasing function of x. Hence, $h_{gamma}(-ln\overline{G}(x);\alpha)$ is a non-decreasing (non-increasing) function of x for $\alpha >$ (<)1.
Thus, if the distribution $G(.)$ has the IFR (DFR) property, then for $\alpha >$ (<)1, $h_F(x)$ is a non-decreasing (non-increasing) function of x, that is the distribution $F(.)$, given by (1) has the IFR (DFR) property.

2.2. Moment Generating Function (MGF)

It is easy to check that a logarithmic transformation of the baseline distribution G transforms the random variable X with density $f(x)$, given by (2), to a gamma variate. Thus, if X has a density (2), then the random variable $Z = -log\overline{F}(x)$ has a gamma distribution with parameter α.

Using this result, a general expression of the moment generating function (mgf) of gamma generated distribution (1) is obtained as

$$\begin{aligned} M_X(t) &= (1/\Gamma(\alpha)) \int_{-\infty}^{\infty} e^{tx} g(x)\{-ln\overline{G}(x)\}^{\alpha-1} dx \\ &= E_Z[e^{tG^{-1}(1-e^{-Z})}] \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} E_Z[G^{-1}(1-e^{-Z})]^r, \end{aligned}$$

where $Z \sim Gamma(\alpha)$, and $G^{-1}(p)$ denotes the value x such that $G(x) = p$. We, therefore, get the moments as

$$\mu'_r = E_X(X^r) = E_Z[G^{-1}(1-e^{-Z})]^r, \quad r \geq 1.$$

2.3. Quantiles and Simulation

The quantile function of a probability distribution is useful in statistical applications and Monte Carlo simulation. The following lemma relates the quantile of a gamma generated distribution to that of the baseline distribution.

LEMMA 4. *The p-th quantile of the gamma generated distribution $F(.)$, given by (1), $0 < p < 1$, is equal to the $(1 - exp(-t_p))$-th quantile of the baseline distribution $G(.)$, where t_p denotes the p-th quantile of the Gamma distribution with shape parameter α.*

PROOF. The p-th quantile $t_F(p)$ of $F(.)$ is such that $F(t_F(p)) = p$, that is

$$F(t_F(p)) = \frac{1}{\Gamma(\alpha)} \int_0^{-log\overline{G}(t_F(p))} t^{\alpha-1} e^{-t} dt = p.$$

Thus, if t_p be the p-th quantile of Gamma(α), we get $t_p = -ln\overline{G}(t_F(p))$, which gives

$$G(t_F(p)) = 1 - exp(-t_p)).$$

Hence the Lemma.

In particular, the median $\mu_e(F) = t_F(0.5)$ of a gamma generated distribution $F(.)$ with baseline distribution $G(.)$ is given by

$$\mu_e(F) = G^{-1}(1 - exp(-t_{0.5})), \tag{4}$$

where $t_{0.5}$ is the median of Gamma(α) distribution.

The median of Gamma(α), and hence that of $F(.)$, cannot be obtained in a closed form. However, for $\alpha \geq 1$, an approximation to the median of Gamma(α) is given by $\alpha\frac{3\alpha-0.8}{3\alpha+0.2}$ (cf. Banneheka and Ekanayake, 2009), which may be used to find $\mu_e(F)$ from (4).

To simulate from the distribution $F(.)$, given by (1), we may first simulate an observation x from the Gamma(α) distribution and then find y such that $G(y) = 1 - exp(-x)$. The simulated observation from $F(.)$ will then be y.

2.4. *Stress-strength reliability*

Suppose X and Y are independently distributed with $X \sim F_1(x)$ and $Y \sim F_2(y)$, where

$$F_1(x) = \frac{1}{\Gamma(\alpha)} \int_0^{-log\overline{G}(x)} t^{\alpha-1}e^{-t}dt,$$
$$F_2(y) = \frac{1}{\Gamma(\beta)} \int_0^{-log\overline{H}(y)} t^{\beta-1}e^{-t}dt.$$

Since $-log\overline{H}(Y)$ is distributed as Gamma(β), we get

$$Pr(X > Y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \{\int_{-ln\overline{G}(\overline{H}^{-1}(e^{-\nu}))}^\infty u^{\alpha-1}e^{-u}du\}\nu^{\beta-1}e^{-\nu}d\nu.$$

When X and Y have the same baseline distribution, that is $G(x) = H(x)$, for $-\infty < x < \infty$, it is easy to show that $Pr(X > Y)$ is equal to the probability that U is greater than V, where U and V are independent Gamma(α) and Gamma(β) variates, respectively. An analytical expression for the same can be obtained as follows:

$$\begin{aligned} Pr(X > Y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \{\int_0^\infty (u+\nu)^{\alpha-1}e^{-(u+\nu)}du\}\nu^{\beta-1}e^{-\nu}d\nu \\ &= \frac{1}{2^{\alpha+\beta}B(\alpha,\beta)} \sum_{j=0}^{\alpha-1} \binom{\alpha-1}{j} 2^{j+1}B(j+1, \alpha+\beta-j-1), \\ &\qquad \text{if } \alpha \text{ is a positive integer} \\ &= \frac{1}{2^{\alpha+\beta}B(\alpha,\beta)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} 2^{j+1}B(j+1, \alpha+\beta-j-1), \\ &\qquad \text{otherwise,} \end{aligned}$$

where $B(a,b) = \int_0^1 z^{a-1}(1-z)^{b-1}dz$.

2.5. Entropy

Entropy indicates the amount of uncertainty associated with a random variable. It is an important concept in many areas of study, like physics, probability and statistics, economics, communication theory. Zografos and Balakrishnan (2009) gave a general expression of a very popular measure of entropy introduced by Shannon (1948). We give an expression of another popular measure of entropy, namely Rényi entropy.

Rényi entropy is defined as

$$I_R(\nu) = (1-\nu)^{-1} log[\int_{-\infty}^{\infty} f(x)^{\nu} dx], \text{where } \nu > 0, \nu \neq 1.$$

From (2),

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1} log[\int_{-\infty}^{\infty} \frac{1}{(\Gamma(\alpha))^{\nu}} \{-log\overline{G}(x)\}^{\nu(\alpha-1)} g(x)^{\nu} dx] \\ &= (1-\nu)^{-1} log\left[\frac{\Gamma(\nu(\alpha-1)+1)}{(\Gamma(\alpha))^{\nu}} E_Z\{g(\overline{G}^{-1}(e^{-z}))^{\nu-1}\}\right] \\ &= (1-\nu)^{-1}\left[log\Gamma(\nu(\alpha-1)+1) - \nu log\Gamma(\alpha) + log E_Z\{g(\overline{G}^{-1}(e^{-z}))\}^{\nu-1}\right], \end{aligned} \tag{5}$$

where Z is distributed as Gamma$(\nu(\alpha-1)+1)$.

Thus, the members of the family of gamma generated distributions (1) can be discriminated among each other by means of the expected value $E_Z\{g(\overline{G}^{-1}(e^{-Z}))\}^{\nu-1}$, which depends on the baseline distribution $G(.)$.

3. Order statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from the $F(.)$ distribution, given by (1), and let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ denote the corresponding ordered observations.

Theorem 5. *The density of the i-th order statistic $X_{(i)}$ is an infinite weighted sum of gamma generated distributions with baseline distribution $G(.)$.*

PROOF. The density of the i-th order statistic $X_{(i)}$ is given by

$$\begin{aligned}
f_{(i)}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i} \\
&= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} [F(x)]^{i+j-1} \\
&= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\frac{1}{\Gamma(\alpha)} \int_0^{-ln\overline{G}(x)} t^{\alpha-1} e^{-t} dt\right]^{i+j-1} \\
&= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \left[\frac{1}{\Gamma(\alpha)} \gamma(\alpha, -ln\overline{G}(x))\right]^{i+j-1},
\end{aligned}$$

where $\gamma(\alpha, z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{\alpha+k}}{k!(\alpha+k)}$.

We can, therefore, write

$$\begin{aligned}
f_{(i)}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{(\Gamma(\alpha))^{i+j-1}} (-ln\overline{G}(x))^{\alpha(i+j-1)} \\
&\quad \left[\sum_{k=0}^{\infty} (-1)^k \frac{(-ln\overline{G}(x))^k}{k!(\alpha+k)}\right]^{i+j-1} \\
&= \frac{n!}{(i-1)!(n-i)!} g(x) \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} \frac{1}{(\Gamma(\alpha))^{i+j}} (-ln\overline{G}(x))^{\alpha(i+j)-1} \\
&\quad \left[\sum_{k=0}^{\infty} (-1)^k \frac{(-ln\overline{G}(x))^k}{k!(\alpha+k)}\right]^{i+j-1}.
\end{aligned}$$

Writing $\alpha_k = \frac{(-1)^k}{k!(\alpha+k)}$, and using the result on power series raised to a positive integer, we get

$$\left[\sum_{k=0}^{\infty} a_k \{-ln\overline{G}(x)\}^k\right]^{i+j-1} = \sum_{k=0}^{\infty} b_{k,i+j-1} \{-ln\overline{G}(x)\}^k,$$

where $b_{0,i+j-1} = a_0^{i+j-1}$ and $b_{k,i+j-1} = \frac{1}{ka_0} \sum_{l=1}^{k} \{(i+j)l - k\} a_l b_{k-l,i+j-1}$.

Using the above expression, we obtain

$$f_{(i)}(x) = \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} \binom{n-i}{j} \frac{(-1)^j b_{k,i+j-1}}{(\Gamma(\alpha))^{i+j}} \Gamma(\alpha(i+j)+k) f(x; \alpha(i+j)+k),$$

where $f(x; \alpha(i+j)+k)$ denotes the density of gamma exponentiated $G(.)$ distribution, the exponentiating distribution being Gamma$(\alpha(i+j)+k)$.

In particular, the density functions of the smallest order statistic $X_{(1)}$ and the largest order statistic $X_{(n)}$ are given by

$$f_{(1)} = n\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\binom{n-1}{j}\frac{(-1)^j b_{k,j}}{(\Gamma(\alpha))^{j+1}}\Gamma((j+1)\alpha+k)f(x;(j+1)\alpha+k),$$

$$f_{(n)} = \frac{n}{\Gamma(\alpha)}\sum_{k=0}^{\infty} b_{k,0}\Gamma(\alpha+k)f(x;\alpha+k).$$

Using Theorem 5, we obtain the r-th order moment of the i-th order statistic as

$$E_F(X_{(i)}^r) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\sum_{k=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j b_{k,i+j-1}}{(\Gamma(\alpha))^{i+j}}\Gamma(\alpha(i+j)+k)$$
$$E([G^{-1}(1-e^{Z_{\alpha(i+j)+k}})]^r),$$

where $Z_{\alpha(i+j)+k} \sim Gamma(\alpha(i+j)+k)$.

Hence,

$$E_F(X_{(1)}^r) = n\sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\binom{n-i}{j}\frac{(-1)^j b_{k,j}}{(\Gamma(\alpha))^{j+1}}\Gamma((j+1)\alpha+k)E([G^{-1}(1-e^{Z_{(j+1)\alpha+k}})]^r),$$

$$E_F(X_{(n)}^r) = \frac{n}{\Gamma(\alpha)}\sum_{k=0}^{\infty} b_{k,0}\Gamma(\alpha+k)E([G^{-1}(1-e^{Z_{\alpha+k}})]^r).$$

4. Some examples

Example 1. Gamma-uniform distribution

Suppose the baseline distribution is uniform in the interval $(0,\theta)$, $\theta > 0$. Then, the pdf and cdf of the baseline distribution are, respectively,

$$g(x) = \frac{1}{\theta}, \text{ and } G(x) = \frac{x}{\theta}, 0 < x < \theta.$$

The hazard rate of the distribution is $h_G(x) = \frac{1}{\theta - x}, 0 < x < \theta$, which is increasing in x.

The gamma-uniform distribution has the density function

$$f(x) = \frac{\theta^{-1}}{\Gamma(\alpha)}\{log\left(\frac{\theta}{\theta-x}\right)\}^{\alpha-1}, 0 < x < \theta, \alpha, \theta > 0,$$

and its hazard rate is given by

$$h_F(x) = \frac{1}{\theta} \cdot \frac{\{log(\theta) - log(\theta - x)\}^{\alpha-1}}{\int_{log(\frac{\theta}{\theta - x})}^{\infty} t^{\alpha-1} e^{-t} dt}.$$

By virtue of Lemma 3, $h_F(x)$ is a non-decreasing function of x when $\alpha > 1$. Thus, the gamma-uniform distribution has the IFR property if $\alpha > 1$.

The mgf of the gamma-uniform distribution is

$$M_X(t) = \sum_{r=0}^{\infty} \frac{(\theta t)^r}{r!} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{1}{(j+1)^{\alpha}}.$$

Hence, $E_X(X^r) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \frac{1}{(j+1)^{\alpha}}, r = 1, 2, \ldots$.

The Rényi entropy is given by

$$I_R(\nu) = (1 - \nu)^{-1} [log\Gamma(\nu(\alpha - 1) + 1) - \nu log\Gamma(\alpha) - (\nu - 1) log\theta].$$

Example 2. Gamma-exponential distribution

Let the baseline distribution be exponential with parameter θ, that is $G(x) = 1 - e^{-\theta x}, x > 0, \theta > 0$. The hazard rate of the distribution is θ, a constant.

Hence, the gamma-exponential distribution, with hazard rate given by $h_F(x) = \frac{\theta(\theta x)^{\alpha-1} e^{-\theta x}}{\int_{\theta x}^{\infty} t^{\alpha-1} e^{-t} dt}$, has the IFR property for $\alpha > 1$, and the DFR property for $\alpha < 1$. The mgf of the distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E_Z \left[\frac{Z}{\theta}\right]^r = \sum_{r=0}^{\infty} \frac{t^r}{r!} \times \frac{1}{\theta^r} \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)}.$$

Hence, $E_X(X^r) = \frac{\Gamma(\alpha+r)}{\theta^r \Gamma(\alpha)}$.

In particular, $E(X) = \frac{\alpha+1}{\theta}$, $Var(X) = \frac{\alpha+1}{\theta^2}$.

The Rényi entropy is given by

$$I_R(\nu) = (1 - \nu)^{-1} [log\Gamma(\nu(\alpha - 1) + 1) - \nu log\Gamma(\alpha) + (\nu - 1) log\theta \\ - (\nu(\alpha - 1) + 1) log\nu].$$

Example 3. Gamma-Pareto distribution

The density function and cdf of the Pareto distribution are given respectively by

$$g(x) = \frac{k\theta^k}{x^{k+1}},$$
$$G(x) = 1 - \left(\frac{\theta}{x}\right)^k, x \geq \theta, \theta, k > 0.$$

Hence, the distribution has the DFR property.

The hazard rate of the gammaPareto distribution is

$$h_F(x) = \frac{k^\alpha \theta^k (log\theta - logx)^{\alpha-1}}{x^{k+1} \int_{log\theta - logx}^{\infty} t^{\alpha-1} e^{-t} dt},$$

and the distribution is DFR for $\alpha < 1$.

The mgf is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{\theta^r t^r}{r!} E_Z\left(e^{\frac{rZ}{k}}\right) = \sum_{r=0}^{\infty} \frac{\theta^r t^r}{r!} (1 - r/k)^{-\alpha}, \text{ provided } r < k.$$

Hence, $E_X(X^r) = \theta^r (1 - r/k)^{-\alpha}$, provided $r < k$.

The Rényi entropy is given by

$$I_R(\nu) = (1-\nu)^{-1}[log\Gamma(\nu(\alpha-1)+1) - \nu log\Gamma(\alpha) + (\nu\alpha+1)logk - (\nu-1)log\theta \\ - (\nu(\alpha-1)+1)log(\nu(k+1)-1)].$$

Example 4. Gamma-Weibull distribution

The Weibull distribution with pdf $g(x) = \beta\lambda x^{\beta-1} e^{-\lambda x^\beta}$, and cdf $G(x) = 1 - e^{-\lambda x^\beta}$, $x > 0$, β, $\lambda > 0$, has the IFR (DFR) property for $\beta > (<)1$.

The hazard rate of the gamma-Weibull distribution is given by

$$h_F(x) = \frac{\lambda^{\alpha-1} \beta \lambda x^{\alpha\beta-1} e^{-\lambda x^\beta}}{\int_{\lambda x^\beta}^{\infty} t^{\alpha-1} e^{-t} dt}.$$

Hence, for $\alpha, \beta > (<)1$ the gamma-Weibull distribution has the IFR (DFR) property.

The mgf is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{\lambda^{r/\beta} r!} E_Z\left(Z^{\frac{r}{\beta}}\right) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{\Gamma(\alpha + r/\beta)}{\lambda^{r/\beta}\Gamma(\alpha)}.$$

Hence, $E_X(X^r) = \frac{\Gamma(\alpha+r/\beta)}{\lambda^{\frac{r}{\beta}}\Gamma(\alpha)}$.

Some simple algebraic manipulation gives

$$E_Z\{g(\overline{G}^{-1}(e^{-Z}))\}^{\nu-1} = \frac{\lambda^{\frac{\nu-1}{\beta}}\beta^{\nu-1}}{\nu^{(\nu-1)(1-1/\beta)+\delta}}\Gamma((\nu-1)(1-1/\beta)+\delta),$$

where $\delta = \nu(\alpha-1)+1$. Rényi entropy can then be easily obtained from (5).

Example 5. Gamma-power function distribution

Consider the power function distribution with pdf $g(x) = k\theta^k x^{k-1}$, and cdf $G(x) = (\theta x)^k$, $0 < x < 1/\theta$, $k > 0$. It can be easily shown that the hazard rate of the distribution is a non-decreasing function of x for $k \geq 2$, and a non-increasing function of x for $k \leq 1$. Hence when $k \geq 2$ and $\alpha > 1$, the gamma-power function distribution has the IFR property, while for $k \leq 1$ and $\alpha < 1$, it has the DFR property.

Now, for $r \geq 1$, $E_Z[G^{-1}(1-e^{-z})]^r = \theta^{-r}\sum_{j=0}^{\infty}(-1)^j\binom{r/k}{j}\frac{1}{(j+1)^\alpha}$ which leads to the mgf and moments of the gamma-power function distribution.

We also have, after some algebraic manipulation,

$$\begin{aligned} E_Z\{g(\overline{G}^{-1}(e^{-z}))\}^{\nu-1} &= (k\theta)^{\nu-1}\sum_{j=0}^{\infty}\binom{a}{j}\frac{(-1)^j}{(j+1)^\delta}, \text{ if } \alpha \geq 0 \\ &= (k\theta)^{\nu-1}\sum_{j=0}^{\infty}\binom{a+j-1}{j}\frac{1}{(j+1)^\delta}, \text{ if } \alpha < 0, \end{aligned}$$

where $\delta = \nu(\alpha-1)+1$, $\alpha = (\nu-1)\frac{k-1}{k}$. This leads to Rényi entropy by the application of (5).

5. Conclusion

The paper investigates some properties of gamma generated distributions, like the ageing property, the moment generating function, the quantiles, entropy, the stress-strength reliability, and the order statistics and their distributions. It is observed that the density curve of such distributions can take different shapes depending on the parametric value of the gamma distribution. Thus, such distributions exhibit more flexibility in modeling real life data. The gamma distribution is commonly used to fit lifetime data, survival data, hydrological data, etc. The class of gamma generated distributions provides more flexible distributions for such applications.

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Summary

Based on standard probability distributions, new families of univariate distributions have been introduced and their properties studied by many authors. The present paper investigates some general properties of a family of gamma generated distributions proposed by Zografos and Balakrishnan (2009).

Keywords: Gamma generated distributions; failure rate; moment generating function; stress-strength reliability; Rényi entropy; order statistics