A NOTE ON THE BOOTSTRAP METHOD FOR TESTING THE EXISTENCE OF FINITE MOMENTS.

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Assumptions that a finite moment of the first, second, fourth or another order exists appear in many theorems in econometrics and statistics. And so it is worthwhile to check if these assumptions are satisfied. One recent method was proposed by Fedotenkov (2013), who suggests applying bootstrap to this problem, and proves consistency of the suggested method. The method employs the idea of Derman and Robbins (1955): if a sample is drawn from a distribution with an infinite mean - under certain general assumptions when the number of observations grows, the arithmetic mean of the sample grows faster than the arithmetic means of the subsamples of a smaller size. This method allows the researcher to test the assumption of the existence of the finite mean, skipping the step of the estimation of a tail index. One could think to use this test as a tail index or moment estimator so as to increase sequentially a power to which the observations are raised until the test starts to reject the hypothesis that the corresponding finite moment exists. The smallest power at which the test starts to reject the null hypothesis gives an inference about the tail index. Although this method of tail index estimation sounds logical and intuitive, our paper holds that it may lead to misleading results. One misunderstanding can already be seen at this stage: if this method skips the calculation of the tail index, why should it be used for tail index estimation? There are many methods designed intentionally for this purpose, several of which are discussed below.

Hypotheses about the existence of finite moments are usually checked assuming that the tail of the distribution $P(X > x)$ behaves as $x^{-\gamma}$, $\gamma > 0$, for large x, estimating the tail index γ. The moments of an order larger or equal to the tail index are infinite. The most popular method for employing this idea is the Hill estimator (Hill 1975), which estimates $1/\gamma$. Deheuvels et al. (1988) presented conditions for consistency of the Hill estimator in the case of independent observations. Resnick and Stărică (1998) proved its consistency for dependent stationary sequences. A common problem of the Hill estimator is that it is not always clear how many observations should be used for the tail index estimation. To improve the method, a number of modifications and improvements were suggested, including Hill plots - where the estimates of the tail index are plotted over the number of

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observations treated as a tail, trusting that the region of stability of the plot would give an inference about the tail index. Once again, to improve this method, Hill plots with a logarithmic axis (Resnick and Stărică (1997), Drees, Resnick, and de Haan (2000)) were developed, and many other methods were proposed. Another weakness of the Hill estimator is that when applied to a sample drawn from a light-tailed distribution with infinitely many finite moments, it provides misleading results. In such cases a more general moment estimator, such as DEdH shall be applied (Dekkers, Einmahl and de Haan (1989)). As we show in this paper, the use of bootstrap for testing the finiteness of higher-order moments if samples are drawn from distributions with infinitely many finite moments may also produce unreliable results.

The rest of this paper is organized as follows: In the next section possible misapplications of the test are described; section 2 presents examples based on Monte-Carlo simulations; section 3 concludes.

1. THE TEST AND ITS POSSIBLE MISAPPLICATIONS

Denote $(X_1, ..., X_n)$ a sample of size n drawn from a distribution $\mathcal{F}$, $X_i > 0, i = 1, ..., n$. $(X_1^*, ..., X_m^*)$ is a bootstrap subsample with replacement of size m, $m \to \infty$, $m = o(n)$ as $n \to \infty$, randomly and independently drawn from $(X_1, ..., X_n)$. The method introduced by Fedotenkov (2013) for testing the null hypothesis that the first moment of the distribution $\mathcal{F}$ is finite, with an alternative that it is infinite, works as follows: First a significance level α is chosen. Next, p-value is calculated:

$$p_n = \frac{1}{B} \sum_{j=1}^{B} I\left(\sum_{s=1}^{m} X_{j,s}^*/m > \xi \sum_{i=1}^{n} X_i/n \right). \tag{1}$$

where I is a unity indicator function, B is the number of bootstrap subsamples and ξ is a constant $0 < \xi < 1$ (close to 1). H_0 is accepted if $p_n > \alpha$ otherwise it is rejected.

As mentioned in Fedotenkov's paper, if a researcher wants to test the existence of a finite k-th order moment, $k > 0$, he/she should raise the observations to the power k. Formally, the hypothesis can be reformulated in the following way:

1. H_0: Distribution $\mathcal{F}$ has a finite k-th moment.

2. H_1: Distribution $\mathcal{F}$ does not have a finite k-th moment.

Then, the p-value is calculated as

$$p_n = \frac{1}{B} \sum_{j=1}^{B} I\left(\sum_{s=1}^{m} X_{j,s}^{*k}/m > \xi \sum_{i=1}^{n} X_i^k/n \right), \tag{2}$$

the rest of the testing procedure remaining the same. The proof of consistency of such a test easily reduces to the proof used by Fedotenkov (2013) for the first moments.

What are the possible applications of such a test? First, if a researcher wants to calculate descriptive statistics, such as mean, variance or skewness, the test

helps to ascertain if such statistics are meaningful. Second, if a researcher wants to use statistical or econometric techniques which require the assumption of the existence of a finite moment of a specific order, this hypothesis can also be tested. However, raising the given sample to an ever larger power until the test starts to reject H_0 so as to estimate the tail index or a number of finite moments is not recommended. The null-hypothesis is usually supposed to be more important than the alternative; the inequality of hypotheses leading to a bias in such an estimation. Furthermore, even if the sample is drawn from a distribution with infinitely many finite moments, at some power k the test starts to reject the hypothesis that the k-th order moment is finite. This is shown in the following statement.

Denote the r-th order statistics of $(X_1, ..., X_n)$ as $X_{(r)}$, i.e. $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$.

PROPOSITION 1. *Suppose that $\mathcal{F}$ is a continuous distribution with a positive support, so that $0 < X_{(1)} < X_{(2)} < ... < X_{(n)}$. Furthermore, suppose that the sample size n is large enough that $m/n < \alpha$ with the fixed significance level α, and the number of bootstrap subsamples B is sufficiently large. Then k_0 exists, $k_0 < \infty$ such that for $k > k_0$ the test rejects the hypothesis that the k-th order moment is finite with probability 1.*

As it is assumed that the sample is drawn from a continuous distribution, the case $X_{(j)} = X_{(j-1)}$ for any j occurs with probability equal to 0, and does not affect the analysis.

PROOF. Denote $\hat{\mathcal{F}}(X) = \sum_{i=1}^{n} I(X_i < X)/n$, and the probability measure corresponding to the empirical distribution function $\hat{\mathcal{F}}$ as Pr^*:

$$Pr^*\left(\frac{1}{m}\sum_{s=1}^{m} X_s^{*k} > \frac{\xi}{n}\sum_{i=1}^{n} X_i^k\right) \leq Pr^*\left(\frac{n}{m\xi}\sum_{s=1}^{m} X_s^{*k} > X_{(n)}^k\right) = \quad (3)$$

$$Pr^*(X_{(n)} \in \{X_1^*, ..., X_m^*\})Pr^*\left(\frac{n}{m\xi}\sum_{s=1}^{m} X_s^{*k} > X_{(n)}^k \Big| X_{(n)} \in \{X_1^*, ..., X_m^*\}\right) +$$

$$Pr^*(X_{(n)} \notin \{X_1^*, ..., X_m^*\})Pr^*\left(\frac{n}{m\xi}\sum_{s=1}^{m} X_s^{*k} > X_{(n)}^k \Big| X_{(n)} \notin \{X_1^*, ..., X_m^*\}\right).$$

First, consider the case when $X_{(n)}$ (the largest element in the sample) is not in the subsample $(X_1^*, ..., X_m^*)$. In this case $X_i^* \leq X_{(n-1)}$, $i = 1...m$,

$$Pr^*\left(\frac{n}{m\xi}\sum_{s=1}^{m} X_s^{*k} > X_{(n)}^k \Big| X_{(n)} \notin \{X_1^*, ..., X_m^*\}\right) \leq$$

$$Pr^*\left(\frac{n}{\xi} X_{(n-1)}^k > X_{(n)}^k \Big| X_{(n)} \notin \{X_1^*, ..., X_m^*\}\right) = 0$$

if $k > \log{(n\xi^{-1})}/\log{(X_{(n)} X_{(n-1)}^{-1})} =: k_0$. Thus, for $k > k_0$ the second term in the right side of equation (3) is equal to zero.

Consider the first term in the right side of the equation (3). Using Bernoulli's inequality, the probability that the largest element of the sample would be drawn into a subsample, can be evaluated as: $Pr^*(X_{(n)} \in \{X_1^*, ..., X_m^*\}) = 1 - (1 - n^{-1})^m \leq m/n < \alpha$. If the number of bootstrap subsamples B grows, p_n, defined in equation (2), converges almost surely to the probability defined on the left side of equation (3) according to the Glivenko-Cantelli lemma; thus, if B is large enough, the test rejects the hypothesis that the moment of order k is finite, $k > k_0$, with probability 1.

It follows from proposition 1 that even if the sample of size n is drawn from a distribution with infinitely many finite moments, a power k_0 can be found so that the hypothesis that the k_0-th (or higher) moment is finite is rejected. According to the definition of k_0, k_0 is itself a random variable, but for a fixed sample this value is fixed and finite. Definitely, if H_1 is valid, when n grows, k_0 also increases, and we may expect that the probability of making the first-order mistake decreases. But with a fixed-sample-size, increase of k may yield misleading results.

Similarly, if the sample is drawn from a distribution with super-heavy tails, such as a log-Pareto or log-Cauchy distribution, with tail indexes equal to zero, (such distributions do not have finite positive moments), misleading results can be achieved. Raising the given sample to an ever-smaller power k until the test starts to accept H_0 in order to estimate the tail index is not a wise strategy either. For distributions with super-heavy tails such an approach would overestimate the tail index. This property is formally formulated and proved in the following proposition.

PROPOSITION 2. *Suppose that $\mathcal{F}$ is a continuous distribution with a positive support, so that $0 < X_{(1)} < X_{(2)} < ... < X_{(n)}$. Furthermore, suppose that the number of bootstrap subsamples B is sufficiently large. Then such k_1 exists, $k_1 > 0$ that for k, $0 < k < k_1$ the test accepts the hypothesis that the k-th order moment is finite with probability one.*

PROOF. Recall that ξ is a constant $0 < \xi < 1$.

$$Pr^*\left(\frac{1}{m}\sum_{s=1}^{m} X_s^{*k} > \frac{\xi}{n}\sum_{i=1}^{n} X_i^k\right) = Pr^*\left(\overline{X^{*k}}/\overline{X^k} > \xi\right) \geq \qquad (4)$$

$$Pr^*\left(\left(X_{(1)}/X_{(n)}\right)^k > \xi\right) = 1,$$

if

$$k < \frac{log(\xi)}{log(X_{(1)}/X_{(2)})} = k_1 > 0. \qquad (5)$$

Analogous to proposition 1, if the number of bootstrap subsamples B grows, p_n, defined in equation (2), converges almost surely to the probability defined in the left side of the equation (4) according to the Glivenko-Cantelli lemma; thus, if B is large enough, the test accepts the hypothesis that the moment of order k, $0 < k < k_1$, is finite with probability 1.
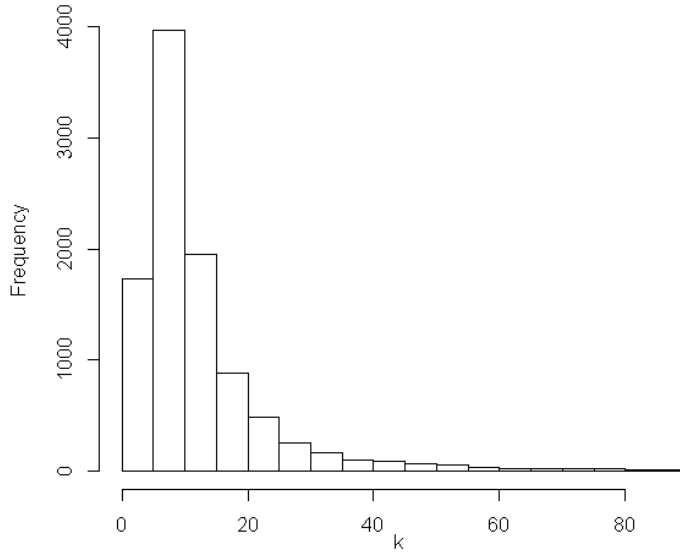
Figure 1 – The smallest moments for which H_0 is rejected (lognormal distribution).

2. MONTE-CARLO EXAMPLE

In this section two Monte-Carlo examples are presented to illustrate the performance of the test in case of distributions with infinitely large number of moments, and no finite positive moments at all. This Monte-Carlo experiment gives some insight about the range of moments for which the test performs well, and also gives indications when the test makes too many mistakes. The first example illustrates test performance for a sample drawn from the standard lognormal distribution of size $n = 200$, when k increases. Lognormal distribution has infinitely many finite moments, thus H_0 is true for each k, $k > 0$. The bootstrap subsample size is set to $m = 5(\approx \log n)$, parameter ξ is equalized to 0.9. 10,000 samples were drawn, from each of them 5,000 bootstrap subsamples with replacements were randomly selected.

Figure 1 is a histogram of the smallest integer moments for which the test rejects the hypothesis that k-th finite moment exists. 1% of the largest values are not included in the histogram; the inclusion of values near 350 would have made the histogram unreadable. As seen in the figure, sometimes the numbers of estimated moments in the samples are not large, indicating that the effects discussed in the paper hold not only for very large moments. The test committed no mistakes for $k = 1$. For higher moments the test did not perform so well. Namely, for $k = 2$ the test was in error in 0.2% of cases. For $k = 3$ the test rejected the null hypothesis in 2.43% of cases, and for $k = 4$ the rate of rejections exceeded the 5% level (6.47%). Therefore, the conclusion can be drawn that, in this case, the test makes a reasonable number of mistakes for $k \leq 3$, and it is better not to use it for higher moments.
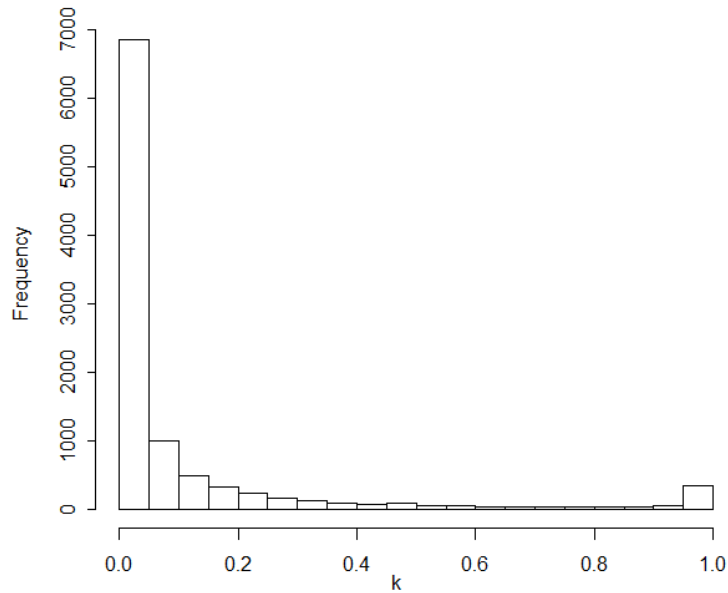
Figure 2 – The largest moments for which H_0 is accepted (log-Pareto distribution).

The second example illustrates test performance in the case of super-heavy tails. Namely, samples were drawn from a log-Pareto distribution with shape and scale parameters equal to 1. k was then gradually reduced from $k = 1$ till $k = 0$ by 0.01 steps till the test accepted the hypothesis that the corresponding moment was finite with the significance level 0.05. The remainder of the experimental design (sample-sizes, number of bootstrap subsamples, etc.) was unchanged from the previous example. The results of this experiment are presented in Figure 2.

In 1.52% of cases, the test accepted the hypothesis that the first moment is finite; 5% error level was achieved checking the finiteness of the $k > 0.67$. In 31.54% of cases, the test accepted the hypothesis of the finiteness of $k > 0.05$ moments; in 32.62% of cases, the test rejected the hypothesis that the $k = 0.01$ moment is finite and made no mistakes. Again it could be concluded that the test performed rather well for testing the finiteness of moments, which are close to 1.

The R code, which was used for simulations, can be found in the online appendix on the author's personal web page.

3. Conclusions

This short note advises caution in using the test for the existence of finite moments, developed by Fedotenkov (2013) to estimate tail indexes or numbers of moments. We found that the test may perform poorly with the hypothesis that high order moments are finite, and even if a sample is drawn from a distribution with infinitely many finite moments, such a k_0 exists that the test rejects the hypothesis about

the k-th $k \geq k_0$ order finite moment existence. As Monte-Carlo simulations show, the test performs well when it is applied for testing the finiteness of the first three moments, with the best results for the testing if the first moment exists. Similarly, a reduction of k, with $k > 0$, may overestimate the tail index if the sample is taken from a distribution with super-heavy tails with the tail indexes equal to zero. But the test performs rather well when it is used for testing if the finite moments of an order close to unity exist.

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Summary

This paper discusses a bootstrap-based test, which checks if finite moments exist, and indicates cases of possible misapplication. It notes, that a procedure for finding the smallest power to which observations need to be raised, such that the test rejects a hypothesis that the corresponding moment is finite, works poorly as an estimator of the tail index or moment estimator. This is the case especially for very low- and high-order moments. Several examples of correct usage of the test are also shown. The main result is derived analytically, and a Monte-Carlo experiment is presented.

Keywords: Bootstrap; finite moments; heavy tails; tail index estimator; test