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CHARACTERIZATION OF BIVARIATE DISTRIBUTIONS USING CONCOMITANTS OF GENERALIZED (K) RECORD VALUES

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1. Introduction

Chandler (1952) introduced the study of record values and documented many of its basic properties. For a detailed discussion on the developments of the theory and applications of record values, see Arnold *et al.* (1998). The applications of order statistics and record values in characterizing univariate distributions are well known. For detailed discussion on these applications, see Ahsanullah (2004), Arnold *et al.* (1998), David and Nagaraja (2003) and Rao and Shanbhag (1998). It is to be noted that in the bivariate setup, concomitants of record values assume its importance instead of the well known record values of the univariate case.

The study of record concomitants was initiated by Houchens (1984). Developments in the theory and applications of concomitants of record values and concomitants of order statistics open the door for analysis of data arising from bivariate distributions in a new perspective. For a description about the theory of concomitants of record values, see Ahsanullah and Nevzorov (2000). For a recent account on the use of concomitants of record values in estimation, see Chacko (2007), Chacko and Thomas (2006, 2008). Veena and Thomas (2008) and Thomas and Veena (2011) have attempted for the first time to characterize some bivariate distributions using the properties of concomitants of order statistics. Also in the available literature it seems Thomas and Veena (2014) is the only paper in which some results on characterizing a class of bivariate distributions by properties of concomitants of record values are discussed.

A difficulty that one encounters in dealing with inference problems based on record values is about the limited occurrence of record data, as the expected

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waiting time of occurrence of every record data after the first is infinite. However one may observe that generally the k-th record values as introduced by Dziubdziela and Kopociński (1976) occur more frequently than those of the classical record values. Suppose $\{X_n\}$ is a sequence of independent and identically distributed (iid) random variables with a common distribution function $F(x)$ which is absolutely continuous. Then for a positive integer $k \geq 1$, we define the sequence $\{{U_m}^{(k)}, m \geq 1\}$ of k^{th} upper record times of $\{X_n, n \geq 1\}$ as follows. ${U_1}^{(k)} = 1, {U_{m+1}}^{(k)} = min\{j > {U_m}^{(k)} : X_{j:j+k-1} > X_{{U_m}^{(k)}:{U_m}^{(k)}+k-1}\}$ where we used $X_{r:n}$ to denote the rth order statistic of a sample of size n. Then Dziubdziela and Kopociński (1976) defined the sequence $\{X_{m,k}\}$, where $X_{m,k} = X_{{U_m}^{(k)}:{U_m}^{(k)}+k-1}$ as the sequence of kth record values. The k-th member of the sequence of the classical record values is also called as k-th record value. This contradicts with the k-th record values as defined in Dziubdziela and Kopociński (1976). Pointing out this conflict in the usage of k-th record values and due to the reason that when $k = 1$ is used, the k-th record values defined by Dziubdziela and Kopociński (1976) generate the classical record values, Minimol and Thomas (2013, 2014) have called the k-th record values as defined in Dziubdziela and Kopociński (1976) as the generalized record values. Agreeing with the contention of Minimol and Thomas (2013, 2014), we also call the above defined k-th upper record values as generalized upper (k) record values (GURVs) all through this paper. In a similar manner, we can also define generalized lower (k) record values (GLRVs). If $\{(X_i, Y_i); i = 1, 2, \dots\}$ is a sequence of bivariate random variables, then from the marginal sequence $\{X_i\}$ of iid univariate random variables, we can construct the sequence of GURVs. Then the accompanying values on the variable Y of the ordered pairs with X variable taking GURVs on the variable X defines the sequence of concomitants of GURVs. This sequence of concomitants of GURVs may be denoted by $\{Y_{[n,k]}\}$. Similarly one can construct the sequence of concomitants of generalized lower (k) record values (GLRVs) as well and is denoted by $\{Y_{(n,k)}\}$. If we interchange the role of X and Y in the above definition then it generates the sequences $\{X_{[n,k]}\}$ and $\{X_{(n,k)}\}$ of concomitants of GURVs and GLRVs respectively on the variable X as well.

Let the sequence $\{(X_i, Y_i)\}$ of iid random variables has the common distribution function $F(x, y)$ with pdf $f(x, y)$. If the common marginal distribution functions of the sequences $\{X_i\}$ and $\{Y_i\}$ are denoted by $F_X(x)$ and $F_Y(y)$ respectively, then the pdf of $Y_{[n,k]}$, the concomitant of the n^{th} GURV is given by (see, Chacko and Mary, 2013)

$$f_{Y_{[n,k]}}(y) = \frac{k^n}{\Gamma(n)} \int_x \{-log[1 - F_X(x)]\}^{n-1}[1 - F_X(x)]^{k-1} f(x, y)\, dx. \tag{1}$$

The pdf of $X_{[n,k]}$, the concomitant of the n^{th} GURV with respect to the variable X is then given by

$$f_{X_{[n,k]}}(x) = \frac{k^n}{\Gamma(n)} \int_y \{-log[1 - F_Y(y)]\}^{n-1}[1 - F_Y(y)]^{k-1} f(x, y)\, dy. \tag{2}$$

In a similar manner the pdf's $f_{Y_{(n,k)}}(y)$ and $f_{X_{(n,k)}}(x)$ of the concomitants of the

GLRVs $Y_{(n,k)}$ and $X_{(n,k)}$ respectively are given by

$$f_{Y_{(n,k)}}(y) = \frac{k^n}{\Gamma(n)} \int_x \{-log[F_X(x)]\}^{n-1}[F_X(x)]^{k-1} f(x,y)\, dx \tag{3}$$

and

$$f_{X_{(n,k)}}(x) = \frac{k^n}{\Gamma(n)} \int_y \{-log[F_Y(y)]\}^{n-1}[F_Y(y)]^{k-1} f(x,y)\, dy. \tag{4}$$

Throughout this paper we will assume that the cdf $F(x,y)$ of the bivariate random variable (X,Y) is absolutely continuous with pdf $f(x,y)$. We will assume that $f(x,y)$ admits the partial derivatives $\frac{\partial}{\partial x}f$, $\frac{\partial}{\partial y}f$, $\frac{\partial^2}{\partial x^2}f$, $\frac{\partial^2}{\partial y^2}f$. We write $f_X(x)$ to denote the pdf of the marginal random variable X with cdf $F_X(x)$. We define $\alpha = inf\{x : f_X(x) > 0\}$ and $\beta = sup\{x : f_X(x) > 0\}$ so that $f_X(x)$ is non-vanishing in the set (α,β) and Lebesgue integrable. Similarly if we write the cdf and pdf of the marginal random variable Y as $F_Y(y)$ and $f_Y(y)$ respectively and define $\gamma = inf\{y : f_Y(y) > 0\}$, $\delta = sup\{y : f_Y(y) > 0\}$, then $f_Y(y)$ is non-vanishing in the set (γ,δ) and Lebesgue integrable. We will also assume in this section that the marginal densities $f_X(x)$ and $f_Y(y)$ admit the first two derivatives. If we consider the family $\mathfrak{F}$ of distributions with the pdf $f(x,y)$ as defined above has a form for $f(x,y)$ given by

$$\begin{aligned} f(x,y) = f_X(x)f_Y(y) + \sum_{i=1}^{t} \alpha_i \{F_X(x)[1-F_X(x)]\}^{m_i}[1-2F_X(x)]f_X(x) \\ \times \{F_Y(y)[1-F_Y(y)]\}^{p_i}[1-2F_Y(y)]f_Y(y), \end{aligned} \tag{5}$$

where t is a positive integer, $m_i, p_i, i = 1,2,\ldots,t$ are non-negative reals and α_i's are constants constrained to lie in suitable intervals about zero, so that the support set of (X,Y) is $(\alpha,\beta)\times(\gamma,\delta)$, then $\mathfrak{F}$ is known as the generalized Morgenstern family of bivariate distributions (GMFBD). For more details regarding this family $\mathfrak{F}$ of distributions, see Veena and Thomas (2008) and Thomas and Veena (2011).

If we put $t = 1, \alpha_1 = \alpha, m_1 = p_1 = 0$, then the form of the pdf of the case reduces to

$$h(x,y) = f_X(x)f_Y(y) + \alpha[1-2F_X(x)]f_X(x)[1-2F_Y(y)]f_Y(y). \tag{6}$$

The sub-family $\mathfrak{M}$ of distributions ($\mathfrak{M} \subset \mathfrak{F}$) with pdf as given above is called the Morgenstern family of bivariate distributions.

In section 2, we prove a theorem characterizing GMFBD with pdf given by (5) and this theorem provides the extension of the results given in Thomas and Veena (2014) relating to the concomitants of record values to the concomitants of generalized record values case. The immediate application of the results of the theorem in modeling a bivariate distribution based on bivariate data sets available from a distribution has been also pointed out in section 2. In section 3, we establish the role of concomitants of GURVs in the unique determination of the parent bivariate distribution. In this section, we have demonstrated how the concomitants of GURVs arising from the Morgenstern family of bivariate distributions and the concomitants of classical record values arising from GMFBD determine the respective parent bivariate distributions uniquely. Section 4 deals with the role of concomitants of GLRVs in the unique determination of the parent bivariate distribution. In this section we have illustrated how the concomitants of GLRVs arising from Morgenstern family of bivariate distributions and the concomitants of classical record values arising from GMFBD determine the respective parent bivariate distributions uniquely. Sections 3 and 4 describe additional new results which provide certain conditions associated with GURVs and GLRVs which determine uniquely the parent bivariate distribution. Some illustrations also have been provided in these sections.

2. Characterization of the generalized Morgenstern family of bivariate distributions

We establish now the following theorem describing certain properties of concomitants of generalized record values which characterize the generalized Morgenstern family $\mathfrak{F}$ of bivariate distributions.

Theorem 1. *For any positive integer* $n \geq 2$, $f_{Y_{[n,k]}}(y) + f_{Y_{(n,k)}}(y) = 2f_Y(y)$, $\forall y \in (\gamma, \delta)$ *and* $f_{X_{[n,k]}}(x) + f_{X_{(n,k)}}(x) = 2f_X(x)$, $\forall x \in (\alpha, \beta)$ *if and only if the parent bivariate distribution belongs to the generalized Morgenstern family* $\mathfrak{F}$ *of bivariate distributions with pdf (5).*

Proof. Suppose the parent bivariate distribution is defined by the pdf $f(x, y)$ as given in (5). Then from (1) and (3) we have

$$f_{Y_{[n,k]}}(y) + f_{Y_{(n,k)}}(y) = \int_x \{\frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!} + \frac{k^n[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!}\}f(x,y)dx.$$

Here

$$\begin{aligned}\int_x & \frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!} f_X(x)f_Y(y)dx \\ &= \int_x \frac{k^n[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!} f_X(x)f_Y(y)dx \\ &= f_Y(y)\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^{t} \alpha_i \{F_Y(y)[1-F_Y(y)]\}^{p_i}[1-2F_Y(y)]f_Y(y) \\ &\times \int_x \left\{ \frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!} \right. \\ &\left. + \frac{k^n[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!} \right\} \{F_X(x)[1-F_X(x)]\}^{m_i} \\ &\qquad \times [1-2F_X(x)]f_X(x)dx = 0,\end{aligned}$$

as $\int_x \{k^n \frac{[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!}\}\{F_X(x)[1-F_X(x)]\}^{m_i}$

$$\times [1-2F_X(x)]f_X(x)dx$$
$$= \int_0^\infty \frac{k^n u^{n-1}}{(n-1)!}(1-e^{-u})^{m_i}(e^{-u})^{m_i+k}[e^{-u}-(1-e^{-u})]du$$

and $\int_x \{\frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!}\}\{F_X(x)[1-F_X(x)]\}^{m_i}$
$$\times [1-2F_X(x)]f_X(x)dx$$
$$= \int_0^\infty \frac{k^n u^{n-1}}{(n-1)!}(1-e^{-u})^{m_i}(e^{-u})^{m_i+k}[(1-e^{-u})-e^{-u}]du.$$

Thus, for any $n \geq 2$, we obtain

$$f_{Y_{[n,k]}}(y) + f_{Y_{(n,k)}}(y) = 2f_Y(y),\ \forall y \in (\gamma, \delta).$$

Similarly, using (2) and (4) we obtain for any $n \geq 2$,

$$f_{X_{[n,k]}}(x) + f_{X_{(n,k)}}(x) = 2f_X(x),\ \forall x \in (\alpha, \beta).$$

Conversely, assume that for $n \geq 2$,

$f_{Y_{[n,k]}}(y) + f_{Y_{(n,k)}}(y) = 2f_Y(y)$, $\forall y \in (\gamma, \delta)$ and

$f_{X_{[n,k]}}(x) + f_{X_{(n,k)}}(x) = 2f_X(x)$, $\forall x \in (\alpha, \beta)$.

Then from the first equation we have

$$\int_x \left\{ \frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!} + \frac{k^n[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!} \right\} f(x,y)dx = 2f_Y(y).$$

That is, $$\int_x \left\{ \frac{k^n[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}}{(n-1)!} + \frac{k^n[-log(1-F_X(x))]^{n-1}[1-F_X(x)]^{k-1}}{(n-1)!} \right\} a(x,y)dx = 0, \tag{7}$$

where $a(x,y) = f(x,y) - f_X(x)f_Y(y)$. Clearly (7) explains that the function within the integral of its left side is integrable and when integrated it gives a value equal to zero. We know that $-logF_X(x) = \sum_{j=1}^{\infty} \frac{[1-F_X(x)]^j}{j}$ and $-log(1-F_X(x)) = \sum_{j=1}^{\infty} \frac{[F_X(x)]^j}{j}$. It follows that $[-logF_X(x)]^{n-1}[F_X(x)]^{k-1}$ and $[-log(1-F_X(x))]^{n-1} \times[1-F_X(x)]^{k-1}$ have similar power series expressions with arguments $1-F_X(x)$ and $F_X(x)$ respectively. Also since $[1-F_X(x)]^i + [F_X(x)]^i$ can be written further as a polynomial in $F_X(x)[1-F_X(x)]$ (Thomas and Veena, 2011,see), we can represent (7) by the following:

$$\int_x \sum_{i=1}^{\infty} c_i\{F_X(x)[1-F_X(x)]\}^i a(x,y)dx = 0, \forall y \in (\gamma, \delta), \tag{8}$$

where c_i's are constants and each term in the integrand is integrable as $a(x,y) = f(x,y) - f_X(x)f_Y(y)$. Now on integrating by parts the integral on the left side of (8) we get

$$\begin{aligned} &\left[\frac{a(x,y)}{[1-2F_X(x)]f_X(x)}\sum_{i=1}^{\infty} c_i \frac{\{F_X(x)[1-F_X(x)]\}^{i+1}}{i+1}\right]_{\alpha}^{\beta} \\ &-\int_{\alpha}^{\beta}\frac{\partial}{\partial x}\left\{\frac{a(x,y)}{[1-2F_X(x)]f_X(x)}\right\}\sum_{i=1}^{\infty} c_i \frac{\{F_X(x)[1-F_X(x)]\}^{i+1}}{i+1}dx = 0 \end{aligned} \tag{9}$$

where the first term of the above expression is clearly equal to zero and the second term should be integrable as (9) is derived from the integrable function given in (8). Clearly for all choices of $a(x,y)$ at the neighbourhood of the median of the marginal random variable X and for given y the function $\frac{\partial}{\partial x}\left[\frac{a(x,y)}{[1-2F_X(x)]f_X(x)}\right]$ may not be bounded as a function of x. However if we substitute $K_1(x,y)[1-2F_X(x)]f_X(x)$ for $a(x,y)$, where for given y, $\frac{\partial}{\partial x}K_1(x,y)$ is bounded for all $x \in (\alpha,\beta)$, then it makes $\frac{\partial}{\partial x}\left[\frac{a(x,y)}{[1-2F_X(x)]f_X(x)}\right]$ bounded as a function of x over (α,β). Also

$$a(x,y) = K_1(x,y)[1-2F_X(x)f_X(x) \tag{10}$$

makes the second term of (9) integrable. Hence any possible solution of $a(x,y)$ in (8) is of the form given by (10). Now using $a(x,y) = K_1(x,y)[1-2F_X(x)]f_X(x)$ within the integral of (9), applying integration by parts once again and simplifying we obtain

$$\int_{\alpha}^{\beta}\frac{\partial}{\partial x}\left\{\frac{\frac{\partial}{\partial x}[K_1(x,y)]}{[1-2F_X(x)]f_X(x)}\right\}\sum_{i=1}^{\infty} d_i\{F_X(x)[1-F_X(x)]\}^{i+2}\,dx = 0, \tag{11}$$

where $d_i = \frac{c_i}{(i+1)(i+2)}, i = 1,2,\ldots,$. Once again following similar arguments as put forwarded already, we can say that $\frac{\partial}{\partial x}[K_1(x,y)]$ can have a representation given by $K_2(x,y)[1-2F_X(x)]f_X(x)$, where for given y, $K_2(x,y)$ is bounded as a function of $x, \forall x \in (\alpha,\beta)$. Clearly if we put

$$K_1(x,y) = M_1(y)\{F_X(x)[1-F_X(x)]\}^m, \tag{12}$$

where $M_1(y)$ is a function of y alone and m is a non-negative real number, then

$$\frac{\partial}{\partial x}[K_1(x,y)] = K_2(x,y)[1-2F_X(x)]f_X(x), \tag{13}$$

where $K_2(x,y) = mM_1(y)\{F_X(x)[1-F_X(x)]\}^{m-1}$. Suppose we take either $K_0(x,y)\{F_X(x)[1-F_X(x)]\}^{m_1}$ as the value of $K_1(x,y)$ or assume that it enters in $K_1(x,y)$ as one of the terms but ceases to be expressed in the form $M_2(y)\{F_X(x)[1-F_X(x)]\}^{m_1}$, such that $M_2(y)$ is a function of y alone and m_1

is a non-negative real number, then

$$\frac{\partial}{\partial x}\{K_0(x,y)\{F_X(x)[1-F_X(x)]\}^{m_1}\}$$
$$=\frac{\partial}{\partial x}\{K_0(x,y)\}\{F_X(x)[1-F_X(x)]\}^{m_1} + K_0(x,y)m_1\{F_X(x)[1-F_X(x)]\}^{m_1-1}[1-2F_X(x)]f_X(x).$$

Clearly the right side of the above equation does not generally lead one to have a representation of the type (13) and this contradicts the integrability of the left side of (11), as in this case $[1-2F_X(x)]$ remains in the denominator as such in one of the terms and hence it takes a value 0, when x takes the median value of the random variable X.

This proves that any form of $K_1(x,y)$ other than that given by (12) ceases to be a solution of $K_1(x,y)$ in (11). In general a linear form of functions of the type $M_i(y)\{F_X(x)[1-F_X(x)]\}^{m_i}$, for i=1,2,... could be taken as a general solution to $K_1(x,y)$ without affecting the boundedness of the functions involved and the integrability of the integrand in the left side of (9). Hence there exists a positive integer t such that the most general representation of $a(x,y)$ as a solution to (8) takes the form

$$a(x,y)=\sum_{i=1}^{t}\alpha_i M_i(y)\{F_X(x)[1-F_X(x)]\}^{m_i}[1-2F_X(x)]f_X(x), \tag{14}$$

where α_i's are constants constrained to lie in suitable intervals about zero, m_i's are non-negative real numbers and $M_i(y)$'s are functions of y alone. Similarly, from the condition
$f_{X_{[n,k]}}(x)+f_{X_{(n,k)}}(x)=2f_X(x)$, $\forall x\in(\alpha,\beta)$, we get

$$a(x,y)=\sum_{i=1}^{t}\alpha_i P_i(x)\{F_Y(y)[1-F_Y(y)]\}^{p_i}[1-2F_Y(y)]f_Y(y), \tag{15}$$

where p_i's are non-negative real numbers and $P_i(x)$'s are functions of x alone.

Since (14) and (15) represents the same function the number of terms in them and the α_i's involved are identically same in both representations. Also as (14) and (15) should be identically equal, we have

$$a(x,y)=\sum_{i=1}^{t}\alpha_i\{F_X(x)[1-F_X(x)]\}^{m_i}[1-2F_X(x)]f_X(x) \times\{F_Y(y)[1-F_Y(y)]\}^{p_i}[1-2F_Y(y)]f_Y(y).$$

This proves that the parent distribution has a pdf $f(x,y)$ given by (5). □

As an immediate consequence of Theorem 1, we have the following corollary.

COROLLARY 2. *Let $F_{Y_{[2,k]}}(y), F_{Y_{(2,k)}}(y), F_{X_{[2,k]}}(x), F_{X_{(2,k)}}(x)$ denote the cumulative distribution functions of $Y_{[2,k]}, Y_{(2,k)}, X_{[2,k]}, X_{(2,k)}$ respectively.*
Then $\frac{1}{2}[F_{Y_{[2,k]}}(y)+F_{Y_{(2,k)}}(y)]=F_Y(y), \forall y$ and $\frac{1}{2}[F_{X_{[2,k]}}(x)+F_{X_{(2,k)}}(x)]=F_X(x)$, $\forall x$ if and only if the parent distribution belongs to generalized Morgenstern family of distributions with pdf $f(x,y)$ as given in (5).

Remark 3. *Suppose we are searching for a suitable bivariate distribution to model a population distribution from which several samples are available. Then, one may observe* $Y_{[2,k]}$, $Y_{(2,k)}$, $X_{[2,k]}$, $X_{(2,k)}$ *for each sample and construct the empirical cumulative distribution functions* $\hat{F}_{Y_{[2,k]}}$, $\hat{F}_{Y_{(2,k)}}$, $\hat{F}_{X_{[2,k]}}$ *and* $\hat{F}_{X_{(2,k)}}$ *from the repeatedly observed bivariate samples. Let* $\hat{F}_X$ *and* $\hat{F}_Y$ *denote the empirical distributions of the marginal* X *observations and marginal* Y *observations respectively. If the graph of* $\frac{1}{2}\{\hat{F}_{Y_{[2,k]}} + \hat{F}_{Y_{(2,k)}}\}$ *is seen identical with that of* $\hat{F}_Y$ *and the graph of* $\frac{1}{2}\{\hat{F}_{X_{[2,k]}} + \hat{F}_{X_{(2,k)}}\}$ *is seen identical with that of* $\hat{F}_X$*, then as a result of Corollary 2, we can limit our search for a distribution from the generalized Morgenstern family of bivariate distributions defined with pdf (5) for constructing an appropriate model to the data.*

Remark 4. *If sufficient number of* $X_{[n,k]}, X_{(n,k)}, Y_{[n,k]}$ *and* $Y_{(n,k)}$*, for any* $n > 2$ *are observable from the repeated samples drawn from the parent bivariate population, then for that* n *also we can construct the empirical distribution functions* $\hat{F}_{X_{[n,k]}}, \hat{F}_{X_{(n,k)}}, \hat{F}_{Y_{[n,k]}}$ *and* $\hat{F}_{Y_{(n,k)}}$ *and ascertain the identical nature of graphs for* $\frac{1}{2}\{\hat{F}_{X_{[n,k]}} + \hat{F}_{X_{(n,k)}}\}$ *and* $\hat{F}_X$ *together with the identical nature of graphs for* $\frac{1}{2}\{\hat{F}_{Y_{[2,k]}} + \hat{F}_{Y_{(2,k)}}\}$ *and* $\hat{F}_Y$ *to suggest the model (5) to the distribution of the population.*

Remark 5. *When compared with classical record values, the occurrence of both GURVs and GLRVs are generally more in number and this makes the applications of the above remarks more versatile than the corresponding result available in Thomas and Veena (2014) for concomitants of classical record values.*

3. Role of Concomitants of GURVs in the Unique Determination of a Bivariate Distribution

We observe from Arnold *et al.* (1998) that the pdf of record values determines uniquely the parent distribution. In this section we obtain similar results on concomitants of generalized (k) record values in the unique determination of a parent bivariate distribution. To establish the results we now define the following.

Definition 6. *Associated with the pdf of* $Y_{[n,k]}$ *given by*

$$f_{Y_{[n,k]}}(y) = \frac{k^n}{\Gamma(n)} \int_x \{-log[1 - F_X(x)]\}^{n-1}[1 - F_X(x)]^{k-1} f(x, y)\, dx \tag{16}$$

we define a family of functions given by

$$f_{Y_{A[\omega,k]}}(y) = \frac{k^\omega}{\Gamma(\omega)} \int_x \{-log[1 - F_X(x)]\}^{\omega-1}[1 - F_X(x)]^{k-1} f(x, y)\, dx \tag{17}$$

for $\omega > 0$. *Clearly* $f_{Y_{A[\omega,k]}}(y)$ *is a pdf for every* $\omega > 0$ *and is defined as the auxiliary density function determined by* $f_{Y_{[n,k]}}(y)$*. Similarly the auxiliary density function determined by* $f_{X_{[n,k]}}(x)$ *is given by*

$$f_{X_{A[\omega,k]}}(x) = \frac{k^\omega}{\Gamma(\omega)} \int_y \{-log[1 - F_Y(y)]\}^{\omega-1}[1 - F_Y(y)]^{k-1} f(x, y)\, dy, \tag{18}$$

for $\omega > 0$.

NOTE 3.1. *Clearly if the parameter* ω *in the auxiliary density functions in (17) and (18) is replaced by* n, *then they are the pdf's of concomitants* $Y_{[n,k]}$ *and* $X_{[n,k]}$ *respectively.*

THEOREM 7. *Let* $h(x, y)$ *be the pdf of a continuous bivariate distribution with marginal pdf's* $f_X(x)$ *and* $f_Y(y)$ *and corresponding marginal distribution functions* $F_X(x)$ *and* $F_Y(y)$ *respectively. Let the pdf of the concomitant of the* n^{th} *GURV be* $f_{Y_{[n,k]}}(y)$. *Then the pdf's* $f_X(x)$ *and* $f_{Y_{A[\omega,k]}}(y)$ *together determine uniquely the bivariate distribution* $h(x, y)$.

PROOF. Clearly the auxiliary density function determined by $f_{Y_{[n,k]}}(y)$ is $f_{Y_{A[\omega,k]}}(y)$, $\omega > 0$.

$$f_{Y_{A[\omega,k]}}(y) = \frac{k^\omega}{\Gamma(\omega)} \int_x \{-log[1 - F_X(x)]\}^{\omega-1}[1 - F_X(x)]^{k-1} f(x, y)\, dx. \tag{19}$$

$$= \int_x \frac{k^\omega\{-log[1 - F_X(x)]\}^{\omega-1}[1 - F_X(x)]^{k-1}}{(\omega - 1)!} f_Y(y) f_{X|Y}(x|y)dx$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} \int_x \{-log[1 - F_X(x)]\}^{\omega-1}[1 - F_X(x)]^{k-1} f_{X|Y}(x|y)dx$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} E\{[-log[1 - F_X(X)]]^{\omega-1}[1 - F_X(x)]^{k-1}|y\}$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} E\{U^{\omega-1}[exp(-U)]^{k-1}|y\},$$

where $U = -log[1 - F_X(X)]$ and the the support set of U is $(0, \infty)$. Hence

$$f_{Y_{A[\omega,k]}}(y) = \tfrac{f_Y(y)k^\omega}{\Gamma(\omega)} \int_0^\infty u^{\omega-1}[e^{-u}]^{k-1} f_{U|Y}(u|y)du.$$

Thus if we write

$$\int_0^\infty u^{\omega-1}[e^{-u}]^{k-1} f_{U|Y}(u|y)du = M_{y,k}(\omega), \tag{20}$$

then for any given y and k, $M_{y,k}(\omega)$ is a Mellin transform. Hence by uniqueness property of Mellin transform, we can determine $[e^{-u}]^{k-1} f_{U|Y}(u|y) f_Y(y) = [e^{-u}]^{k-1} h_{U,Y}(u, y)$ by inversion and thereby determine $h_{U,Y}(u, y)$.
But $u = -log[1 - F_X(x)]$ is a monotone function. By transformation of variable we can then determine $h(x, y)$ uniquely. Hence the theorem. □

COROLLARY 8. *Let* $h(x, y)$ *be the pdf of a continuous bivariate distribution with marginal pdf's* $f_X(x)$ *and* $f_Y(y)$ *and corresponding marginal distribution functions* $F_X(x)$ *and* $F_Y(y)$ *respectively. Let the pdf of the concomitant of the* n^{th} *GURV be* $f_{X_{[n,k]}}(x)$. *Then the pdf's* $f_Y(y)$ *and* $f_{X_{A[\omega,k]}}(x)$ *together determine uniquely the bivariate distribution* $h(x, y)$.

The proof of the above corollary is omitted as it is just similar to the proof of Theorem 7.

3.1. Inversion technique of determining the parent bivariate distribution using the distribution of concomitant of the GURV

EXAMPLE 3.1. *Suppose $f_X(x)$ is the pdf of the first marginal random variable X of a bivariate random vector (X,Y). If*

$$f_{Y_{[n,k]}}(y) = f_Y(y) + \rho[1-2F_Y(y)]f_Y(y)(\frac{2k^n}{(k+1)^n} - 1), -1 < \rho < 1$$

represents the pdf of $Y_{[n,k]}$, the concomitant of the GURV arising from the given parent bivariate distribution with pdf $f_Y(y)$ and cdf $F_Y(y)$ on the other marginal random variable Y, then the pdf $h(x,y)$ of the parent bivariate distribution is

$$h(x,y) = f_X(x)f_Y(y)\{1+\rho[1-2F_X(x)][1-2F_Y(y)]\} \tag{21}$$

PROOF. Since $f_{Y_{[n,k]}}(y)$ is the pdf of the concomitant $Y_{[n,k]}$ we write

$f_{Y_{A[\omega,k]}}(y) = f_Y(y) + \rho[1-2F_Y(y)]f_Y(y)(\frac{2k^\omega}{(k+1)^\omega} - 1)$.

From the proof of theorem 7, it is clear that

$$f_{Y_{A[\omega,k]}}(y) = \frac{k^\omega}{\Gamma(\omega)} f_Y(y) \int_0^\infty v^{\omega-1}(e^{-v})^{k-1} f_{V|Y}(v|y)dv, \tag{22}$$

where $v = -log[1-F_X(x)]$ and $\int_0^\infty v^{\omega-1}(e^{-v})^{k-1} f_{V|Y}(v|y)dv = M_{y,k}(\omega)$ is the Mellin transform of $(e^{-v})^{k-1} f_{V|Y}(v|y)$. Also from (22) we have

$M_{y,k}(\omega) = \frac{\Gamma(\omega) f_{Y_{A[\omega,k]}}(y)}{k^\omega f_Y(y)} = \frac{\Gamma(\omega)}{k^\omega} + \rho[1-2F_Y(y)](\frac{2\Gamma(\omega)}{(k+1)^\omega} - \frac{\Gamma(\omega)}{k^\omega})$.

From Bateman (1954), we have

$(e^{-v})^{k-1} f_{V|Y}(v|y) = (e^{-v})^k + \beta[1-2F_Y(y)][2(e^{-v})^{k+1} - (e^{-v})^k]$.

Hence

$f_{V|Y}(v|y) = e^{-v} + \beta[1-2F_Y(y)][2(e^{-v})^2 - e^{-v}]$.
As $v = -log[1-F_X(x)]$ is a monotone function, by applying transformation of variables we get

$f_{X|Y}(x|y) = f_X(x) + \rho[1-2F_Y(y)]f_X(x)[1-2F_X(x)]$.

We then have

$$h(x,y) = f_X(x)f_Y(y)\{1+\rho[1-2F_X(x)][1-2F_Y(y)]\}. \tag{23}$$

□

Clearly $h(x, y)$ as given in (23) is the well known Morgenstern bivariate distribution determined by the marginal distribution functions $F_X(x)$ and $F_Y(y)$.

It is to be noted that theorem 7 is true for every positive integer k. Hence when we put $k = 1$ in the theorem, it becomes the statement of the unique determination of parent bivariate distribution based on the concomitants of classical record values. The following example illustrates the application of theorem 7 in this case.

EXAMPLE 3.2. *Suppose $f_X(x)$ is the pdf of the first marginal random variable of a bivariate random vector (X, Y). If for $m_i > 0, q_i > 0$ and suitable α_i for $i = 1, 2, \ldots, t$ are such that*

$$f_{Y_{[n]}}(y) = f_Y(y) + \textstyle\sum_{i=1}^{t} \alpha_i [F_Y(y)]^{q_i}[1 - F_Y(y)]^{q_i}[1 - 2F_Y(y)] f_Y(y) \times \textstyle\sum_{r=0}^{m_i-1} \begin{pmatrix} m_i - 1 \\ r \end{pmatrix} (-1)^r \left(\frac{2}{(m_i+r+2)^n} - \frac{1}{(m_i+r+1)^n}\right)$$

represents the pdf of concomitant of the nth upper record value arising from the given parent bivariate distribution where $f_Y(y)$ and $F_Y(y)$ are the pdf and cdf of an arbitrary random variable Y, then the pdf $h(x, y)$ of the parent bivariate distribution is

$$h(x, y) = f_X(x) f_Y(y) + f_X(x) f_Y(y) \sum_{i=1}^{t} \alpha_i [F_X(x)]^{m_i}[1 - F_X(x)]^{m_i} \times [1 - 2F_X(x)][F_Y(y)]^{q_i}[1 - F_Y(y)]^{q_i}[1 - 2F_Y(y)]. \tag{24}$$

PROOF. Since $f_{Y_{[n]}}(y)$ is the pdf of the concomitant $Y_{[n]}$ we write

$$f_{Y_{A[\omega]}}(y) = f_Y(y) + \textstyle\sum_{i=1}^{t} \alpha_i [F_Y(y)]^{q_i}[1 - F_Y(y)]^{q_i}[1 - 2F_Y(y)] f_Y(y) \times \textstyle\sum_{r=0}^{m_i-1} \begin{pmatrix} m_i - 1 \\ r \end{pmatrix} (-1)^r \left(\frac{2}{(m_i+r+2)^\omega} - \frac{1}{(m_i+r+1)^\omega}\right).$$

From the proof of theorem 7, it is clear that

$$f_{Y_{A[\omega]}}(y) = \frac{f_Y(y)}{\Gamma(\omega)} \int_0^\infty v^{\omega-1}(e^{-v})^{k-1} f_{V|Y}(v|y) dv, \tag{25}$$

where $v = -log[1 - F(x)]$ and $\displaystyle\int_0^\infty v^{\omega-1}(e^{-v})^{k-1} f_{V|Y}(v|y) dv = M_{y,k}(\omega)$ is the Mellin transform of $(e^{-v})^{k-1} f_{V|Y}(v|y)$. Also from (25) we have

$$\begin{aligned} M_{y,k}(\omega) &= \tfrac{\Gamma(\omega) f_{Y_{A[\omega]}}(y)}{f_Y(y)} \\ &= \Gamma(\omega) + \textstyle\sum_{i=1}^{t} \alpha_i [F_Y(y)]^{q_i}[1 - F_Y(y)]^{q_i}[1 - 2F_Y(y)] \\ &\quad \times \textstyle\sum_{r=0}^{m_i-1} \begin{pmatrix} m_i - 1 \\ r \end{pmatrix} (-1)^r \left(\frac{2\Gamma(\omega)}{(m_i+r+2)^\omega} - \frac{\Gamma(\omega)}{(m_i+r+1)^\omega}\right). \end{aligned}$$

From Bateman (1954), we have

$$(e^{-v})^{k-1} f_{V|Y}(v|y) = (e^{-v})^k + \rho \sum_{i=1}^{t} \alpha_i [F_Y(y)]^{q_i}[1 - F_Y(y)]^{q_i}$$

$$\times[1-2F_Y(y)]\sum_{r=0}^{m_i-1}\binom{m_i-1}{r}(-1)^r[2(e^{-v})^{m_i+r+2}-(e^{-v})^{m_i+r+1}].$$

$$= (e^{-v})^k+\sum_{i=1}^{t}\alpha_i[F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)]$$
$$\times (e^{-v})^{m_i}(1-e^{-v})^{m_i}[2(e^{-v})^{k+1}-(e^{-v})^k].$$

Hence

$$f_{V|Y}(v|y) = e^{-v}+\sum_{i=1}^{t}\alpha_i[F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)]$$
$$\times (e^{-v})^{m_i}(1-e^{-v})^{m_i}[2(e^{-v})-1].$$

As $v=-log[1-F_X(x)]$ is a monotone function, by applying transformation of variables we get

$$f_{X|Y}(x|y) = f_X(x)+\sum_{i=1}^{t}\alpha_i[F_X(x)]^{m_i}[1-F_X(x)]^{m_i}[1-2F_X(x)]f_X(x)$$
$$\times [F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)].$$

We then have

$$h(x,y) = f_X(x)f_Y(y)+f_X(x)f_Y(y)\sum_{i=1}^{t}\alpha_i[F_X(x)]^{m_i}[1-F_X(x)]^{m_i}$$
$$\times[1-2F_X(x)][F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)]. \quad (26)$$

□

It is to be noted that (26) is the well known generalized Morgenstern bivariate distribution as defined in (5).

4. Role of Concomitants of GLRVs in the Unique Determination of a Bivariate Distribution

We observe from Arnold *et al.* (1998) that the pdf of record values determines uniquely the parent distribution. In this section we obtain similar results on concomitants of GLRVs in the unique determination of a parent bivariate distribution. To establish the results we now define the following.

Definition 9. *Associated with the pdf of* $Y_{(n,k)}$ *given by*

$$f_{Y_{(n,k)}}(y) = \frac{k^n}{\Gamma(n)}\int_x\{-log[F_X(x)]\}^{n-1}[F_X(x)]^{k-1}f(x,y)\,dx \quad (27)$$

we define a family of functions given by

$$f_{Y_{A(\omega,k)}}(y) = \frac{k^\omega}{\Gamma(\omega)}\int_x\{-log[F_X(x)]\}^{\omega-1}[F_X(x)]^{k-1}f(x,y)\,dx \quad (28)$$

for $\omega > 0$. *Clearly* $f_{Y_{A(\omega,k)}}(y)$ *is a pdf for every* $\omega > 0$ *and is defined as the auxiliary density function determined by* $f_{Y_{(n,k)}}(y)$. *Similarly the auxiliary density function determined by* $f_{X_{(n,k)}}(x)$ *is given by*

$$f_{X_{A(\omega,k)}}(x) = \frac{k^\omega}{\Gamma(\omega)} \int_y \{-log[F_Y(y)]\}^{\omega-1}[F_Y(y)]^{k-1} f(x,y)\,dy, \omega > 0. \tag{29}$$

NOTE 4.1. *Clearly if the parameter* ω *in the auxiliary density functions in (28) and (29) is replaced by* n, *then they are the pdf's of concomitants* $Y_{(n,k)}$ *and* $X_{(n,k)}$ *respectively.*

THEOREM 10. *Let* $h(x,y)$ *be the pdf of a continuous bivariate distribution with marginal pdf's* $f_X(x)$ *and* $f_Y(y)$ *and corresponding marginal distribution functions* $F_X(x)$ *and* $F_Y(y)$ *respectively. Let the pdf of the concomitant of the* n^{th} *GLRV be* $f_{Y_{(n,k)}}(y)$. *Then the pdf's* $f_X(x)$ *and* $f_{Y_{A(\omega,k)}}(y)$ *together determine uniquely the bivariate distribution* $h(x,y)$.

PROOF. Clearly the auxiliary density function determined by $f_{Y_{(n,k)}}(y)$ is $f_{Y_{A(\omega,k)}}(y)$, $\omega > 0$.

$$f_{Y_{A(\omega,k)}}(y) = \frac{k^\omega}{\Gamma(\omega)} \int_x \{-log[F_X(x)]\}^{\omega-1}[F_X(x)]^{k-1} f(x,y)\,dx. \tag{30}$$

$$= \int_x \frac{k^\omega\{-log[F_X(x)]\}^{\omega-1}[F_X(x)]^{k-1}}{(\omega-1)!} f_Y(y) f_{X|Y}(x|y)dx$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} \int_x \{-log[F_X(x)]\}^{\omega-1}[F_X(x)]^{k-1} f_{X|Y}(x|y)dx$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} E\{[-log[F_X(X)]]^{\omega-1}[F_X(x)]^{k-1}|y\}$$

$$= \frac{f_Y(y)k^\omega}{\Gamma(\omega)} E\{U^{\omega-1}[exp(-U)]^{k-1}|y\},$$

where $U = -log[F_X(X)]$ and the the support set of U is $(0,\infty)$. Hence

$$f_{Y_A(\omega,k)}(y) = \tfrac{f_Y(y)k^\omega}{\Gamma(\omega)} \int_0^\infty u^{\omega-1}[e^{-u}]^{k-1} f_{U|Y}(u|y)du.$$

Thus if we write

$$\int_0^\infty u^{\omega-1}[e^{-u}]^{k-1} f_{U|Y}(u|y)du = M_{y,k}(\omega), \tag{31}$$

then for any given y and k, $M_{y,k}(\omega)$ is a Mellin transform. Hence by uniqueness property of Mellin transform, we can determine $[e^{-u}]^{k-1} f_{U|Y}(u|y) f_Y(y) = [e^{-u}]^{k-1} h_{U,Y}(u,y)$ by inversion and thereby determine $h_{U,Y}(u,y)$.
But $u = -log[F_X(x)]$ is a monotone function. By transformation of variable we can then determine $h(x,y)$ uniquely. Hence the theorem. □

COROLLARY 11. *Let $h(x,y)$ be the pdf of a continuous bivariate distribution with marginal pdf's $f_X(x)$ and $f_Y(y)$ and corresponding marginal distribution functions $F_X(x)$ and $F_Y(y)$ respectively. Let the pdf of the concomitant of the n^{th} GLRV be $f_{X_{(n,k)}}(x)$. Then the pdf's $f_Y(y)$ and $f_{X_{A(\omega,k)}}(x)$ together determine uniquely the bivariate distribution $h(x,y)$.*

The proof of the above corollary is omitted as it is just similar to the proof of Theorem 10.

4.1. Inversion technique of determining the parent bivariate distribution using the distribution of concomitant of the GLRV

EXAMPLE 4.1. *Suppose $f_X(x)$ is the pdf of the first marginal random variable X of a bivariate random vector (X,Y). If*

$$f_{Y_{(n,k)}}(y) = f_Y(y) + \rho[1-2F_Y(y)]f_Y(y)(1-\frac{2k^n}{(k+1)^n}), -1 < \rho < 1$$

represents the pdf of $Y_{(n,k)}$, the concomitant of the GLRV arising from the given parent bivariate distribution with pdf $f_Y(y)$ and cdf $F_Y(y)$ on the other marginal random variable Y, then the pdf $h(x,y)$ of the parent bivariate distribution is

$$h(x,y) = f_X(x)f_Y(y)\{1+\rho[1-2F_X(x)][1-2F_Y(y)]\} \tag{32}$$

The proof follows similarly as in the case of Example 3.1.

It is to be noted that theorem 10 is true for every positive integer k. Hence when we put $k=1$ in the theorem, it becomes the statement of the unique determination of parent bivariate distribution based on the concomitants of classical lower record values. The following example illustrates the application of theorem 10 in this case.

EXAMPLE 4.2. *Suppose $f_X(x)$ is the pdf of the first marginal random variable of a bivariate random vector (X,Y). If for $m_i > 0, q_i > 0$ and suitable α_i for $i = 1,2,\ldots,t$ are such that*

$$f_{Y_{(n)}}(y) = f_Y(y) + \textstyle\sum_{i=1}^{t} \alpha_i [F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)]f_Y(y)$$
$$\times \textstyle\sum_{r=0}^{m_i-1} \begin{pmatrix} m_i-1 \\ r \end{pmatrix} (-1)^r (\frac{1}{(m_i+r+1)^n} - \frac{2}{(m_i+r+2)^n})$$

represents the pdf of concomitant of the nth lower record value arising from the given parent bivariate distribution where $f_Y(y)$ and $F_Y(y)$ are the pdf and cdf of an arbitrary random variable Y, then the pdf $h(x,y)$ of the parent bivariate distribution is

$$h(x,y) = f_X(x)f_Y(y) + f_X(x)f_Y(y)\sum_{i=1}^{t}\alpha_i[F_X(x)]^{m_i}[1-F_X(x)]^{m_i}$$
$$\times[1-2F_X(x)][F_Y(y)]^{q_i}[1-F_Y(y)]^{q_i}[1-2F_Y(y)]. \tag{33}$$

The proof follows similarly as in the case of Example 3.2.

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Summary

In this paper we have derived some properties of concomitants of generalized (k) record values which characterize the generalized Morgenstern family of bivariate distributions. The role of concomitants of generalized (k) record values in the unique determination of the parent bivariate distribution has been established. We have also illustrated how the concomitants of generalized (k) record values characterize the Morgenstern family of bivariate distributions.

Keywords: Characterization of bivariate distributions; Concomitants of generalized (k) record values; Generalized Morgenstern family of bivariate distributions.