# SOME RESULTS ON A GENERALIZED RESIDUAL ENTROPY BASED ON ORDER STATISTICS

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## 1. INTRODUCTION

To quantify the statistical nature of lost information in communication channels mathematically, Shannon (1948) introduced a concept (known as the Shannon entropy) analogous to the entropy described in statistical thermodynamics. Let $X$ be a non-negative random variable representing the lifetime of a component with an absolutely continuous cumulative distribution function (CDF) $F(x)$, probability density function (PDF) $f(x)$, survival function (SF) $\bar{F}(x) = 1 - F(x)$ and hazard rate (HR) $\lambda_F(x) = f(x)/\bar{F}(x)$. The Shannon entropy of $X$ is given by $S(X) = -\int_0^\infty f(x) \ln f(x) dx$, which has received considerable popularity in a wide variety of contexts. For a detailed account of importance and applications of the notion of entropy in various disciplines one may refer to Cover and Thomas (2006). The Shannon entropy $S(X)$ can be seen as a negative logarithm moment as it is expectation of logarithm of the fundamental measure with a negative sign. As different moments divulge different information on the distribution, it is expected that other entropies (generalized) may divulge different characteristics of the data set. Due to this reason, several general families of entropy measures have been introduced in the literature. In this direction we refer to Renyi (1961); Varma (1966); Kapur (1967); Tsallis (1988). Note that these generalized measures have several properties such as smoothness, large dynamic range with respect to certain conditions that make them useful for specific applications (see for example, Varma, 1966; Renyi, 2012; Tsallis, 1988; Kapur, 1994). Pharwaha and Singh (2009) determined randomness of mammograms based on some non-Shannon measures as these have higher dynamic range than the Shannon measure over a variety of scattering conditions and therefore, useful in the problems of estimating scatter density and regularity (see Smolikovd *et al.*, 2002). For a non-negative random variable $X$, the generalized entropy (see Varma, 1966) of order $(\alpha, \beta)$ is given by

$$V_{\alpha,\beta}(X) = \left(\frac{1}{\beta - \alpha}\right) \ln \int_0^\infty f^{\alpha + \beta - 1}(x) dx, \ 0 \le \beta - 1 < \alpha < \beta. \tag{1}$$

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Note that as $\beta$ equals to 1, (1) reduces to the Renyi entropy of order $\alpha$ (see Renyi, 1961) of a non-negative random variable $X$.

In the context of reliability and life testing studies when the present age of a component needs to be incorporated, the measure given in (1) is not appropriate. In this situation one may be interested in studying the residual lifetime of a component which is still working at time $t \geq 0$. We denote it by $X_t = [X - t | X \geq t]$. Accordingly, the measure $V_{\alpha,\beta}(X)$ given in (1) needs to be modified. For some references on residual entropy and generalized residual entropy, we refer to Ebrahimi and Pellerey (1995); Ebrahimi (1996); Ebrahimi and Kirmani (1996); Belzunce *et al.* (2004); Abraham and Sankaran (2005); Nanda and Paul (2006); Zarezadeh and Asadi (2010); Li and Zhang (2011). In analogy to Ebrahimi (1996), a generalized residual entropy (GRE) of order $(\alpha, \beta)$ of $X$ is given by

$$V_{\alpha,\beta}(X;t) = \left(\frac{1}{\beta - \alpha}\right) \ln \int_t^\infty \frac{f^{\alpha+\beta-1}(x)}{\bar{F}^{\alpha+\beta-1}(t)} dx, \ 0 \leq \beta - 1 < \alpha < \beta, \ t \geq 0. \qquad (2)$$

It is noted that when $t = 0$, $V_{\alpha,\beta}(X;t)$ reduces to $V_{\alpha,\beta}(X)$. The measure (2) plays an important role as a measure of complexity and uncertainty in different areas such as physics, electronics and engineering to describe many chaotic systems. In actuarial science, GRE given in (2) can be presented as the pre-payment entropy of claims (losses) with a deductible $t$. For some results on information measures based on (1), we refer to Kayal and Vellaisamy (2011); Kumar and Taneja (2011); Kayal (2015a,b). In this paper, we investigate some properties of GRE of order statistics. It is noted that order statistics specially deal with the properties and applications of ordered random variables and their functions. In the study of many real life problems related to flood, breaking strength, atmospheric temperature, atmospheric pressure etc., the future possibilities in the recurrence of extreme situations are of much importance and accordingly, the problem of interest in these cases reduces to that of the extreme observations. Besides these, the order statistics have been applied in several domains, such as in robust statistical estimation and detection of outliers, digital image processing, characterization of probability distributions, entropy estimation, reliability analysis. For detail, one may refer to David and Nagaraja (2003) and the references therein.

In Section 2, we present some preliminary results. We establish that both increasing generalized residual entropy (IGRE) and decreasing generalized residual entropy (DGRE) properties of a stochastically smaller random variable is preserved by the larger one in Section 3. It is shown that IGRE and DGRE properties are preserved by the formation of a parallel system, but not under the formation of a series system. We obtain bounds for GRE of order statistics. Under some conditions, we establish that the GRE of parallel and series systems are monotone functions of the number of observations of a sample. Finally, in Section 4, we present some computational results to verify some of the results derived in Section 3. We also provide estimators in estimating the GRE of exponential distribution. The maximum likelihood estimator is derived in this purpose.

Throughout the paper, we assume that the random variables are non-negative and have absolutely continuous CDF. The terms decreasing and increasing are used for non-increasing and non-decreasing, respectively. We denote $\gamma = \alpha + \beta - 1$,

where $0 \leq \beta - 1 < \alpha < \beta$.

## 2. BACKGROUND

Here, we present some preliminaries which will be required to obtain some results in the rest of the paper. Let $X_1, X_2, \ldots, X_r, \ldots, X_n$ be $n$ independent copies of a random variable $X$ with absolutely continuous CDF $F(x)$ and PDF $f(x)$. The order statistics of $\{X_1, X_2, \ldots, X_r, \ldots, X_n\}$ are defined as an arrangement of the observations in increasing order. We denote $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{r:n} \leq \ldots \leq X_{n:n}$, where $X_{r:n}$ represents the $r$-th order statistic. Order statistics have various applications. For example, a policy for a couple pays out when the first of the spouses dies. Here one's interest may be on studing the distribution of $X_{1:n}$, which is the random variable defined to be the minimum of two lifespans of the couple. Also, let an insurance company hold 500 policies of which somebody has cash-at-hand to pay 250. In this set up, one may be interested to know the distribution of the variable $X_{250:500}$, the 250th occurrence of a pay-out. The PDF and SF of $X_{r:n}$ are respectively given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} F^{r-1}(x) \bar{F}^{n-r}(x) f(x) \tag{3}$$

and

$$\bar{F}_{r:n}(x) = \sum_{i=0}^{r-1} \binom{n}{i} F^i(x) \bar{F}^{n-i}(x) = \frac{B_{F(x)}(r, n-r+1)}{B(r, n-r+1)}, \tag{4}$$

where $B(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1} du, a > 0, b > 0$ and $B_{F(x)}(a,b) = \int_{F(x)}^1 u^{a-1}(1-u)^{b-1} du, a > 0, b > 0$. Here, $B(a,b)$ and $B_{F(x)}(a,b)$ are known as beta function and incomplete beta function, respectively. For $a > 0$ and $b > 0$, we denote the truncated beta distribution as $B^t(a,b)$. Let $Y$ be a random variable following $B^t(a,b)$ distribution (denoted by $Y \sim B^t(a,b)$), then the PDF of $Y$ is

$$f_Y(x) = \frac{x^{a-1}(1-x)^{b-1}}{B_t(a,b)}, \ t < x < 1, a > 0, b > 0. \tag{5}$$

The HR of $X_{r:n}$ can be written as

$$\lambda_{F_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)! \sum_{i=0}^{r-1} \binom{n}{i} (F(x)/\bar{F}(x))^{i-(r-1)}} \lambda_F(x). \tag{6}$$

Let $X$ be a random variable following uniform distribution with PDF

$$f(x|a,b) = \frac{1}{b-a}, \ 0 < a < x < b.$$

For convenience, we denote $X \sim U(a,b)$.

LEMMA 1. *(Thapliyal and Taneja, 2012) Let $U_{r:n}$ be the r-th order statistic of a random sample of size n from $U(0,1)$. Then*

$$V_{\alpha,\beta}(U_{r:n};t) = \left(\frac{1}{\beta-\alpha}\right)\ln B_t(\gamma(r-1)+1,\gamma(n-r)+1)$$
$$-\left(\frac{\gamma}{\beta-\alpha}\right)\ln B_t(r,n-r+1). \quad (7)$$

LEMMA 2. *(Thapliyal and Taneja, 2012) Let X be a random variable with CDF $F(x)$ and PDF $f(x)$. Then GRE of the r-th order statistic can be represented in terms of that of the r-th order statistic from $U(0,1)$ as*

$$V_{\alpha,\beta}(X_{r:n};t) = V_{\alpha,\beta}(U_{r:n};F(t)) + \left(\frac{1}{\beta-\alpha}\right)\ln E[f^{\gamma-1}(F^{-1}(Y_r))], \quad (8)$$

*where $Y_r \sim B^{F(t)}(\gamma(r-1)+1,\gamma(n-r)+1)$.*

Let $X$ and $Y$ denote the random variables with respective CDFs $F(x)$ and $G(x)$, PDFs $f(x)$ and $g(x)$, SFs $\bar{F}(x)$ and $\bar{G}(x)$ and HRs $\lambda_F(x)$ and $\lambda_G(x)$.

DEFINITION 3. *A random variable X has decreasing (increasing) generalized residual entropy of order $(\alpha,\beta)$, abbreviated by DGRE (IGRE) if $V_{\alpha,\beta}(X;t)$ is decreasing (increasing) in $t \geq 0$.*

DEFINITION 4. *A random variable X is said to be stochastically less than or equal to Y, abbreviated by $X \overset{st}{\leq} Y$, if $\overline{F}(t) \leq \overline{G}(t)$, for all $t \geq 0$.*

DEFINITION 5. *A random variable X is said to be less than or equal to Y in likelihood ratio ordering, abbreviated by $X \overset{lr}{\leq} Y$, if $f(t)/g(t)$ is decreasing in $t \geq 0$.*

Note that when the supports of $X$ and $Y$ have a common left end point, we have the following result between the stochastic order and the likelihood ratio order described in Definition 4 and Definition 5, respectively.

$$X \overset{lr}{\leq} Y \Rightarrow X \overset{st}{\leq} Y.$$

One may refer to Shaked and Shanthikumar (2007) for more results on usual stochastic orderings.

## 3. MAIN RESULTS

We begin with the following lemma which will be used in obtaining new results in this section.

LEMMA 6. *The GRE of a random variable X with CDF $F(x)$ and PDF $f(x)$ can be expressed as*

$$V_{\alpha,\beta}(X;t) = \left(\frac{1-\gamma}{\beta-\alpha}\right)\ln\bar{F}(t) + \left(\frac{1}{\beta-\alpha}\right)\ln E[f^{\gamma-1}(X)|X \geq t]$$
$$= -\left(\frac{\ln\gamma}{\beta-\alpha}\right) + \left(\frac{1}{\beta-\alpha}\right)\ln E[\lambda_F^{\gamma-1}(X_\gamma)|X_\gamma \geq t],$$

*where $\lambda_F(x) = f(x)/\bar{F}(x)$ is the hazard rate function of $X$ and $X_\gamma$ has the survival function $\bar{F}^\gamma(x)$.*

PROOF. From (2) we have

$$
\begin{aligned}
V_{\alpha,\beta}(X;t) &= \ln \bar{F}(t) + \left(\frac{1}{\beta - \alpha}\right) \ln \int_t^\infty \left[\frac{f^\gamma(x)}{\bar{F}^{2\beta-1}(t)}\right] dx \\
&= \left(\frac{1-\gamma}{\beta - \alpha}\right) \ln \bar{F}(t) + \left(\frac{1}{\beta - \alpha}\right) \ln \int_t^\infty f^{\gamma-1}(x) \frac{f(x)}{\bar{F}(t)} dx \\
&= \left(\frac{1-\gamma}{\beta - \alpha}\right) \ln \bar{F}(t) + \left(\frac{1}{\beta - \alpha}\right) \ln E[f^{\gamma-1}(X)|X \geq t],
\end{aligned}
$$

since the distribution of $[X|X \geq t]$ is $f(x)/\bar{F}(t)$. To prove the second part, again from (2)

$$
\begin{aligned}
V_{\alpha,\beta}(X;t) &= -\left(\frac{\ln \gamma}{\beta - \alpha}\right) + \left(\frac{1}{\beta - \alpha}\right) \ln \int_t^\infty \frac{f^{\gamma-1}(x_\gamma)}{\bar{F}^{\gamma-1}(x_\gamma)} f(x_\gamma | x_\gamma \geq t) dx_\gamma \\
&= -\left(\frac{\ln \gamma}{\beta - \alpha}\right) + \left(\frac{1}{\beta - \alpha}\right) \ln E[\lambda_F^{\gamma-1}(X_\gamma)|X_\gamma \geq t].
\end{aligned}
$$

This completes the proof. $\square$

In the following we obtain expressions of $V_{\alpha,\beta}(X;t)$, $V_{\alpha,\beta}(X_{1:n};t)$ and $V_{\alpha,\beta}(X_{n:n};t)$ for exponential and Pareto distributions.

EXAMPLE 1. *Exponential distribution has prominent applications in various real life problems. For example, service times, inter-arrival times, etc. are usually observed to be exponentially distributed. Let $X$ be a random variable following exponential distribution with CDF $F(x|\sigma) = 1 - \exp(-\sigma x)$, $x > 0$, $\sigma > 0$. It is easy to obtain that $V_{\alpha,\beta}(X;t) = -(\frac{1}{\beta-\alpha}) \ln \gamma + (\frac{\gamma-1}{\beta-\alpha}) \ln \sigma$. Now, to compute $V_{\alpha,\beta}(X_{r:n};t)$, we have $f(F^{-1}(x)) = \sigma(1-x)$ and $E[f^{\gamma-1}(F^{-1}(Y_1))] = (((n-1)\gamma + 1)\sigma^{\gamma-1}/n\gamma) \exp(-\sigma t(\gamma - 1))$. Again $V_{\alpha,\beta}(X_{1:n};t) = -(\frac{1}{\beta-\alpha}) \ln \gamma + (\frac{\gamma-1}{\beta-\alpha}) \ln(n\sigma)$. For $r = n$, from Lemma 2, we obtain $V_{\alpha,\beta}(X_{n:n};t) = (\frac{\gamma-1}{\beta-\alpha}) \ln \sigma - (\frac{\gamma}{\beta-\alpha}) \log(\frac{1}{n}(1 - (F(t|\sigma))^n)) + (\frac{1}{\beta-\alpha}) \log \int_{F(t|\sigma)}^1 x^{\gamma(n-1)}(1-x)^{\gamma-1} dx$.*

EXAMPLE 2. *Pareto distribution plays a central role in various applications. It is used in the investigation of city population, occurrence of natural resources, insurance risk and business failures. It has been an important model in many socio-economic studies. Let $X$ be a random variable following Pareto distribution with CDF $F(x|\theta, \delta) = 1 - (\delta/x)^\theta$, $x \geq \delta > 0$, $\theta > 0$. Now, it is easy to obtain that $f(F^{-1}(x)) = \theta(1-x)^{1+\frac{1}{\theta}}/\beta$ and $E[f^{\gamma-1}(F^{-1}(Y_1))] = [(\gamma(n-1) + 1)/(\gamma(n\theta + 1) - 1)](\delta/t)^{(\theta+1)(\gamma-1)} \theta^\gamma \delta^{1-\gamma}$. We obtain $V_{\alpha,\beta}(X;t) = -(\frac{\gamma-1}{\beta-\alpha}) \ln t + (\frac{\gamma}{\beta-\alpha}) \ln \theta - (\frac{1}{\beta-\alpha}) \ln[\gamma(\theta+1) - 1]$, provided $\gamma > (\theta+1)^{-1}$. Again, $V_{\alpha,\beta}(X_{1:n};t) = -(\frac{\gamma-1}{\beta-\alpha}) \ln t + (\frac{\gamma}{\beta-\alpha}) \ln(n\theta) - (\frac{1}{\beta-\alpha}) \ln[\gamma(n\theta+1) - 1]$, provided $\gamma > (n\theta+1)^{-1}$. Also, from Lemma 2, $V_{\alpha,\beta}(X_{n:n};t) = -(\frac{\gamma-1}{\beta-\alpha}) \ln(\theta/\delta) - (\frac{\gamma}{\beta-\alpha}) \ln((1 - (1 - (\delta/t)^\theta)^n)/n) + (\frac{1}{\beta-\alpha}) \ln \int_{1-(\delta/t)^\theta}^1 (1-x)^{\frac{(\theta+1)(\gamma-1)}{\theta}} x^{\gamma(n-1)} dx$.*

Now, we present some results on GRE in terms of ordering properties of distributions. Several aging properties can be characterized in reliability theory using stochastic comparison between total and its residual lifetime. For example, new better than used property holds if and only if $X_t \overset{st}{\leq} X$, for all $t \geq 0$. In the following theorem we prove a similar characterization of DGRE distributions.

THEOREM 7. *A random variable $X$ with PDF $f(x)$ and SF $\bar{F}(x)$ is DGRE if and only if $V_{\alpha,\beta}(X_s; t) \leq V_{\alpha,\beta}(X; t)$, for all $s, t \geq 0$.*

PROOF. A random variable $X$ is DGRE if and only if $V_{\alpha,\beta}(X; s+t) \leq V_{\alpha,\beta}(X; t)$, for all $s, t \geq 0$. Moreover, the PDF and SF of $X_s$ are given as $f_s(x) = f(s+x)/\bar{F}(s)$ and $\bar{F}_s(x) = \bar{F}(s + x)/\bar{F}(s)$, respectively. Therefore, for all $s, t \geq 0$,

$$V_{\alpha,\beta}(X_s; t) = \left(\frac{1}{\beta - \alpha}\right) \ln \int_t^\infty \frac{f_s^\gamma(x)}{\bar{F}_s^\gamma(t)} dx \quad = \quad \left(\frac{1}{\beta - \alpha}\right) \ln \int_{s+t}^\infty \frac{f^\gamma(x)}{\bar{F}^\gamma(s + t)} dx$$
$$= \quad V_{\alpha,\beta}(X; s + t).$$

This completes the proof. □
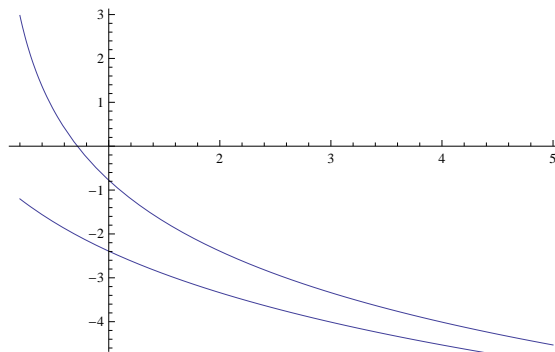


(a)

*Figure 1* – Plot of $V_{\alpha,\beta}(X; s + t)$ (bottom curve) and $V_{\alpha,\beta}(X; t)$ (top curve) versus $t$ (along horizontal axis) for Pareto distribution as described in Example 2. Here we assume $\alpha = 1.2$, $\beta = 1.5$, $\theta = 2$ and $s = 1$.

From Fig. 1, it is easy to see that the Pareto random variable is DGRE for $\alpha = 1.2$, $\beta = 1.5$ and $\theta = 2$. Here, it should be mentioned that we observe similar behaviour of Pareto distribution for any $\theta > 0$ and $\alpha + \beta > 2$. Our next theorem is a key result to obtain some subsequent results.

THEOREM 8. *Let $X$ and $Y$ be two random variables with CDFs $F(x)$ and $G(x)$, PDFs $f(x)$ and $g(x)$, SFs $\bar{F}(x)$ and $\bar{G}(x)$ and HRs $\lambda_F(x)$ and $\lambda_G(x)$, respectively. Also let $X \overset{lr}{\leq} Y$; and $\lambda_G(t)/\lambda_F(t)$ be an increasing function in $t \geq 0$. If $X$ is IGRE (DGRE), then $Y$ is also IGRE (DGRE).*

PROOF. Assume $\xi(t) = \lambda_G(t)/\lambda_F(t)$. Then from Lemma 6, the GRE of $Y$ can be written as

$$V_{\alpha,\beta}(Y;t) = -\left(\frac{\ln\gamma}{\beta-\alpha}\right) + \left(\frac{1}{\beta-\alpha}\right)\ln E[\lambda_F^{\gamma-1}(Y_\gamma)\xi^{\gamma-1}(Y_\gamma)|Y_\gamma \geq t]. \qquad (9)$$

Under the given assumption, it is required to show that $\eta(t) = E[\lambda_F^{\gamma-1}(Y_\gamma)\xi^{\gamma-1}(Y_\gamma)$ $|Y_\gamma \geq t]$ is increasing (decreasing) in $t \geq 0$, that is, $\eta'(t) = \gamma\lambda_G(t)[\eta(t) - \lambda_F^{\gamma-1}(t)\xi^{\gamma-1}(t)] \geq (\leq)0$, where $'$ denote the derivative. The PDF of $[Y_\gamma|Y_\gamma \geq t]$ can be obtained as

$$f(y_\gamma|y_\gamma \geq t) = \frac{\gamma\bar{G}^{\gamma-1}(y_\gamma)g(y_\gamma)}{\bar{G}^\gamma(t)}, \quad y_\gamma \geq t.$$

Moreover,

$$\begin{aligned}
\eta(t) - \lambda_F^{\gamma-1}(t)\xi^{\gamma-1}(t) &= \int_t^\infty f(y_\gamma|y_\gamma \geq t)[\lambda_F^{\gamma-1}(y_\gamma)\xi^{\gamma-1}(y_\gamma) - \lambda_F^{\gamma-1}(t)\xi^{\gamma-1}(t)]dy_\gamma \\
&= \int_t^\infty f(y_\gamma|y_\gamma \geq t)\xi^{\gamma-1}(y_\gamma)[\lambda_F^{\gamma-1}(y_\gamma) - \lambda_F^{\gamma-1}(t)]dy_\gamma \\
&\quad + \int_t^\infty f(y_\gamma|y_\gamma \geq t)\lambda_F^{\gamma-1}(t)[\xi^{\gamma-1}(y_\gamma) - \xi^{\gamma-1}(t)]dy_\gamma \\
&= I_1 + I_2,
\end{aligned}$$

where $I_1 = \int_t^\infty f(y_\gamma|y_\gamma \geq t)\xi^{\gamma-1}(y_\gamma)[\lambda_F^{\gamma-1}(y_\gamma)-\lambda_F^{\gamma-1}(t)]dy_\gamma$ and $I_2 = \int_t^\infty f(y_\gamma|y_\gamma \geq t)\lambda_F^{\gamma-1}(t)[\xi^{\gamma-1}(y_\gamma) - \xi^{\gamma-1}(t)]dy_\gamma$. Let $X$ be IGRE (DGRE), implies $V_{\alpha,\beta}(X;t)$ be an increasing (decreasing) function in $t \geq 0$. Thus, $h(t) = E[\lambda_F^{\gamma-1}(X_\gamma)|X_\gamma \geq t]$ is also an increasing (decreasing) function in $t \geq 0$. Therefore, we have $h(t) \geq (\leq)\lambda_F^{\gamma-1}(t)$, which implies that

$$\int_t^\infty \gamma f(x_\gamma)\bar{F}^{\gamma-1}(x_\gamma)[\lambda_F^{\gamma-1}(x_\gamma) - \lambda_F^{\gamma-1}(t)]dx_\gamma \geq (\leq)0, \qquad (10)$$

for all $t \geq 0$. Again under the assumption $X \overset{lr}{\leq} Y$, and using Lemma 7.1(a) of Barlow and Proschan (1981), it can be shown that

$$I_1 = \frac{1}{\bar{G}^\gamma(t)}\int_t^\infty \frac{g^\gamma(y_\gamma)}{f^\gamma(y_\gamma)}\gamma\bar{F}^{\gamma-1}(y_\gamma)f(y_\gamma)[\lambda_F^{\gamma-1}(y_\gamma) - \lambda_F^{\gamma-1}(t)]dy_\gamma \geq (\leq)0.$$

Moreover, as $\xi(t)$ is an increasing function in $t \geq 0$, therefore it can also be shown that $I_2 \geq (\leq)0$. This completes the proof. $\square$

In recent years, researchers in reliability theory have shown considerable attention in the study of stochastic and reliability properties of various technical systems. The $(n - r + 1)$-out-of-$n$ system structure is a very popular type of redundancy. This system functions if and only if at least $(n - r + 1)$ out of $n$ components are operating ($r \leq n$). In particular, when $r = 1$ and $r = n$, it corresponds to series and parallel system, respectively. The GRE of $X_{r:n}$ measures the

entropy of the residual lifetime of a $(n - r + 1)$-out-of-$n$ working system at time $t \geq 0$. Hence, it can be important for system designers to get information about the entropy of used $(n - r + 1)$-out-of-$n$ systems at any time $t \geq 0$. Asadi and Ebrahimi (2000) established that decreasing uncertainty residual life is preserved under the formation of parallel system. Recently, Mahmoudi and Asadi (2010) and Li and Zhang (2011) showed that decreasing Renyi entropy residual life property is preserved under the formation of the same system. It is not difficult to show that $f_{n:n}(x)/f(x)$ and

$$\frac{\lambda_{F_{n:n}}(x)}{\lambda_F(x)} = \frac{n}{\sum_{i=0}^{n-1} \binom{n}{i} \left(\frac{F(x)}{\bar{F}(x)}\right)^{i-(n-1)}}$$

are increasing in $x \geq 0$. Hence, as a consequence of Theorem 8, we get the following corollary which shows that IGRE and DGRE properties of the life distributions are preserved under the formation of parallel system.

COROLLARY 9. *Let $X$ be a random variable as described in Theorem 8. Then the $n$-th order statistic $X_{n:n}$ is IGRE (DGRE) if $X$ is.*

As an application of Corollary 9, we consider the example presented in Fig. 2 which shows that IGRE and DGRE properties of Pareto distributions are preserved under the formation of parallel system.
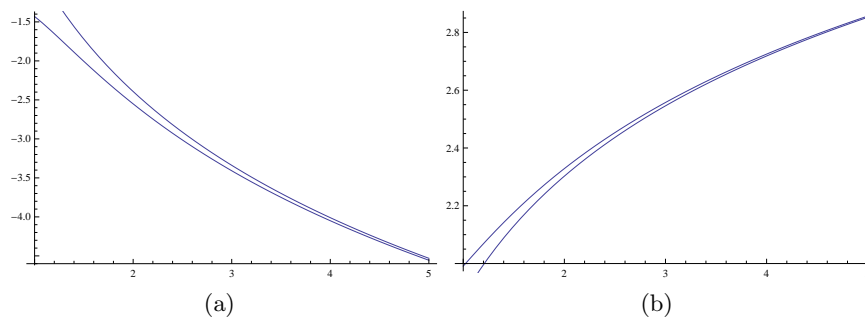


(a)                                                                (b)

*Figure 2* – (a) Plot of $V_{\alpha,\beta}(X_{n:n};t)$ (bottom curve) and $V_{\alpha,\beta}(X;t)$ (top curve) versus $t$ (along horizontal axis) for Pareto distribution as described in Example 2. Here we assume $\alpha = 1.2$, $\beta = 1.5$, $\theta = 2$, $\delta = 0.2$ and $n = 5$. (b) Plot of $V_{\alpha,\beta}(X_{n:n};t)$ (top curve) and $V_{\alpha,\beta}(X;t)$ (bottom curve) versus $t$ (horizontal axis) for Pareto distribution. Assume $\alpha = 0.2$, $\beta = 1.2$, $\theta = 2$, $\delta = 0.2$ and $n = 5$.

The following example shows that DGRE and IGRE properties do not hold under the formation of a series system.

EXAMPLE 3. *Suppose a random variable $X$ has the SF*

$$\bar{F}(x) = \begin{cases} 1 - \dfrac{x^2}{2}, & 0 \le x < 1, \\ \dfrac{2}{3} - \dfrac{x^2}{6}, & 1 \le x < 2, \\ 1, & x \ge 2. \end{cases} \tag{11}$$

The GRE of $X$ and the first order statistic $X_{1:n}$ can be obtained as

$$V_{\alpha,\beta}(X;t) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right) \ln\left[\left(\dfrac{2}{2 - t^2}\right)^\gamma \left(\dfrac{1 - t^{\gamma+1}}{\gamma + 1} + \dfrac{2^{\gamma+1} - 1}{(\gamma+1)3^\gamma}\right)\right], \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0 \le t < 1, \\ \left(\dfrac{1}{\beta - \alpha}\right) \ln\left[\left(\dfrac{2}{4 - t^2}\right)^\gamma \left(\dfrac{2^{\gamma+1} - t^{\gamma+1}}{\gamma + 1}\right)\right], \\ \qquad\qquad\qquad\qquad\qquad \text{if } 1 \le t < 2, \end{cases} \tag{12}$$

and

$$V_{\alpha,\beta}(X_{1:n};t) = \begin{cases} \left(\dfrac{1}{\beta - \alpha}\right) \ln\left[\dfrac{(2n)^\gamma}{(2 - t^2)^{n\gamma}} \displaystyle\int_t^1 \left(x(2 - x^2)^{n-1}\right)^\gamma dx \right. \\ \qquad\qquad \left. + \dfrac{(3n)^\gamma}{(3(2 - t^2))^{n\gamma}} \displaystyle\int_1^2 \left(x(4 - x^2)^{n-1}\right)^\gamma dx\right], \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0 \le t < 1, \\ \left(\dfrac{1}{\beta - \alpha}\right) \ln\left[\dfrac{(2n)^\gamma}{(4 - t^2)^{n\gamma}} \displaystyle\int_t^2 \left(x(4 - x^2)^{n-1}\right)^\gamma dx\right], \\ \qquad\qquad\qquad\qquad\qquad \text{if } 1 \le t < 2, \end{cases} \tag{13}$$
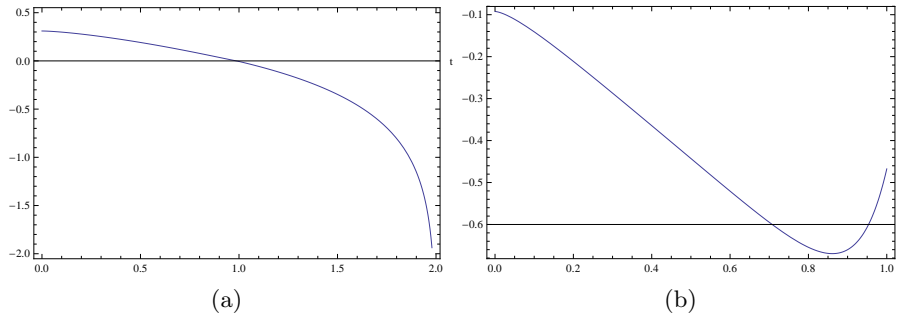
respectively.



*Figure 3* – (a) $V_{0.4,1.2}(X;t)$ Vs $t$, (b) $V_{0.4,1.2}(X_{1:n};t)$ Vs $t$ when $n = 10$.

From Fig. 3(a), it is clear that $V_{\alpha,\beta}(X;t)$ given in (12) is decreasing in $t$, for $\alpha = 0.4$ and $\beta = 1.2$, that is, $X$ is DGRE, but $V_{\alpha,\beta}(X_{1:n};t)$ (see Fig. 3(b)) is not monotone in $0 \le t < 1$, for the same values of $\alpha$ and $\beta$, when $n = 10$. Hence, we conclude that $X_{1:n}$ is not DGRE. Also Fig. 4(a) shows that $V_{\alpha,\beta}(X;t)$ is increasing in $t$ for $\alpha = 1.1$ and $\beta = 1.2$, that is, $X$ is IGRE, whereas $V_{\alpha,\beta}(X_{1:n};t)$
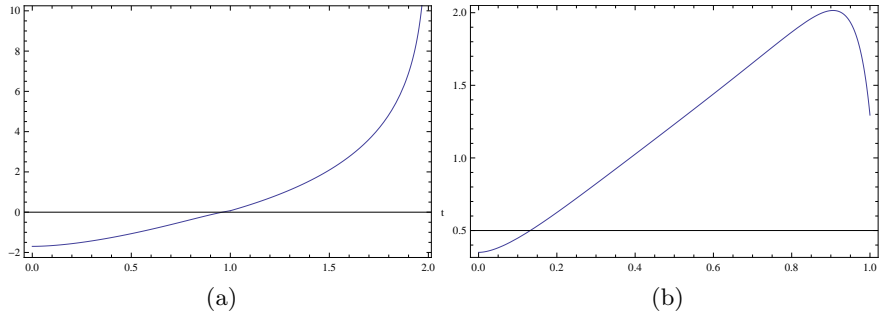
*Figure 4* – (a) $V_{1.1,1.2}(X;t)$ Vs $t$, (b) $V_{1.1,1.2}(X_{1:n};t)$ Vs $t$ when $n = 10$.

is not monotone in $0 \le t < 1$, for the same values of $\alpha$, $\beta$ and $n = 10$ (see Fig. 4(b)).

Asadi and Ebrahimi (2000) proved that if $X_{r:n}$ is decreasing uncertainty residual life, then $X_{r+1:n}$, $X_{r:n-1}$ and $X_{r+1:n+1}$ are also decreasing uncertainty residual life. Li and Zhang (2011) proved it for the Renyi entropy. In the following corollary, we establish a similar result for GRE given in (2).

COROLLARY 10. *Let $X$ be a random variable as described in Theorem 8. Then the order statistics $X_{r+1:n}$, $X_{r:n-1}$ and $X_{r+1:n+1}$ are IGRE (DGRE) if $X_{r:n}$ is.*

PROOF. Using the results of Nagaraja (1990) and the Theorem 8, the proof follows.                                                                                                   □

In the following we obtain bounds for the GRE of the $r$-th order statistic in terms of the GRE of each components and that of the $r$-th order statistic from $U(0,1)$. Note that the bounds are useful when the GRE of order statistics does not have a closed form expression, or the expectation in Lemma 2 can not be easily evaluated.

THEOREM 11. *Let $X$ be a random variable as described in Theorem 8. Also let $V_{\alpha,\beta}(X;t)$ and $V_{\alpha,\beta}(X_{r:n};t)$ denote the GREs of $X$ and $X_{r:n}$, respectively. Then the following results hold.*
*(a) Let $V_{\alpha,\beta}(X;t)$ be finite. If $M_r = max\{F(t), \frac{r-1}{n-1}\}$, then*

$$V_{\alpha,\beta}(X_{r:n};t) \le V_{\alpha,\beta}(X;t) + \left(\frac{\gamma}{\beta - \alpha}\right) \ln \bar{F}(t) + A_r(t), \tag{14}$$

*where $A_r(t) = \left(\frac{\gamma}{\beta-\alpha}\right)[(r-1)\ln M_r + (n-r)\ln(1 - M_r) - \ln B_{F(t)}(r, n-r+1)]$.*
*(b) Let $M = f(m)$ be finite, where $m = sup\{x : f(x) \le M\}$ is mode of the distribution. Then*

$$V_{\alpha,\beta}(X_{r:n};t) \le (\ge)V_{\alpha,\beta}(U_{r:n};F(t)) + \left(\frac{\gamma - 1}{\beta - \alpha}\right) \ln M, \ for \ \alpha + \beta > (<)2. \tag{15}$$

PROOF. (a): From (8) we have

$$
\begin{aligned}
V_{\alpha,\beta}(X_{r:n};t) &= V_{\alpha,\beta}(U_{r:n};F(t)) \\
&\quad + \Big(\frac{1}{\beta-\alpha}\Big)\ln\int_{F(t)}^{1}\frac{y^{\gamma(r-1)}(1-y)^{\gamma(n-r)}f^{\gamma-1}(F^{-1}(y))}{B_{F(t)}(\gamma(r-1)+1,\gamma(n-r)+1)}dy \\
&\leq V_{\alpha,\beta}(U_{r:n};F(t)) + \Big(\frac{1}{\beta-\alpha}\Big)\ln\Big(M_r\int_{F(t)}^{1}f^{\gamma-1}(F^{-1}(y))dy\Big) \\
&= V_{\alpha,\beta}(U_{r:n};F(t)) + \Big(\frac{1}{\beta-\alpha}\Big)\ln\int_{t}^{1}f^{\gamma}(v)dv + \Big(\frac{1}{\beta-\alpha}\Big)\ln M_r \\
&= V_{\alpha,\beta}(U_{r:n};F(t)) + V_{\alpha,\beta}(X;t) + \Big(\frac{1}{\beta-\alpha}\Big)\ln(\bar{F}^{\gamma}(t)M_r) \\
&= V_{\alpha,\beta}(X;t) + \Big(\frac{\gamma}{\beta-\alpha}\Big)\ln\bar{F}(t) + A_r(t).
\end{aligned}
$$

This completes the proof of Part (a). The proof of Part (b) follows as $f^{\gamma-1}(F^{-1}(y)) \leq (\geq)M^{\gamma-1}$, for $\alpha+\beta > (<)2$. $\qquad\square$

In the following we obtain bounds of GRE of the $r$-th order statistic for Pareto and generalized exponential distributions.

EXAMPLE 4. *(a) Let $X$ be a random variable with PDF $f(x|\theta,\delta) = \frac{\theta\delta^{\theta}}{x^{\theta+1}}$, $x \geq \delta > 0, \theta > 0$. Then $V_{\alpha,\beta}(X_{r:t};t) \leq A_r(t) + (\frac{1}{\beta-\alpha})\ln[\frac{(\theta\delta^{\theta})^{\gamma}}{(\gamma(\theta+1)-1)t^{\gamma(\theta+1)-1}}]$ and $V_{\alpha,\beta}(X_{r:t};t) \leq (\geq)V_{\alpha,\beta}(U_{r:n};F(t)) + (\frac{\gamma-1}{\beta-\alpha})\ln(\frac{\theta}{\delta})$, for $\alpha+\beta > (<)2$.*
*(b) For a random variable with PDF $f(x|\theta) = \theta e^{-x}(1-e^{-x})^{\theta-1}$, $x > 0, \theta > 0$, we have $V_{\alpha,\beta}(X_{r:t};t) \leq A_r(t) + (\frac{1}{\beta-\alpha})\ln[\theta^{\gamma}\bar{B}_{1-e^{-t}}(\gamma(\theta-1)+1,\gamma)]$ and $V_{\alpha,\beta}(X_{r:t};t) \leq (\geq)V_{\alpha,\beta}(U_{r:n};F(t)) + (\frac{\gamma-1}{\beta-\alpha})\ln(1-\frac{1}{\theta})^{\theta-1}, \theta > 1$, for $\alpha+\beta > (<)2$.*

In this part of the paper we obtain some monotone properties of GRE of parallel and series systems. First we present the following theorem, which will be helpful to obtain further results.

THEOREM 12. *Let $u(x)$ and $v_{\lambda}(x)$, $\lambda > 0$ be non-negative functions, where $u(x)$ is increasing. Assume that $0 \leq t < c \leq \infty$ and $W_{\lambda}$ has a density function $f_{W_{\lambda}}(w)$, where*

$$
f_{W_{\lambda}}(w) = \frac{u^{p\lambda}(w)v_{\lambda}(w)}{\int_{t}^{c}u^{p\lambda}(x)v_{\lambda}(x)dx}, \ t < w < c, \ p \in R. \tag{16}
$$

*For $p \in R$, we define a function $h_{\gamma}(.)$ as*

$$
h_{\gamma}(p) = \Big(\frac{1}{\beta-\alpha}\Big)\ln\frac{\int_{t}^{c}u^{p\gamma}(x)v_{\gamma}(x)dx}{\Big(\int_{t}^{c}u^{p}(x)v_{1}(x)dx\Big)^{\gamma}}. \tag{17}
$$

*Then for $W_{\gamma} \overset{st}{\leq} (\overset{st}{\geq})W_{1}$, $h_{\gamma}(p)$ is a decreasing (increasing) function of $p$.*

PROOF. Assume that $h_\gamma(p)$ is differentiable. Therefore we have

$$\frac{\partial h_\gamma(p)}{\partial p} = \frac{\gamma\left(\int_t^c u^p(x)v_1(x)dx\right)^{-(\gamma+1)}}{(\beta-\alpha)g_\gamma(p)}\left[\int_t^c \ln u(x)u^{p\gamma}(x)v_\gamma(x)dx \int_t^c u^p(x)v_1(x)dx\right.$$
$$\left. -\int_t^c \ln u(x)u^p(x)v_1(x)dx \int_t^c u^{p\gamma}(x)v_\gamma(x)dx\right], \qquad (18)$$

where $g_\gamma(p) = \int_t^c u^{p\gamma}(x)v_\gamma(x)dx/\left(\int_t^c u^p(x)v_1(x)dx\right)^\gamma$. Using the fact that $W_\gamma \overset{st}{\leq}(\overset{st}{\geq})W_1$ and $ln$ is increasing function we get (see Shaked and Shanthikumar, 2007)

$$E[\ln u(W_\gamma)] \leq (\geq)E[\ln u(W_1)], \qquad (19)$$

which implies that (18) is non-positive (non-negative). Hence $h_\gamma(p)$ is a decreasing (increasing) function of $p$. This completes the proof. □

THEOREM 13. *Under the assumptions of Theorem 12, if $u(x)$ is decreasing, then for $W_\gamma \overset{st}{\leq}(\overset{st}{\geq})W_1$, $h_\gamma(p)$ is an increasing (decreasing) function of $p$.*

PROOF. The proof is similar to that of the Theorem 12, hence omitted. □

In consequence of the Theorem 12 and Theorem 13, we have the following corollary.

COROLLARY 14. *Consider a parallel (series) system consists of $n$ components from $U(0,1)$. Then the GRE of the system lifetime $(U_{n:n}(U_{1:n}))$ is increasing (decreasing) function of the number of components for $\alpha + \beta > (<)2$.*

PROOF. From (7), we have $V_{\alpha,\beta}(U_{n:n};t) = (\frac{1}{\beta-\alpha})\ln(\int_t^1 x^{\gamma(n-1)}dx/(\int_t^1 x^{n-1}dx)^\gamma)$, which can be written in the form of (17), where $u(x) = x$ and $v_\gamma(x) = x^{-\gamma}$. Without loss of generality, one may assume that $n \geq 1$ is a continuous variable. Also, it can be showed that the ratio $\int_t^1 x^{\gamma(n-1)}dx/\int_t^1 x^{n-1}dx$ is increasing (decreasing) for $\alpha + \beta > (<)2$. Thus, for the chosen functions $u(x) = x$ and $v_\gamma(x) = x^{-\gamma}$, it is easy to prove that $W_\gamma \overset{st}{\geq}(\overset{st}{\leq})W_1$, for $\alpha + \beta > (<)2$, where the density function of $W_\lambda$ is given in (16). Therefore, from Theorem 12, the proof follows for parallel system. The proof for a series system follows similarly. This completes the proof. □

COROLLARY 15. *Let $U_{r:n}$ denote the $r$-th order statistic of $U(0,1)$. If $r_1 \leq r_2 \leq n$ are integers then for $t \geq \frac{r_2-1}{n-1}$, $V_{\alpha,\beta}(U_{r_1:n};t) \leq V_{\alpha,\beta}(U_{r_2:n};t)$.*

PROOF. From (7), $V_{\alpha,\beta}(U_{r:n};t)$ can be written as

$$V_{\alpha,\beta}(U_{r:n};t) = \left(\frac{1}{\beta-\alpha}\right)\ln\left(\int_t^1 u^{r\gamma}(x)v_\gamma(x)dx\Big/\left[\int_t^1 u^r(x)v_1(x)dx\right]^\gamma\right), \qquad (20)$$

where $u(x) = x/(1-x)$ and $v_\gamma(x) = (1-x)^{r\gamma}/x^\gamma$. Again, for $\alpha + \beta > (<)2$ and $t \geq \frac{r-1}{n-1}$, it is not hard to see that $W_\gamma \overset{st}{\geq}(\overset{st}{\leq})W_1$. Therefore, for $r_1 \leq r_2 \leq n$ and $t \geq \frac{r_2-1}{n-1}$, we have $V_{\alpha,\beta}(U_{r_1:n};t) \leq V_{\alpha,\beta}(U_{r_2:n};t)$. This completes the proof. □

In the following we obtain some results for the random variables having monotone density functions. Note that the class of CDFs with decreasing PDF contains exponential, Pareto, mixture of exponential and Pareto distributions, etc. There are also CDFs with increasing PDF. For example the power distribution with PDF $f(x|a) = ax^{a-1}, 0 < x < 1, a > 0$ has increasing density function.

THEOREM 16. *Consider a parallel (series) system consists of n components with CDF $F(x)$ and PDF $f(x)$. Further, assume that $f(x)$ is increasing (decreasing) in its support. Then the GRE of a system lifetime is increasing (decreasing) in n for $\alpha + \beta > (<)2$.*

PROOF. First we prove the result for parallel system. The other part follows analogously. Assume $Y_n \sim B^{F(t)}(\gamma(n-1)+1, 1)$. Then it can be easily shown that $Y_n \overset{st}{\leq} Y_{n+1}$. Also, for $\alpha + \beta > (<)2$, $f^{\gamma-1}(F^{-1}(x))$ is increasing (decreasing) in $x$. Hence for $\alpha + \beta > (< 2)$,

$$\left(\frac{1}{\beta - \alpha}\right) \ln \frac{E(f^{\gamma-1}(F^{-1}(Y_{n+1})))}{E(f^{\gamma-1}(F^{-1}(Y_n)))} \geq (\leq)0. \tag{21}$$

Moreover, from Lemma 2 we have

$$V_{\alpha,\beta}(X_{n+1:n+1}; t) - V_{\alpha,\beta}(X_{n:n}; t) = \delta_{\alpha,\beta}(n; t) - \left(\frac{1}{\beta - \alpha}\right) \ln \left[\frac{E[f^{\gamma-1}(F^{-1}(Y_n))]}{E[f^{\gamma-1}(F^{-1}(Y_{n+1}))]}\right],$$

where $\delta_{\alpha,\beta}(n; t) = V_{\alpha,\beta}(U_{n+1:n+1}; F(t)) - V_{\alpha,\beta}(U_{n:n}; F(t))$. Now using Corollary 14 and the inequality (21) the proof follows. This completes the proof. $\square$



(a)                    (b)
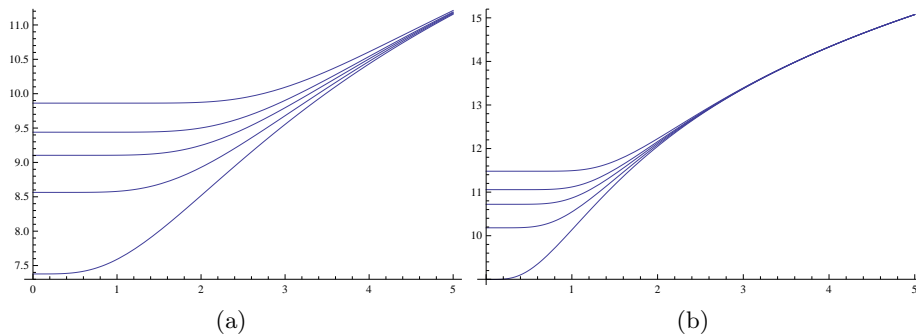
*Figure 5* – Fig. (a) and (b) represent plot of $V_{\alpha,\beta}(X_{n:n}; t)$ of exponential distribution with mean 1 and 0.5, respectively. Assume $\alpha = 1.2$ and $\beta = 1.5$. Curves corresponds to sample sizes $5, 10, 15, 20, 30$ are from bottom to top.

## 4. COMPUTATION RESULTS

In this section, we show that some of the results derived in the earlier sections of this paper are consistent with the numerical study. We present the estimated value

of $V_{\alpha,\beta}(X;t)$, $V_{\alpha,\beta}(X_{1:n};t)$ and $V_{\alpha,\beta}(X_{n:n};t)$ of an exponential distribution with mean $1/\sigma$, $\sigma > 0$. It should be mentioned that the statistical data are generated based on the Monte-Carlo simulation. The estimated values are computed based on 5000 samples with different sample sizes and different values of the parameters. The generalized residual entropy of exponential distribution with mean $1/\sigma$ can be obtained as

$$V_{\alpha,\beta}(X;t) = \left(\frac{1}{\beta-\alpha}\right)\ln(\sigma^{\gamma-1}/\gamma). \tag{22}$$

To estimate $V_{\alpha,\beta}(X;t)$ given in (22) from simulated exponential data, we use the maximum likelihood estimator (MLE) of $\sigma$. Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from exponential distribution with mean $1/\sigma$. Then the maximum likelihood estimator of $\sigma$ is given by $\hat{\sigma}_{ml} = n/\sum_{i=1}^{n} X_i = 1/\bar{X}$. As MLE is invariant, therefore for exponential distribution, $V_{\alpha,\beta}(X;t)$ can be estimated via the MLE which is given by

$$\hat{V}_{\alpha,\beta}(X;t) = \left(\frac{1}{\beta-\alpha}\right)\ln\left(\frac{(1/\bar{X})^{\gamma-1}}{\gamma}\right). \tag{23}$$

The generalized residual entropy of $X_{1:n}$ and $X_{n:n}$ of exponential distribution can be obtained as

$$V_{\alpha,\beta}(X_{1:n};t) = \left(\frac{1}{\beta-\alpha}\right)\ln\left(\frac{(n\sigma)^{\gamma-1}}{\gamma}\right) \tag{24}$$

and

$$\begin{aligned}
V_{\alpha,\beta}(X_{n:n};t) &= \left(\frac{\gamma-1}{\beta-\alpha}\right)\ln\sigma - \left(\frac{\gamma}{\beta-\alpha}\right)\log\left(\frac{1}{n}\left(1-(F(t))^n\right)\right) \\
&+ \left(\frac{1}{\beta-\alpha}\right)\log\int_{F(t)}^{1} x^{\gamma(n-1)}(1-x)^{\gamma-1}dx,
\end{aligned} \tag{25}$$

respectively, where $F(t) = 1 - exp(-\sigma t)$. The estimators of $V_{\alpha,\beta}(X_{1:n};t)$ and $V_{\alpha,\beta}(X_{n:n};t)$ given in (23) and (24), are given by

$$\hat{V}_{\alpha,\beta}(X_{1:n};t) = \left(\frac{1}{\beta-\alpha}\right)\ln\left(\frac{(n/\bar{X})^{\gamma-1}}{\gamma}\right) \tag{26}$$

and

$$\begin{aligned}
\hat{V}_{\alpha,\beta}(X_{n:n};t) &= \left(\frac{\gamma-1}{\beta-\alpha}\right)\ln(1/\bar{X}) - \left(\frac{\gamma}{\beta-\alpha}\right)\log\left(\frac{1}{n}\left(1-(\hat{F}(t))^n\right)\right) \\
&+ \left(\frac{1}{\beta-\alpha}\right)\log\int_{\hat{F}(t)}^{1} x^{\gamma(n-1)}(1-x)^{\gamma-1}dx,
\end{aligned} \tag{27}$$

respectively, where $\hat{F}(t) = 1 - exp(-t/\bar{X})$. For the sake of the space, we only present few values. However, similar trends have been observed for other values of the parameters and sample sizes. In the following we consider an example dealing with the real data-set.

TABLE 1
$\alpha = 1.2, \beta = 1.5.$

| $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X;t)$ | $V_{\alpha,\beta}(X;t)$ | $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X;t)$ | $V_{\alpha,\beta}(X;t)$ |
|---|---|---|---|---|---|---|---|
| 1 | 5 | $-1.7246$ | $-1.7687$ | 2 | 5 | $-0.1072$ | $-0.1514$ |
| | 10 | $-1.7394$ | $-1.7687$ | | 10 | $-0.1221$ | $-0.1514$ |
| | 15 | $-1.6021$ | $-1.7687$ | | 15 | $-0.1052$ | $-0.1514$ |
| | 20 | $-1.7103$ | $-1.7687$ | | 20 | $-0.0929$ | $-0.1514$ |
| | 30 | $-1.7303$ | $-1.7687$ | | 30 | $-0.1131$ | $-0.1514$ |

TABLE 2
$\alpha = 0.2, \beta = 1.2.$

| $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X;t)$ | $V_{\alpha,\beta}(X;t)$ | $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X;t)$ | $V_{\alpha,\beta}(X;t)$ |
|---|---|---|---|---|---|---|---|
| 1 | 5 | $0.9049$ | $0.9163$ | 2 | 5 | $0.4890$ | $0.5004$ |
| | 10 | $0.9087$ | $0.9163$ | | 10 | $0.4928$ | $0.5004$ |
| | 15 | $0.8734$ | $0.9163$ | | 15 | $0.4575$ | $0.5004$ |
| | 20 | $0.9012$ | $0.9163$ | | 20 | $0.4853$ | $0.5004$ |
| | 30 | $0.9064$ | $0.9163$ | | 30 | $0.4905$ | $0.5004$ |

TABLE 3
$\alpha = 1.2, \beta = 1.5.$

| $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X_{1:n};t)$ | $V_{\alpha,\beta}(X_{1:n};t)$ | $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X_{1:n};t)$ | $V_{\alpha,\beta}(X_{1:n};t)$ |
|---|---|---|---|---|---|---|---|
| 1 | 5 | $2.0307$ | $1.9865$ | 2 | 5 | $3.6481$ | $3.6039$ |
| | 10 | $3.6332$ | $3.6039$ | | 10 | $5.2506$ | $5.2213$ |
| | 15 | $4.7166$ | $4.5500$ | | 15 | $6.3339$ | $6.1673$ |
| | 20 | $5.2797$ | $5.2212$ | | 20 | $6.8971$ | $6.8386$ |
| | 30 | $6.2057$ | $6.1673$ | | 30 | $7.8231$ | $7.7847$ |

TABLE 4
$\alpha = 0.2, \beta = 1.2.$

| $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X_{1:n};t)$ | $V_{\alpha,\beta}(X_{1:n};t)$ | $\sigma$ | $n$ | $\hat{V}_{\alpha,\beta}(X_{1:n};t)$ | $V_{\alpha,\beta}(X_{1:n};t)$ |
|---|---|---|---|---|---|---|---|
| 1 | 5 | $-0.0607$ | $-0.0493$ | 2 | 5 | $-0.4766$ | $-0.4653$ |
| | 10 | $-0.4727$ | $-0.4652$ | | 10 | $-0.8887$ | $-0.8811$ |
| | 15 | $-0.7514$ | $-0.7085$ | | 15 | $-1.1673$ | $-1.1244$ |
| | 20 | $-0.8962$ | $-0.8811$ | | 20 | $-1.3121$ | $-1.2971$ |
| | 30 | $-1.1343$ | $-1.1244$ | | 30 | $-1.5502$ | $-1.5403$ |

TABLE 5
$\alpha = 1.2, \beta = 1.5$

| $n$ | $\sigma$ | $t$ | $V_{\alpha,\beta}(X_{n:n};t)$ | $\hat{V}_{\alpha,\beta}(X_{n:n};t)$ | $n$ | $\sigma$ | $t$ | $V_{\alpha,\beta}(X_{n:n};t)$ | $\hat{V}_{\alpha,\beta}(X_{n:n};t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 0.2 | -3.0801 | -3.0359 | 5 | 2 | 0.2 | -1.4441 | -1.3984 |
| | | 0.5 | -3.0361 | -3.0047 | | | 0.5 | -1.1324 | -1.0940 |
| | | 1.0 | -2.7497 | -2.5301 | | | 1.0 | -0.5492 | -0.3285 |
| | | 1.5 | -2.4115 | -2.3307 | | | 1.5 | -0.2982 | -0.2291 |
| | | 2.0 | -2.1665 | -2.1154 | | | 2.0 | -0.2053 | -0.1635 |
| 10 | 1 | 0.2 | -3.2067 | -3.1626 | 10 | 2 | 0.2 | -1.5893 | -1.5451 |
| | | 0.5 | -3.2062 | -3.1769 | | | 0.5 | -1.5395 | -1.5067 |
| | | 1.0 | -3.1568 | -2.9678 | | | 1.0 | -0.9517 | -0.6914 |
| | | 1.5 | -2.9122 | -2.8284 | | | 1.5 | -0.4723 | -0.3912 |
| | | 2.0 | -2.5690 | -2.5090 | | | 2.0 | -0.2716 | -0.2256 |

TABLE 6
$\alpha = 0.2, \beta = 1.2$

| $n$ | $\sigma$ | $t$ | $V_{\alpha,\beta}(X_{n:n};t)$ | $\hat{V}_{\alpha,\beta}(X_{n:n};t)$ | $n$ | $\sigma$ | $t$ | $V_{\alpha,\beta}(X_{n:n};t)$ | $\hat{V}_{\alpha,\beta}(X_{n:n};t)$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 1 | 0.2 | 1.1017 | 1.0902 | 5 | 2 | 0.2 | 0.6742 | 0.6623 |
| | | 0.5 | 1.0817 | 1.0736 | | | 0.5 | 0.6160 | 0.6072 |
| | | 1.0 | 1.0319 | 0.9820 | | | 1.0 | 0.5460 | 0.5170 |
| | | 1.5 | 0.9901 | 0.9724 | | | 1.5 | 0.5174 | 0.5011 |
| | | 2.0 | 0.9619 | 0.9505 | | | 2.0 | 0.5067 | 0.4964 |
| 10 | 1 | 0.2 | 1.1334 | 1.1221 | 10 | 2 | 0.2 | 0.7163 | 0.7048 |
| | | 0.5 | 1.1304 | 1.1228 | | | 0.5 | 0.6875 | 0.6789 |
| | | 1.0 | 1.1034 | 1.0541 | | | 1.0 | 0.5936 | 0.5394 |
| | | 1.5 | 1.0552 | 1.0364 | | | 1.5 | 0.5374 | 0.5198 |
| | | 2.0 | 1.0094 | 0.9969 | | | 2.0 | 0.5143 | 0.5036 |

*TABLE 7*

*Annual wage data (in multiplies of* $10,000$ *U.S. dollars).*

| 1.12 | 1.54 | 1.19 | 1.08 | 1.12 | 1.56 | 1.23 | 1.03 | 1.15 | 1.07 |
|------|------|------|------|------|------|------|------|------|------|
| 1.25 | 1.19 | 1.28 | 1.32 | 1.07 | 1.51 | 1.03 | 1.04 | 1.16 | 1.40 |
| 1.08 | 1.05 | 1.58 | 1.04 | 1.19 | 1.11 | 1.01 | 1.57 | 1.12 | 1.15 |

EXAMPLE 5. *Consider annual wage data (in multiplies of* $10,000$ *U.S. dollars) of a random sample of 30 production-line workers in a large industrial firm due to Dyer (1981). The data are presented in Table* 7. *He verified that Pareto distribution provids a good fit for the data. Here based on this data the maximum likelihood estimates of the scale parameter* $\delta$ *and shape parameter* $\theta$ *are computed as* $\hat{\delta}_{ml} = 1.01$ *and* $\hat{\theta}_{ml} = 5.92$. *Hence for* $\alpha = 1.2$, $\beta = 1.5$, $n = 5$ *and* $t = 2$, *we obtain* $\hat{V}_{\alpha,\beta}(X;t) = 0.54$, $\hat{V}_{\alpha,\beta}(X_{1:n};t) = 4.47$ *and* $\hat{V}_{\alpha,\beta}(X_{n:n};t) = 0.48$.

## ACKNOWLEDGEMENTS

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SUMMARY

*Some Results on a Generalized Residual Entropy based on Order Statistics*

In the present paper, we discuss some monotone properties of the GRE of order $(\alpha, \beta)$ in order statistics under various assumptions. It is shown that monotone properties are preserved under the formation of a parallel system but not under the formation of a series system. A counter example is presented. Bounds of the GRE of order statistics are obtained. The GRE of parallel and series systems are shown to be monotone function of the number of observations of a given sample. Numerical simulation is carried out for verification of the theoretical results. Maximum likelihood estimators of GRE of $X$, $X_{1:n}$ and $X_{n:n}$ are obtained when independent data are drawn from exponential distribution.

*Keywords:* generalized residual entropy; order statistics; parallel and series systems; maximum likelihood estimator.