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ESTIMATION OF PARAMETERS OF A POWER FUNCTION DISTRIBUTION AND ITS CHARACTERIZATION BY K-TH RECORD VALUES

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1. INTRODUCTION

Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them: *e.g.* Olympic records or world records in sports. Motivated by extreme weather conditions, Chandler (1952) introduced record values and record value times. Feller (1966) gave some examples of record values with respect to gambling problems. Resnick (1973) discussed the asymptotic theory of records. Theory of record values and its distributional properties have been extensively studied in the literature. See Ahsanullah (1988, 1995), Arnold and Balakrishnan (1989), Arnold *et al.* (1992, 1998), Nevzorov (1987) and Kamps (1995) for reviews on various developments in the area of records.

We shall now consider the situations in which the record values (*e.g.* successive largest insurance claims in non-life insurance, highest water-levels or highest temperatures) themselves are viewed as 'outliers'and hence the second or third largest values are of special interest. Insurance claims in some non-life insurance can be used as one of the examples. Observing successive k-th largest values in a sequence, Dziubdziela and Kopociński (1976) proposed the following model of k-th record values, where k is some positive integer.

Let X_1, X_2,......be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Let $X_{j:n}$ denote the j-th order statistic of a sample $(X_1,X_2,...,X_n)$. For a fixed $k \geq 1$, we define the sequence $U_1^{(k)}$, $U_2^{(k)}$,......of k-th upper record times of X_1, X_2,......as follows:

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\{j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_n^{(k)}+k-1}\},$$

$n = 1,2,......$

The sequence $\{Y_n^{(k)}, n \geq 1\}$, where $Y_n^{(k)} = X_{U_n^{(k)}}$ is called the sequence of k-th upper record values of the sequence $\{X_n, n \geq 1\}$. For convenience, we define $Y_0^{(k)} = 0$. Note that for $k = 1$ we have $Y_n^{(1)} = X_{U_n}$, $n \geq 1$, which are record values of $\{X_n, n \geq 1\}$ (Ahsanullah, 1995).

In this paper, we shall make use of the properties of the k-th upper record values to develop inferential procedures such as point estimation. We shall obtain the best linear unbiased estimates of the parameters of the power function distribution in terms of k-th upper record values. (Ahsanullah, 1986) considered the problem of estimation of parameters for rectangular distribution based on upper record values ($k = 1$). At the end we give the characterization of the power function distribution using k-th upper record values.

In order to derive the estimates for the parameters of the power function distribution and to give its characterization, we need some recurrence relations for single and product moments of k-th upper record values which have been established in the next section.

2. RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS

A random variable X is said to have a power function distribution if its pdf is given by

$$f(x) = \frac{\gamma}{\beta - \alpha}\left(\frac{\beta - x}{\beta - \alpha}\right)^{\gamma - 1} \quad , \qquad \alpha < x < \beta, \gamma > 0, \tag{1}$$

and the cdf is given by

$$F(x) = 1 - \left(\frac{\beta - x}{\beta - \alpha}\right)^{\gamma} \quad , \qquad \alpha < x < \beta, \gamma > 0. \tag{2}$$

The power function distribution is a member of Beta distributions. The Beta distribution is one of the most frequently employed to fit theoretical distributions. It is also used as a prior distribution for a binomial proportion. Now, let us consider some examples of the distribution (1).

For $\gamma = 1$, (1) is the pdf of a two-parameter rectangular distribution. The consumption of fuel by an airplane during a flight may be assumed to be rectangular with parameters α and β. The thickness of steel produced by the rolling machines of steel plants may be considered as rectangular distribution with parameters α and β (Ahsanullah, 1986).

It can be seen that for the power function distribution defined in (1)

$$(\beta - x) f(x) = \gamma(1 - F(x)), \qquad \alpha < x < \beta, \gamma > 0. \tag{3}$$

The relation in (3) will be employed in this paper to derive recurrence relations for the moments of k-th upper record values from the power function distribution.

Let $\{Y_n^{(k)}, n \geq 1\}$ be a sequence of k-th upper record values from (1). Then, the pdf of $Y_n^{(k)}$ $(n \geq 1)$ as given by Dziubdziela and Kopociński (1976) is as follows:

$$f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[H(x)]^{n-1}[1-F(x)]^{k-1} f(x). \tag{4}$$

Also the joint density function of $Y_m^{(k)}$ and $Y_n^{(k)}$ $(1 \leq m < n)$, $n = 2,3,\ldots$, as discussed by Grudzień (1982) is given by:

$$\begin{aligned} f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = {} & \frac{k^n}{(m-1)!\,(n-m-1)!}[H(x)]^{m-1}[H(y)-H(x)]^{n-m-1}h(x) \\ & .[1-F(y)]^{k-1} f(y), \qquad x < y, \end{aligned} \tag{5}$$

where $H(x) = -\log[1-F(x)]$, log is the natural logarithm and $h(x) = H'(x)$.

Theorem 1: Fix a positive integer $k \geq 1$. For $n \geq 1$ and $r = 0,1,2,\ldots$,

$$E(Y_n^{(k)})^{r+1} = \frac{(r+1)}{(r+1+k\gamma)}\left[\beta E(Y_n^{(k)})^r + \frac{k\gamma}{r+1}E(Y_{n-1}^{(k)})^{r+1}\right]. \tag{6}$$

Proof: For $n \geq 1$ and $r = 0,1,2,\ldots\ldots$, we have from (4) and (3)

$$\beta E(Y_n^{(k)})^r - E(Y_n^{(k)})^{r+1} = \frac{k^n\gamma}{(n-1)!}\int_\alpha^\beta x^r[H(x)]^{n-1}[1-F(x)]^k dx.$$

Integrating by parts, taking x^r as the part to be integrated and the rest of the integrand for differentiation, we get (6).

Theorem 2: For $1 \leq m \leq n-2$, $r, s = 0,1,2,\ldots\ldots$,

$$\begin{aligned} & E\left[(Y_m^{(k)})^r(Y_n^{(k)})^{s+1}\right] = \\ & = \frac{(s+1)}{(s+1+k\gamma)}\left[\beta E\{(Y_m^{(k)})^r(Y_n^{(k)})^s\} + \frac{k\gamma}{(s+1)}E\{(Y_m^{(k)})^r(Y_{n-1}^{(k)})^{s+1}\}\right] \end{aligned} \tag{7}$$

and for $m \geq 1$, $r, s = 0,1,2,\ldots\ldots$,

$$E\left[(Y_m^{(k)})^r (Y_{m+1}^{(k)})^{s+1}\right]= \tag{8}$$
$$=\frac{(s+1)}{(s+1+k\gamma)}\left[\beta E\{(Y_m^{(k)})^r (Y_{m+1}^{(k)})^s\}+\frac{k\gamma}{(s+1)}E(Y_m^{(k)})^{r+s+1}\right].$$

Proof: From (5), for $1 \le m \le n-1$ and r, s = 0,1,2,......, we obtain

$$\beta E\left[(Y_m^{(k)})^r (Y_n^{(k)})^s\right]-E\left[(Y_m^{(k)})^r (Y_n^{(k)})^{s+1}\right]= \tag{9}$$
$$=\frac{k^n}{(m-1)!\,(n-m-1)!}\int_\alpha^\beta x^r\,[H(x)]^{m-1}.h(x)\; I(x)dx\,,$$

where

$$I(x)=\gamma\int_x^\beta y^s\,[H(y)-H(x)]^{n-m-1}[1-F(y)]^k\,dy\,,$$

on using the relation in (3). Upon integrating by parts, treating y^s for integration, we get

$$I(x)=-\frac{(n-m-1)}{(s+1)}\int_x^\beta y^{s+1}[H(y)-H(x)]^{n-m-2}[1-F(y)]^{k-1}\; f(y)dy$$
$$+\frac{k\gamma}{(s+1)}\int_x^\beta y^{s+1}[H(y)-H(x)]^{n-m-1}[1-F(y)]^{k-1}\,f(y)dy\,.$$

Substituting the above expression for $I(x)$ in (9) and simplifying , it leads to (7). Proceeding in a similar manner for the case $n = m+1$, the recurrence relation given in (8) can easily be established.

3. ESTIMATION OF THE PARAMETERS α AND β WHEN SHAPE PARAMETER γ IS KNOWN

It can easily be shown, on using (1), (2) and (4), that

$$E(Y_n^{(k)})=\frac{k^n\gamma^n}{(n-1)!}\int_\alpha^\beta x\left[-\log\left(\frac{\beta-x}{\beta-\alpha}\right)\right]^{n-1}\left(\frac{\beta-x}{\beta-\alpha}\right)^{k\gamma-1}\frac{1}{\beta-\alpha}\,dx\,,$$

which on further simplification gives

$$E(Y_n^{(k)})=\beta-(\beta-\alpha)\left(\frac{k\gamma}{k\gamma+1}\right)^n. \tag{10}$$

Similarly, one can obtain

$$E(Y_n^{(k)})^2 = \beta^2 + (\beta-\alpha)^2\left(\frac{k\gamma}{k\gamma+2}\right)^n - 2\beta(\beta-\alpha)\left(\frac{k\gamma}{k\gamma+1}\right)^n .$$

Hence

$$\mathrm{Var}(Y_n^{(k)}) = (\beta-\alpha)^2\left[\left(\frac{k\gamma}{k\gamma+2}\right)^n - \left(\frac{k\gamma}{k\gamma+1}\right)^{2n}\right]. \qquad (11)$$

Further, on using the recurrence relations given in equations (6) and (7) in the relation

$$\mathrm{Cov}(Y_m^{(k)}, Y_n^{(k)}) = E(Y_m^{(k)} Y_n^{(k)}) - E(Y_m^{(k)})E(Y_n^{(k)}),$$

we get

$$\mathrm{Cov}(Y_m^{(k)}, Y_n^{(k)}) = \frac{k\gamma}{k\gamma+1}\mathrm{Cov}(Y_m^{(k)}, Y_{n-1}^{(k)}), \qquad n > m.$$

Applying it recursively , it can easily be verified that

$$\mathrm{Cov}(Y_m^{(k)}, Y_n^{(k)}) = \left(\frac{k\gamma}{k\gamma+1}\right)^{n-m}\mathrm{Var}(Y_m^{(k)}), \qquad n > m. \qquad (12)$$

Let us consider the following transformation

$$Z_1^{(k)} = Y_1^{(k)} ,$$

$$Z_i^{(k)} = \left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{i-1}{2}}\left(Y_i^{(k)} - \frac{k\gamma}{k\gamma+1}Y_{i-1}^{(k)}\right), \qquad i = 2,3,\ldots,n .$$

Then on using (10), we obtain

$$E(Z_1^{(k)}) = \frac{\beta}{k\gamma+1} + \left(\frac{k\gamma}{k\gamma+1}\right)\alpha, \qquad (13)$$

$$E(Z_i^{(k)}) = \left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{i-1}{2}}\frac{\beta}{k\gamma+1}, \qquad i = 2,3,\ldots,n. \qquad (14)$$

Similarly, on using (11) , we obtain

$$\mathrm{Var}(Z_i^{(k)}) = \frac{k\gamma(\beta-\alpha)^2}{(k\gamma+2)(k\gamma+1)^2}, \qquad i = 1,2,.....,n. \tag{15}$$

Also

$$\mathrm{Cov}(Z_i^{(k)}, Z_j^{(k)}) = 0, \quad \textit{for } i \neq j, \qquad 1 \leq i < j \leq n. \tag{16}$$

Let $\mathbf{Z}' = (Z_1^{(k)}, Z_2^{(k)}, ..., Z_n^{(k)})$. Then

$$\mathrm{E}(\mathbf{Z}) = \mathbf{A}\,\boldsymbol{\theta}\ , \tag{17}$$

where

$$\mathbf{A} = \begin{pmatrix} \frac{k\gamma}{k\gamma+1} & \frac{1}{k\gamma+1} \\ 0 & \left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{1}{2}} \frac{1}{k\gamma+1} \\ . & . \\ . & . \\ . & . \\ . & . \\ 0 & \left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{n-1}{2}} \frac{1}{k\gamma+1} \end{pmatrix}, \qquad \theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

The best linear unbiased estimates $\hat{\alpha}$, $\hat{\beta}$ of α and β, respectively (when γ is known), based on $Y_1^{(k)}$, $Y_2^{(k)}$,......, $Y_n^{(k)}$ are given by

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (\mathbf{A}'\mathbf{A})^{-1}\,\mathbf{A}'\mathbf{Z}\ . \tag{18}$$

We have

$$(\mathbf{A}'\mathbf{A}) = \begin{pmatrix} \left(\frac{k\gamma}{k\gamma+1}\right)^2 & \frac{k\gamma}{(k\gamma+1)^2} \\ \frac{k\gamma}{(k\gamma+1)^2} & S \end{pmatrix},$$

where

$$S=\left(\frac{1}{k\gamma+1}\right)^2+\left(\frac{k\gamma+2}{k\gamma}\right)\left(\frac{1}{k\gamma+1}\right)^2+\left(\frac{k\gamma+2}{k\gamma}\right)^2\left(\frac{1}{k\gamma+1}\right)^2+\ldots\ldots\ldots$$
$$+\left(\frac{k\gamma+2}{k\gamma}\right)^{n-1}\left(\frac{1}{k\gamma+1}\right)^2$$
$$=\frac{1}{(k\gamma+1)^2}\left[1+\frac{k\gamma+2}{2}\left\{\left(\frac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right\}\right] \quad .$$

Now

$$(\mathbf{A}'\mathbf{A})^{-1}=\frac{(k\gamma+1)^4}{(k\gamma)^2\left[\frac{(k\gamma+2)}{2}\left\{\left(\frac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right\}\right]}\begin{pmatrix} S & -\frac{k\gamma}{(k\gamma+1)^2} \\ -\frac{k\gamma}{(k\gamma+1)^2} & \left(\frac{k\gamma}{k\gamma+1}\right)^2\end{pmatrix}.$$

Substituting for $(\mathbf{A}'\mathbf{A})^{-1}$ in (18) and simplifying the resulting expression, we obtain

$$\hat{\alpha}=\frac{2(k\gamma+1)^4}{(k\gamma+2)(k\gamma)^2\left[\left(\frac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right]}\left[\left[\left(S\frac{k\gamma}{k\gamma+1}-\frac{k\gamma}{(k\gamma+1)^3}\right)Z_1^{(k)}-\frac{k\gamma}{(k\gamma+1)^3}\right.\right.$$
$$\left.\left.\cdot\left\{\left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{1}{2}}Z_2^{(k)}+\ldots..+\left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{n-1}{2}}Z_n^{(k)}\right\}\right]\right]$$

and

$$\hat{\beta}=\frac{2(k\gamma+1)^4}{(k\gamma+2)(k\gamma)^2\left[\left(\frac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right]}\left[\frac{(k\gamma)^2}{(k\gamma+1)^3}\left\{\left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{1}{2}}Z_2^{(k)}+\ldots\ldots\right.\right.$$
$$\left.\left.+\left(\frac{k\gamma+2}{k\gamma}\right)^{\frac{n-1}{2}}Z_n^{(k)}\right\}\right].$$

Hence

$$\mathrm{Var}(\hat{\alpha}) = \frac{2(\beta-\alpha)^2(k\gamma+1)^2}{(k\gamma+2)^2(k\gamma)} \frac{S}{\left[\left(\dfrac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right]},$$

$$\mathrm{Var}(\hat{\beta}) = \frac{2(\beta-\alpha)^2 k\gamma}{(k\gamma+2)^2\left[\left(\dfrac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right]}$$

and

$$\mathrm{Cov}(\hat{\alpha},\hat{\beta}) = -\frac{2(\beta-\alpha)^2}{(k\gamma+2)^2\left[\left(\dfrac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right]}.$$

The generalized variance $\hat{\Sigma}$ of $\hat{\alpha}$ and $\hat{\beta}$ ($\hat{\Sigma} = \mathrm{Var}(\hat{\alpha})\mathrm{Var}(\hat{\beta}) - (\mathrm{Cov}(\hat{\alpha},\hat{\beta}))^2$) is

$$\frac{2(\beta-\alpha)^4}{(k\gamma+2)^3\left(\left(\dfrac{k\gamma+2}{k\gamma}\right)^{n-1}-1\right)}.$$

On considering the two k-th upper record values $Y_s^{(k)}$ and $Y_r^{(k)}$ ($s > r$) it follows from (10) and (12) that the best linear unbiased estimates of α and β based on these two k-th record values are given by

$$\alpha^* = \left(\frac{k\gamma+1}{k\gamma}\right)^r Y_r^{(k)} - \left\{\left(\frac{k\gamma+1}{k\gamma}\right)^r - 1\right\}\beta^*,$$

$$\beta^* = \frac{Y_s^{(k)} - \left(\dfrac{k\gamma+1}{k\gamma}\right)^{r-s} Y_r^{(k)}}{1-\left(\dfrac{k\gamma+1}{k\gamma}\right)^{r-s}}.$$

The variances and covariances of α^* and β^* are

$$\mathrm{Var}(\alpha^*) = \frac{\{(k\gamma+1)^{2r} - (k\gamma)^r(k\gamma+2)^r\}}{(k\gamma)^r(k\gamma+2)^r}(\beta-\alpha)^2 + \left\{\left(\frac{k\gamma+1}{k\gamma}\right)^r - 1\right\}^2 \mathrm{Var}(\beta^*),$$

$$\operatorname{Var}(\beta^*) = \frac{\left\{\left(\frac{k\gamma+2}{k\gamma}\right)^{r-s} - \left(\frac{k\gamma+1}{k\gamma}\right)^{2(r-s)}\right\}}{\left(\frac{k\gamma+2}{k\gamma}\right)^{r}\left\{1-\left(\frac{k\gamma+1}{k\gamma}\right)^{(r-s)}\right\}^2}(\beta-\alpha)^2$$

and

$$\operatorname{Cov}(\alpha^*,\beta^*) = -\left\{\left(\frac{k\gamma+1}{k\gamma}\right)^{r} - 1\right\}\operatorname{Var}(\beta^*).$$

It can be shown that the generalized variance Σ^* of α^* and β^* $(\Sigma^* = \operatorname{Var}(\alpha^*)\ \operatorname{Var}(\beta^*) - (\operatorname{Cov}(\alpha^*,\beta^*))^2)$ is minimum when s = n and r = 1. Hence the best linear unbiased estimates of α and β based on two selected k-th record values are

$$\tilde{\alpha} = \frac{k\gamma+1}{k\gamma}\mathrm{Y}_1^{(k)} - \frac{\tilde{\beta}}{k\gamma},$$

$$\tilde{\beta} = \frac{\mathrm{Y}_n^{(k)} - \left(\frac{k\gamma+1}{k\gamma}\right)^{1-n}\mathrm{Y}_1^{(k)}}{1-\left(\frac{k\gamma+1}{k\gamma}\right)^{1-n}}.$$

Also

$$\operatorname{Var}(\tilde{\alpha}) = \frac{(\beta-\alpha)^2}{k\gamma(k\gamma+2)} + \frac{Var(\tilde{\beta})}{(k\gamma)^2},$$

$$\operatorname{Var}(\tilde{\beta}) = \frac{\{(k\gamma+1)^{2(n-1)} - (k\gamma)^{n-1}(k\gamma+2)^{n-1}\}}{\left\{\left(\frac{k\gamma+1}{k\gamma}\right)^{n-1} - 1\right\}^2 (k\gamma)^{n-2}(k\gamma+2)^n}(\beta-\alpha)^2$$

and

$$\operatorname{Cov}(\tilde{\alpha},\tilde{\beta}) = -\frac{\operatorname{Var}(\tilde{\beta})}{k\gamma}.$$

Let

$$e_1 = \frac{\operatorname{Var}(\hat{\alpha})}{\operatorname{Var}(\tilde{\alpha})}, \quad e_2 = \frac{\operatorname{Var}(\hat{\beta})}{\operatorname{Var}(\tilde{\beta})} \quad \text{and} \quad e_{12} = \frac{\operatorname{Cov}(\hat{\alpha},\hat{\beta})}{\operatorname{Cov}(\tilde{\alpha},\tilde{\beta})} .$$

Thus the generalized variance $\tilde{\Sigma}$ of $\tilde{\alpha}$ and $\tilde{\beta}$ is

$$\frac{\{(k\gamma+1)^{2(n-1)} - (k\gamma)^{n-1}(k\gamma+2)^{n-1}\}}{(k\gamma)^{n-1}(k\gamma+2)^{n+1}\left\{\left(\frac{k\gamma+1}{k\gamma}\right)^{n-1} - 1\right\}^2}(\beta-\alpha)^4 .$$

Further, it can be seen that $e_{12} = e_2$.

In Table 1 and Table 2, we tabulate the values of e_1 and e_2 for n = 2, 4, 6, 10, 15, 20 and 30, γ = 1 and k = 2 and 3, respectively . It can be seen from the tables that the efficiency of the best linear unbiased estimate of α based on two k-th record values are very high compared to the corresponding estimate based on a complete set of k-th record values.

TABLE 1

Values of e_1 and e_2 for k = 2

n	e_1	e_2
2	1.0000	1.0000
4	0.9965	0.9506
6	0.9977	0.8743
10	0.9996	0.7272
15	1.0000	0.6148
20	1.0000	0.5592
30	1.0000	0.5170

TABLE 2

Values of e_1 and e_2 for k = 3

n	e_1	e_2
2	1.0000	1.0000
4	0.9968	0.9688
6	0.9970	0.9148
10	0.9989	0.7846
15	0.9998	0.6492
20	1.0000	0.5613
30	1.0000	0.4725

Remark: Although β can be accurately estimated by taking large n, the estimate of α does not improve with increasing n (asymptotically for k = 2, $\operatorname{Var}(\hat{\alpha}) = 1/8$ and for k = 3, $\operatorname{Var}(\hat{\alpha}) = 1/15$).

4. UNBIASED ESTIMATE OF γ WHEN α AND β ARE KNOWN

The joint density function of the first *n* *k*-th record values is given by:

$$f_{Y_1^{(k)},\ldots,Y_n^{(k)}}(y_1,\ldots,y_n)=k^n\prod_{i=1}^{n-1}\frac{f(y_i)}{1-F(y_i)}(1-F(y_n))^{k-1}f(y_n),\quad y_1<\ldots<y_n, \tag{19}$$

(Kamps, 1995). Using (19), the likelihood function in this case is given by

$$L=\frac{k^n\gamma^n}{(\beta-\alpha)^n}\left\{\prod_{i=1}^{n-1}\left(\frac{\beta-y_i}{\beta-\alpha}\right)^{-1}\right\}\left(\frac{\beta-y_n}{\beta-\alpha}\right)^{k\gamma-1},\qquad \alpha<y_1<y_2<\ldots<y_n<\beta.$$

Hence

$$\log L=n\log\gamma+\log C+(k\gamma-1)\log\left(\frac{\beta-y_n}{\beta-\alpha}\right),$$

where

$$C=\frac{k^n}{(\beta-\alpha)^n}\prod_{i=1}^{n-1}\left(\frac{\beta-y_i}{\beta-\alpha}\right)^{-1}.$$

Then the maximum likelihood estimate of γ is

$$\tilde{\tilde{\gamma}}=\frac{n}{k\log\left(\frac{\beta-y_n}{\beta-\alpha}\right)}.$$

Further

$$\mathrm{E}(\tilde{\tilde{\gamma}})=\frac{k^{n-1}\gamma^n}{(n-1)!}\int_\alpha^\beta\frac{n}{\log\left(\frac{\beta-x}{\beta-\alpha}\right)}\left[-\log\left(\frac{\beta-x}{\beta-\alpha}\right)\right]^{n-1}\left(\frac{\beta-x}{\beta-\alpha}\right)^{k\gamma-1}\frac{1}{\beta-\alpha}\,dx$$

$$=\frac{n\,k^{n-1}}{(n-1)!}\int_\alpha^\beta\gamma^n z^{n-2}e^{-k\gamma z}dz$$

$$=\frac{n}{(n-1)}\gamma.$$

Note that $\tilde{\tilde{\gamma}}$ is a biased estimator for γ. The unbiased estimator γ^{**} for γ is given by

$$\gamma^{**} = \frac{(n-1)}{n}\tilde{\tilde{\gamma}} = \frac{(n-1)}{k\log\left(\dfrac{\beta - y_n}{\beta - \alpha}\right)},$$

and it can easily be verified that

$$\mathrm{Var}(\gamma^{**}) = \frac{\gamma^2}{(n-2)} \qquad , \quad n > 2.$$

5. CHARACTERIZATION

This section contains characterization of the power function distribution. We shall use the following result of (Lin,1986).

Theorem 3 (Lin): Let n_o be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$, and $g(x) \geq 0$ be an absolutely continuous function with $g'(x) \neq 0$ a.e. on (a, b) . Then the sequence of functions $\{(g(x))^n \, e^{-g(x)}, \, n \geq n_o\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).

Theorem 4: A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$E(Y_n^{(k)})^{r+1} = \frac{(r+1)}{(r+1+k\gamma)}\left[\beta E(Y_n^{(k)})^r + \frac{k\gamma}{r+1}E(Y_{n-1}^{(k)})^{r+1}\right], \tag{20}$$

for $n = 1,2,......$ and $r = 0,1,2,......$, where $k \geq 1$ is any fixed positive integer.

Proof: The necessary part follows from (6). On the other hand if the recurrence relation (20) is satisfied, we get

$$\frac{(r+1+k\gamma)}{(r+1)}\frac{k^n}{(n-1)!}\int_\alpha^\beta x^{r+1}[H(x)]^{n-1}[1-F(x)]^{k-1} f(x)dx = \frac{\beta k^n}{(n-1)!}\int_\alpha^\beta x^r [H(x)]^{n-1}$$
$$.\,[1-F(x)]^{k-1} f(x)dx$$
$$+ \frac{k^n\gamma}{(n-2)!(r+1)}\int_\alpha^\beta x^{r+1}[H(x)]^{n-2}$$
$$.\,[1-F(x)]^{k-1} f(x)dx\,.$$

Integrating the last integral on the right hand side of the above equation by parts, we get

$$\frac{k^n(r+1+k\gamma)}{(n-1)!\,(r+1)}\int_\alpha^\beta x^{r+1}[H(x)]^{n-1}[1-F(x)]^{k-1}f(x)\,dx=$$

$$=\frac{\beta k^n}{(n-1)!}\int_\alpha^\beta x^r[H(x)]^{n-1}f(x).[1-F(x)]^{k-1}\,dx+$$

$$+\frac{k^{n+1}\gamma}{(n-1)!(r+1)}\int_\alpha^\beta x^{r+1}[H(x)]^{n-1}.[1-F(x)]^{k-1}f(x)\,dx-$$

$$-\frac{k^{n`}\gamma}{(n-1)!}\int_\alpha^\beta x^r[H(x)]^{n-1}.[1-F(x)]^k\,dx\,,$$

which after simplification reduces to

$$\int_\alpha^\beta x^r[H(x)]^{n-1}[1-F(x)]^{k-1}\left[\frac{(r+1+k\gamma)}{r+1}x\,f(x)+\gamma(1-F(x))-\beta\,f(x)\right.$$
$$\left.-\frac{k\gamma x}{r+1}f(x)\right]dx=0\quad .$$

Using *Theorem 3* with $g(x)=-\log[1-F(x)]=H(x)$, it follows that

$$(\beta-x)\,f(x)=\gamma(1-F(x)),$$

which proves that $f(x)$ has the form (1).

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RIASSUNTO

La stima dei parametri di una funzione di potenza e la sua caratterizzazione attraverso i k *valori estremi*

Nel lavoro si derivano stime lineari e corrette dei parametri di una funzione di potenza sulla base dei *k* valori estremi.

SUMMARY

Estimation of parameters of a power function distribution and its characterization by k-th record values

In this paper linear unbiased estimates of the parameters of a power function distribution based on *k*-th record values have been derived.