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RELIABILITY ESTIMATION FOR POISSON-EXPONENTIAL MODEL UNDER PROGRESSIVE TYPE-II CENSORING DATA WITH BINOMIAL REMOVAL DATA

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1. Introduction

In the statistical literature one can find numerous distributions for modelling lifetime data. Perhaps the oldest and most extensively used life time distribution is exponential distribution. However, its usefulness is often questioned on the ground that it is only covering those situations where failure rate is constant. In practical field, a number of situations arise where can not be constant. For these cases, models having non constant failure rate are developed and available in the literature e.g., Gamma, Weibull, exponentiated exponential, etc. These distributions are generalization of exponential distribution and possess increasing, decreasing or constant failure rate depending on the value of the shape parameters and reduce to exponential distribution for specific choices of the shape parameter. For example, Gupta and Kundu [14] proposed the use of a generalized exponential distribution. To get a decreasing failure rate distribution Kus [19] modified the exponential distribution by finding the distribution of the minimum of n independently, identically and exponentially distributed random variables where n is random following zero truncated poisson distribution. Since the distribution is obtained through the compounding of poisson and exponential.

Further Barreto-Souza and Cribari-Neto [8] generalized the distribution proposed by Kus [19] by including a power parameter. Louzada-Neto et al. [24] proposed a new family of PE distribution having increasing failure rate. The distribution has been obtained by finding the distribution of the maximum of n

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independently, identically and exponentially distributed random variables where n is random following zero truncated Poisson distribution. The motivation for this family of distribution can also be traced in the study of complementary risk (CR) problems in presence of latent risks (see, Louzada-Neto [22]) i.e., for those situations when only life-time values are observed but no information is available about the factors responsible for component failures. For other details regarding CR and related models, the readers may refer Basu and Klein [9], Adamidis and Loukas [2] and Louzada-Neto [23] etc.

In reliability studies, experimenters wish to observe the failure times of items (units) placed on test. But due to time and cost constraints or various other reasons, experimenters are unable to observe life time of all items. This results to availability of censored data. Type-I and type-II are the most common and popular censoring schemes discussed in statistical literature. However, in medical/engineering survival analysis, removal of items may occur at intermediate steps also, due to various reasons which are beyond the control of the experimenter. For such a situation, progressive censoring scheme is an appropriate censoring scheme as it allows the removal of surviving items before the termination point of the test. For details and its applicability regarding PT-II CBRs see (Singh et al.[32], Balakrishnan and Aggarwala [4]). If Life test experiment starts with n units. At the first failure time $X_{1:m:n}, r_1$ $(0 \leq r_1 \leq n-m)$ units are removed from the surviving units. At second failure time $X_{2:m:n}, r_2$ $(0 \leq r_2 \leq n-m-r_1)$ units from remaining units are removed, and this process continues; till m^{th} failure time is observed i.e. at m^{th} failure all the remaining $r_m = n-m-r_1-r_2 \cdots r_{m-1}$ units are removed. Note that, m is pre-fixed and $r_i^{\prime}s$ are random. Further, for the sake of simplicity, we assume here that the probability of removal of a unit at every stage is p for each remaining unit. Thus, the number of units removed at i^{th} failure r_i will follow a binomial distribution (see, Tse et al. [34]) i.e, $r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1, 2, 3, \cdots m-1$ and $r_0 = 0$. This censoring scheme is known as progressive type-II censoring with binomial removals, denoted now onwards as PT-II CBR. It may be noted that, if $r_1 = r_2 = \cdots = r_m = 0$, PT-II CBR scheme reduces to complete sampling scheme and if $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n-m$ this scheme reduce to conventional right type-II censoring scheme. In last few years, the estimation of parameters of different life time distribution based on progressive censored samples have been studied by several authors such as Balakrishnan [7], Childs and Balakrishnan []11, Balakrishnan and Kannan [6], Mousa and Jheen [25], Ng et al. [26] and Krishna and Kumar [18]. The progressive type-II censoring with binomial removal has been considered by Tse et al. 34 for Weibull distribution and Wu and Chang [35] for Exponential distribution. Under the progressive type-II censoring with random removals, Wu and Chang [36] and Yuen and Tse [37] have developed the estimation procedure for the Pareto distribution and Weibull distribution respectively, when the number of units removed at each failure time has a discrete uniform distribution. Cramer and Iliopoulos [12] have proposed an adaptive progressive type-II censoring procedure which covers the cases of fixed censoring scheme and random censoring according to some probability distribution.

The problem of point estimation for the parameter of the PED has been discussed by a number of authors, but most of these have confined themselves to maximum likelihood estimation (MLE) or Bayes estimation under symmetric and asymmetric loss functions respectively, see Louzada-Neto et al. [24], Singh et al.[31]-[32] etc. But none has discussed the methods of estimation, namely MLE and least square methods are used to estimate two parameters and comparison between these methods are calculated under PT-II CBR. Thus, our aim is to obtain the MLEs based on asymptotic normal approximations and the least square estimation of the parameters of PED under PT-II CBR.

The organization of rest of the paper is as follows. Section 2 provides a brief description of PED model, the likelihood function under PT-II CBR . In section 3 deals with MLEs and least square estimation of the parameters of PED. A numerical study is performed to compare the effects of variation of effective sample sizes on these estimates under PT-II CBR censoring schemes in Section 4. A real data examples are given to illustrate the use of PED as a lifetime model and its reliability estimation with PT-II CBR in Section 5. Finally Conclusions are given in last Section.

2. The Model

The Cumulative density function (cdf) of PED (λ, θ) is

$$F(x) = 1 - \left[\frac{1 - e^{-\theta e^{-\lambda x}}}{1 - e^{-\theta}}\right], \quad x > 0, \quad \lambda > 0, \quad \theta > 0,$$

and the probability density function (pdf) is given as,

$$f(x) = \frac{\theta \lambda e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}}, \quad x > 0, \quad \lambda > 0, \quad \theta > 0. \tag{1}$$

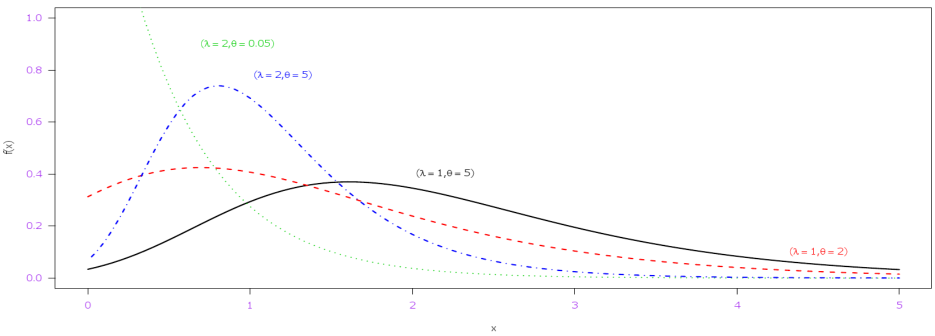

Figure 1 – The pdf plot of PED for different combinations of λ and θ.

The parameters λ and θ are the scale and shape parameters respectively of the distribution. Figure 1 shows pdfs of PED for different values of θ and λ. Its pdf is decreasing if $0 < \theta < 1$ and unimodal for $\theta \geq 1$. The modal value λe^{-1} is obtained at $x = \frac{1}{\lambda} \log\left(\frac{\theta}{\lambda}\right)$.

A general expression for the rth raw moment can be given as

$$\begin{aligned} \mu_r^{'} &= E\left[X^r\right] \\ &= \int x^r f(x) dx \qquad (2) \\ &= \int x^r \frac{\theta \lambda e^{-\lambda x - \theta e^{-\lambda x}}}{1 - e^{-\theta}} dx. \qquad (3) \end{aligned}$$

The above expression can not be solved in usual form; however it can be represented in the form of special function. Following [13], it can be expressed as follows:

$$\mu_r^{'} = \frac{\theta \Gamma(r+1)}{\lambda^r \left[1 - e^{-\theta}\right]} F_{r+1,r+1}\left(\left[1, \cdots, 1\right], \left[2, \cdots, 2\right], -\theta\right), \qquad (4)$$

where, $F_{p,q}(a, b, \theta)$ is the generalized hypergeometric function defined below:

$$F_{p,q}\left(a, b, \theta\right) = \sum_{j=0}^{\infty} \frac{\left[\theta^j \prod_{i=1}^{p} \Gamma(a_i + j) \Gamma(a_i)^{-1}\right]}{\left[\Gamma(j+1) \prod_{i=1}^{q} \Gamma(b_i + j) \Gamma(b_i)^{-1}\right]}, \qquad (5)$$

where, $a = \left[a_1, \cdots, a_p\right]$; p is the number of terms of a and $b = \left[b_1, \cdots, b_q\right]$; q is the terms of b. The proof of the Equation (4) is obtained by direct integration; see Louzada-Neto et al. [13]. It can also be noted from Louzada-Neto et al. [13] that the mean and variance of the distribution can be obtained as

$$\begin{aligned} E\left[X\right] &= \frac{\theta}{\lambda \left[1 - e^{-\theta}\right]} F_{2,2}\left(\left[1, 1\right], \left[2, 2\right], -\theta\right) \, and \\ Var\left[X\right] &= \frac{\theta}{\lambda^2 \left[1 - e^{-\theta}\right]} \left[F_{3,3}\left(\left[1, 1, 1\right], \left[2, 2, 2\right], -\theta\right)\right. \\ &\quad \left. - \frac{\theta}{\left[1 - e^{-\theta}\right]} F_{2,2}\left(\left[1, 1\right], \left[2, 2\right], -\theta\right)^2\right] \end{aligned}$$

respectively. It is interesting to note here that the skewness and kurtosis are independent of scale parameter λ and depend on shape parameter θ. It is observed that skewness and kurtosis both are decreasing function of the shape parameter θ; see Tomazella et al. [33].

2.1. Reliability characteristics

(i) Since mean does not take a very nice closed form, we shall consider here the median time to system failure (MdTSF) which is given by

$$MdTSF = \frac{log(\theta - log(-log(0.5 + 0.5e^{-\theta})))}{\lambda}$$

(ii) The corresponding reliability function is given by

$$R(x) = \left[\frac{1-e^{-\theta e^{-\lambda x}}}{1-e^{-\theta}}\right], \quad x>0, \quad \lambda>0, \quad \theta>0.$$

(iii) The associated hazard function can easily be obtained as

$$h(x) = \frac{\theta\lambda e^{-\lambda x-\theta e^{-\lambda x}}}{1-e^{-\theta e^{-\lambda x}}}, \quad x>0, \quad \lambda>0, \quad \theta>0.$$

It may be noted that the initial and long term failure values are finite and are given as

$$h(0) = \lim_{x\to 0}\frac{\lambda\theta e^{-\theta x}}{(1-e^{-\theta})} = \frac{\lambda\theta}{(e^{\theta}-1)}$$

and

$$h(\infty) = \lim_{x\to\infty}\frac{\theta\lambda e^{-\lambda x}}{e^{\theta e^{-\lambda x}}-1} = \lim_{x\to\infty}\frac{\theta\lambda}{\theta+\theta^2 e^{-\lambda x}+\cdots} = \lambda$$

respectively.

It can be seen from Figure 2, that the failure function is increasing. For other details about PE distribution see Ristic and Nadarajah [27].

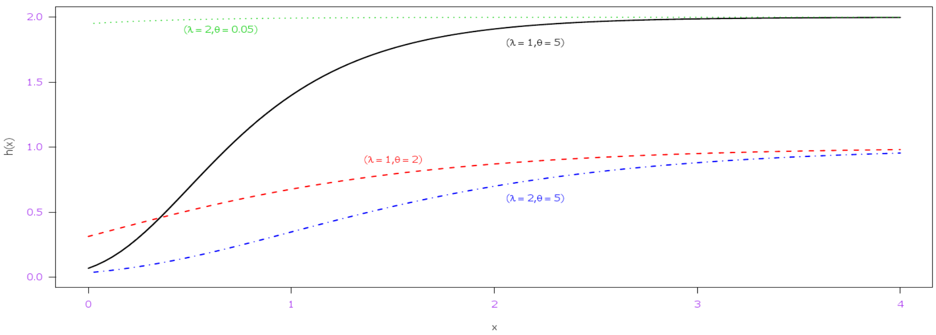

Figure 2 – The failure rate plot of PED for different combinations of λ and θ.

2.2. *Data collection*

Let us suppose that n items, the life time of which follow PED, are put on life test. Further assume that R_1 items after first failure $X_{1:m:n}$ and R_2 items after failure $X_{2:m:n}, \cdots, R_m$ items after m^{th} failure $X_{m:m:n}$ are randomly removed from the test. Let the sample obtained in this way is denoted by $(X_{1:m:n}, R_1), (X_{2:m:n}, R_2), (X_{3:m:n}, R_3), \cdots, (X_{m:m:n}, R_m)$. Where $X_{1:m:n} < X_{2:m:n} < X_{3:m:n}, \cdots < X_{m:m:n}$. It may be noted here that the number of items removed at i^{th} stage, R_i is random variable following binomial distribution (as explained in section 1). If the number of removals, say $R_1 = r_1, R_2 = r_2, R_3 = r_3, \cdots, R_m = r_m$, are assumed to be fixed, the conditional likelihood function can be written as (see Cohen [10], Kamps and Cramer [17], Balakrishnan et al. [5]):

$$\begin{aligned} L(\alpha,\theta;x|R=r) &= f_{(X_{1:m:n},\cdots,X_{m:m:n})}(x_1,\cdots,x_m) \\ &= c\prod_{i=1}^{m} f\left(x_i\right)\left[1-F\left(x_i\right)\right]^{r_i}, \quad -\infty < x_1 < \cdots < x_m < \infty, \end{aligned} \tag{6}$$

Where $n = m+\sum_{i=1}^{m} r_i$, $n, m \in N, r_i \in N_0, 1 \leq i \leq m$, $r_i \sim B(n-m-\sum_{l=0}^{i-1} r_l, p)$ for $i = 1, 2, 3, \cdots m-1$ and $r_0 = 0$ and $c = \prod_{i=1}^{m} \gamma_i$ with $\gamma_i = \sum_{j=i}^{m}(r_j+1)$ and for $\gamma_1 = n$.

Substituting (1) and (3.1) into (3.1), we get

$$L(\alpha,\lambda;x|R=r) = c\prod_{i=1}^{m} \frac{\theta\lambda e^{-\lambda x_i - \theta e^{-\lambda x_i}}}{1-e^{-\theta}} \left\{\frac{1-e^{-\theta e^{-\lambda x_i}}}{1-e^{-\theta}}\right\}^{r_i}. \tag{7}$$

As mentioned earlier also, the number of items removed are random and independent of each other with probability p for each unit at every stage. Thus, the number of the units R_i removed at i^{th} failure $X_{i:m:n}$; $i = 1, 2, \cdots (m-1)$, follows a binomial distribution with parameters $\left(n-m-\sum_{l=1}^{i-1} r_i, p\right)$. Therefore,

$$P\left(R_1 = r_1; p\right) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \tag{8}$$

and for $i = 2, 3, \cdots, m-1$,

$$\begin{aligned} P\left(R_i; p\right) &= P\left(R_i = r_i | R_{i-1} = r_{i-1}, \cdots R_1 = r_1\right) \\ &= \binom{n-m-\sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=0}^{i-1} r_l}. \end{aligned} \tag{9}$$

We further assume that $R_i s$ are independent of $X_{i:m:n}$ for all i. Then full likelihood function takes the following form

$$L\left(\theta,\lambda,p;x\right) = L\left(\theta,\lambda;x|R=r\right) P\left(R=r;p\right), \tag{10}$$

where

$$\begin{aligned} P\left(R=r;p\right) = {} & P\left(R_1=r_1\right) P\left(R_2=r_2|R_1=r_1\right) P\left(R_3=r_3|R_2=r_2, R_1=r_1\right) \\ & \cdots P\left(R_{m-1}=r_{m-1}|R_{m-2}=r_{m-2}, \cdots R_1=r_1\right). \end{aligned} \tag{11}$$

Substituting (8) and (9) into (11), we get

$$P\left(R=r;p\right) = \frac{(n-m)!p^{\sum_{i=1}^{m-1} r_i}\left(1-p\right)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n-m-\sum_{l=1}^{i-1} r_i\right)!\prod_{i=1}^{m-1} r_i!}. \tag{12}$$

Now, using (7),(10) and (12), we can write the full likelihood function in the following form

$$L(\lambda, \theta, p; x) = AL_1(\lambda, \theta) L_2(p), \tag{13}$$

where

$$A = \frac{c(n-m)!}{\left(n-m-\sum_{l=1}^{i-1} r_i\right)! \prod_{i=1}^{m-1} r_i!},$$

$$L_1(\theta; \lambda) = \prod_{i=1}^{m} \frac{\theta\lambda e^{-\lambda x_i - \theta e^{-\lambda x_i}}}{1-e^{-\theta}} \left\{ \frac{1-e^{-\theta e^{-\lambda x_i}}}{1-e^{-\theta}} \right\}^{r_i}, \tag{14}$$

and

$$L_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}. \tag{15}$$

3. Parameter estimation of λ, θ and reliability characteristics

For this PED(λ, θ) we use two known methods for estimating the parameter λ and θ, namely ML method and LS method. It is noted that the estimate can not be expresses in nice closed form when both of the parameters are unknown in either two methods.

3.1. Maximum likelihood estimators

In this section, we have obtained the MLEs of the parameters λ, θ and p based on PT-II CBRs. We observe from (13), (14) and (15) that likelihood function is multiplication of three terms, namely, A, L_1 and L_2. Out of these A does not dependent on the parameters θ, λ and p, thus, it behaves as constant for given data set. L_1 does not involved p and can be treated as function of θ and λ only, where as L_2 involves p only. Therefore, the MLEs of θ and λ can be derived by maximizing L_1. Similarly the MLE of p can be obtained by maximizing L_2.
Taking log of both sides of (14), we have

$$\begin{aligned} \ln L_1(\lambda; \theta) = m \ln(\lambda) + m \ln(\theta) - \lambda \sum_{i=1}^{m} x_i - \theta \sum_{i=1}^{m} e^{-\lambda x_i} \\ + \sum_{i=1}^{m} r_i \ln(1 - e^{-\theta e^{-\lambda x_i}}) - n \ln(1-e^{-\theta}). \end{aligned} \tag{16}$$

The normal equations can be obtained by differentiating (16) with respect to θ and λ and equating these to zero. Thus, MLEs of θ and λ can be obtained by simultaneously solving the following normal equations :

$$\frac{m}{\theta} - \sum_{i=1}^{m} e^{-\lambda x_i} + \sum_{i=1}^{m} \frac{r_i e^{-\lambda x_i - \theta e^{-\lambda x_i}}}{1-e^{-\theta e^{-\lambda x_i}}} - \frac{ne^{-\theta}}{1-e^{-\theta}} = 0 \tag{17}$$

and

$$\frac{m}{\lambda} - \sum_{i=1}^{m} x_i + \theta \sum_{i=1}^{m} x_i e^{-\lambda x_i} - \theta \sum_{i=1}^{m} \frac{r_i x_i e^{-\lambda x_i - \theta e^{-\lambda x_i}}}{1 - e^{-\theta e^{-\lambda x_i}}} = 0. \tag{18}$$

It may be noted that (17) and (18) can not be solved simultaneously to provide a nice closed form for the estimators. Therefore, we propose the use of fixed point iteration method for solving these equations. For further details see Jain et al. [15], Rao [28] and Singh et al. [30]. Hence, the MLEs of MdTSF, $R(t)$ and $h(t)$ can be evaluated using invariance property of MLEs as

$$\hat{M}dTSF = \frac{log(\hat{\theta} - log(-log(0.5 + 0.5e^{-\hat{\theta}})))}{\hat{\lambda}}$$

$$\hat{R}(t) = \left[\frac{1 - e^{-\hat{\theta}e^{-\hat{\lambda}t}}}{1 - e^{-\hat{\theta}}}\right], \quad t > 0$$

$$\hat{h}(t) = \frac{\hat{\theta}\hat{\lambda}e^{-\hat{\lambda}t - \hat{\theta}e^{-\hat{\lambda}t}}}{1 - e^{-\hat{\theta}e^{-\hat{\lambda}t}}}, \quad t > 0.$$

The MLE of parameter p can be obtained by maximizing (15). Thus we find immediately,

$$\hat{p}_{mle} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1} r_i (m-i-1)}.$$

3.2. Least square estimation

In this section, we drive The least squares estimators (LSEs) of the two parameters of PED, with modifications necessary to make for PT-II CBR sample. The LSEs were originally proposed by Swain et al. [29] for Beta distribution.

Let $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ be a PT-II CBR sample with sample size n, failure information m and censoring scheme with random $R = (R_1, R_2, \cdots, R_m)$ from a population with cdf $F(.)$. Then, for $i = 1, 2, \cdots, m$ see (Balakrishnan and Aggarwala [4][pp. 22-23], Balakrishnan et al. [5]).

$$E\left(F\left(X_{i:m:n}\right)\right) = 1 - \prod_{j=m-i+1}^{m} \alpha_j$$

where

$$\alpha_j = \tfrac{a_i}{(1+a_i)} \text{ and } a_i = i + \sum_{j=m-i+1}^{m} r_j$$

Now, the LSE of the parameters can be obtained by minimising

$$Q = \sum_{i=1}^{m} \left[F\left(X_{i:m:n}\right) - E\left(F\left(X_{i:m:n}\right)\right)\right]^2 \tag{19}$$

with respect to the parameters.

In the case of PED, the LSEs of (λ, θ) can be obtained by minimizing,

$$Q\left(\lambda, \theta\right) = \sum_{i=1}^{m} \left[\prod_{j=m-i+1}^{m} \alpha_j - \left\{ \frac{1 - e^{-\theta e^{-\lambda x_i}}}{1 - e^{-\theta}} \right\} \right]^2 \tag{20}$$

with respect to the parameters λ and θ. Thus, the LSEs of (λ, θ) are the solutions of the following simultaneous equations:

$$\frac{\partial Q\left(\lambda, \theta\right)}{\partial \lambda} = \sum_{i=1}^{m} \left[\left[\prod_{j=m-i+1}^{m} \alpha_j - \left\{ \frac{1 - e^{-\theta e^{-\lambda x_i}}}{1 - e^{-\theta}} \right\} \right] \theta x_i \left\{ \frac{e^{-\left(\theta e^{-\lambda x_i} + \lambda x_i\right)}}{1 - e^{-\theta}} \right\} \right] = 0, \tag{21}$$

$$\frac{\partial Q\left(\lambda, \theta\right)}{\partial \theta} = \sum_{i=1}^{m} \left[\left[\prod_{j=m-i+1}^{m} \alpha_j - \left\{ \frac{1 - e^{-\theta e^{-\lambda x_i}}}{1 - e^{-\theta}} \right\} \right] \frac{e^{-\theta}(1 - e^{-\theta e^{-\lambda x_i}}) - (1 - e^{-\theta}) e^{-\left(\theta e^{-\lambda x_i} + \lambda x_i\right)}}{(1 - e^{-\theta})^2} \right] = 0. \tag{22}$$

The solutions of the simultaneous Equations (21) and (22) can be evaluated numerically by some suitable iterative procedure such as Newton-Raphson method, for given the values of $(n, m, R, X_{i:m:n})$ for $i = 1, 2, 3, \cdots, m$.

3.3. Asymptotic confidence intervals

The first derivatives of the log likelihood of PED with respect to λ and θ are given by equation (17) and (18) and hence the second derivatives are

$$\frac{\partial^2 \ln L_1(\lambda;\theta)}{\partial \theta^2} = -\sum_{i=1}^{m} \frac{r_i e^{-\theta e^{-\lambda x_i} - \lambda x_i} \left\{ e^{-\lambda x_i} \left(e^{-\theta e^{-\lambda x_i}} - 1 \right) - e^{-\theta e^{-\lambda x_i}} \right\}}{\left(1 - e^{-\theta e^{-\lambda x_i}}\right)^2} - \frac{m}{\theta^2} + \frac{n}{\left(1 - e^{-\theta}\right)^2} \tag{23}$$

$$\frac{\partial^2 \ln L_1(\lambda;\theta)}{\partial \lambda^2} = -\theta \sum_{i=1}^{m} \frac{r_i x_i e^{-\lambda x_i - \theta e^{-\lambda x_i}} \left\{ \left(1 - e^{-\theta e^{-\lambda x_i}}\right) (\theta x_i e^{-\lambda x_i} - x_i) + \left(\theta x_i e^{-\lambda x_i - \theta e^{-\lambda x_i}}\right) \right\}}{\left(1 - e^{-\theta e^{-\lambda x_i}}\right)^2} - \frac{m}{\lambda^2} - \sum_{i=1}^{m} x_i^2 e^{-\lambda x_i} \tag{24}$$

$$\frac{\partial^2 \ln L_1(\lambda;\theta)}{\partial \lambda \partial \theta} = \frac{\partial^2 \ln L_1(\lambda;\theta)}{\partial \theta \partial \lambda} = \sum_{i=1}^{m} x_i e^{-\lambda x_i} + \theta \sum_{i=1}^{m} \frac{r_i e^{-\lambda x_i - \theta e^{-\lambda x_i}} \left\{ \left(1 - e^{-\theta e^{-\lambda x_i}}\right) (\theta x_i e^{-\lambda x_i} - x_i) + \left(\theta x_i e^{-\lambda x_i - \theta e^{-\lambda x_i}}\right) \right\}}{\left(1 - e^{-\theta e^{-\lambda x_i}}\right)^2} \tag{25}$$

If we denote the MLE of $\zeta = (\lambda, \theta)$ by $(\hat{\lambda}_{ML}, \hat{\theta}_{ML})$, the observed information matrix is then given by

$$I(\delta) = \begin{bmatrix} \frac{\partial^2 lnL_1(\lambda;\theta)}{\partial\lambda^2} & \frac{\partial^2 lnL_1(\lambda;\theta)}{\partial\lambda\partial\theta} \\ \frac{\partial^2 lnL_1(\lambda;\theta)}{\partial\theta\partial\lambda} & \frac{\partial^2 lnL_1(\lambda;\theta)}{\partial\theta^2} \end{bmatrix}_{\lambda=\hat{\lambda}_{ML},\theta=\hat{\theta}_{ML}} \tag{26}$$

And hence the variance covariance matrix would be $I^{-1}(\delta)$. The approximate $(1-\alpha)100\%$ confidence intervals (CIs) for the parameters λ and θ are $\hat{\lambda}_{ML} \pm \zeta_{\alpha/2}\sqrt{V(\hat{\lambda}_{ML})}$ and $\hat{\theta}_{ML} \pm \zeta_{\alpha/2}\sqrt{V(\hat{\theta}_{ML})}$, respectively where $V(\hat{\lambda}_{ML})$ and $V(\hat{\theta}_{ML})$ are variances of $\hat{\lambda}_{ML}$ and $\hat{\theta}_{ML}$, which are given by the first and the second, diagonal element of $I^{-1}(\delta)$, and $\zeta_{\alpha/2}$ is the upper $(\alpha/2)$ percentile of standard normal distribution.

4. Numerical study

The estimators $\hat{\lambda}_{ML}$ and $\hat{\theta}_{ML}$ denote the MLEs of the parameters λ and θ respectively while $\hat{\lambda}_{LS}$ and $\hat{\theta}_{LS}$ are corresponding LSEs. Also $((\lambda_{L^c}\ \lambda_{U^c})$ and $(\theta_{L^c}\ \theta_{U^c}))$ represent average CI. We have obtained the estimates and reliability characteristics from MLEs and estimates from LSEs. The comparisons are based on MSE through the simulated sample of estimates. It may be mentioned here that the exact expressions for the estimates can not be expressed in closed form. Therefore, the estimates are estimated on the basis of Monte-Carlo simulation study of 10000 samples. We have taken $(\lambda = 2, \theta = 5)$ and $(\lambda = 2, \theta = 6)$ for three different sample sizes (i) small sample size $n = 20$, (ii) moderate sample size $n = 30$, and (iii) large sample size $n = 50$. Also, each sample size has 4 censoring schemes. For the arbitrary chosen value of this parameter $(\lambda = 2, \theta = 5)$ has been all ready studied in bayesian paradigm under PT-II CBR by Singh et al. [32]. So, We study this values in classical frame under MLE and LSE. The study contains the following steps:

(a) Take the values of the parameters λ, θ and mission time t.

(b) Compute the actual values of MdTSF, $R(t)$ and $h(t)$.

(c) Generate a PT-II CBR sample of size n with m failures using the algorithm given by Balakrishnan and Shandhu [3].

(d) For each value of $(n = 20, 30, 50)$, four values of m are considered, so that, the percentage of failure information $(m/n) \times 100$ is $40, 50, 80$ and 100%.

(e) Compute the MLEs of $\lambda, \theta, MdTSF, R(t)$ and $h(t)$ according to Subsection 3.1. Also, compute CI for both λ and θ (using normal approximation) and corresponding compute LSEs of λ and θ according to Subsection 3.2.

(f) Repeat steps $(c)-(d)$, $N = 10,000$ times for chosen values of $(\lambda = 2, \theta = 5, t = 1.8, p = 0.5)$ and $(\lambda = 2, \theta = 6, t = 1.8, p = 0.5)$ with each of 12 censoring schemes. Compute the expected value (EV), mean square error (MSE) of the estimates obtained in step (d) using the following formulae EV $= \frac{1}{N}\sum_{i=1}^{N}\hat{\phi}(\delta)$ and MSE$= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\phi}(\delta) - \phi(\delta)\right)^2$, where $\delta = (\lambda, \theta)$, $\phi(.)$ is function of the set of model parameter, while $\hat{\phi}(\delta)$ is an estimate of $\phi(\delta)$.

4.1. Comparison of simulation study

The simulation study was performed for various values of effective sample m for both parameters are unknown. Briefly, we are giving here the simulation (Tables 1−3), for both parameters are unknown with ($\lambda = 2, \theta = 5$) and ($\lambda = 2, \theta = 6$) under different percentage of failure information. The conclusions for these studies are given below.

(a) Tables 1 shows the LSE of λ and θ and the corresponding reliability characteristics respectively. From Tables 1, one can note that the MSE and bais decrease as the effective sample size(m) get near to n.

(b) The MLEs are presented in Tables (1-2). It is also observed similar pattern as Tables 1.

(c) One can see, based on Tables 1 and Tables 2, the LSEs of λ, θ perform better as compare to MLE in terms of bais and MSE.

(d) In Tables 2-4, 6 and 5, 7, (($\lambda_{L^c}\ \lambda_{U^c}$) and ($\theta_{L^c}\ \theta_{U^c}$)) are presented an average limit of lower and upper limit of λ and θ respectively. It can be seen from these tables that as m increases the average length of the intervals decreases.

5. Real data analysis

In this section, we consider two types of real data example. First data set is a result of a test on endurance of deep groove ball bearings and was originally discussed by Lieblein and Zelen [21] and also given in Lawless [20]. Second data set in this example was extracted from Ed Fuller of the NICT Ceramics Division. It contains polished window strength data. A paper by Pepi [16] describes the all-glass airplane window design. Since the data were used in a case of study is a form of reliability analysis Under PT-II CBRs. The second data sets are as follows: $18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381$. In order to have an idea about the associated failure rate for both example, we considered, a graphical method based on TTT (Total Time on Test) plot as a crude indicator see Aarset [1]. The empirical TTT is given as

$$T(\tfrac{r}{n}) = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}},$$

where $r = 1, 2, \cdots, n$ and $x_{(r)}$ is the order statistics of the sample. Figure (5,8) shows the TTT plot, which is concave indicating that data relates to an increasing failure rate. Thus, It can be properly accommodated by a PED. The fitting of PED was checked using CDF-plot and PP-plot given in Figure (3-4), Figure (6-7) and Kolmogorov-Smirnov(KS) test. Value of the test statistics $0.115023 < 0.275(KS_{(Tabulated)})$ and $0.1052121 < 0.2903226(KS_{(Tabulated)})$, which shows that PED provides a satisfactory fit to the considered two data sets respectively. On the basis of these data set the MLEs, LSEs, intervals of λ and θ are presented in Table 5 and Table 7 respectively. But for the purpose of illustrating the method discussed in this paper, a PT-II CBR is generated from these data sets under different schemes. The number of removals are shown in Table 4 Table 6 under different schemes. Using the formulae given in section (3) under different degree of censoring, the LSEs and MLEs of λ and θ are presented in Table 5 and Table 7 respectively.

6. Conclusion

In this paper, we have considered the problem of classical estimation of parameters of PED under PT-II CBR sample. We have found that in most of the considered method of estimations, LSEs provides the precise estimate with smaller MSE as well as bais. Finally, we may conclude that the LSEs discussed in this article can be recommended for use of PED parameter estimation under PT-II CBRs sample.

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Appendix

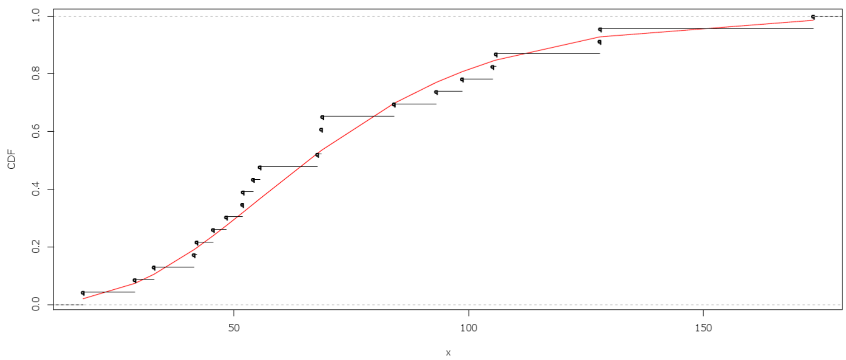

Figure 3 – CDF plots for endurance of deep groove ball bearings data.

TABLE 1
The LSEs, $\hat{\lambda}_{LS}, \hat{\theta}_{LS}$ with N=10000.

		$(\lambda = 2, \theta = 5)$				$(\lambda = 2, \theta = 6)$			
		$\hat{\lambda}_{LS}$		$\hat{\theta}_{LS}$		$\hat{\lambda}_{LS}$		$\hat{\theta}_{LS}$	
n	m	EV	MSE	EV	MSE	EV	MSE	EV	MSE
20	8	2.257866	0.1432075	5.217637	0.07505436	2.257793	0.1388416	6.211979	0.06751352
	10	2.253057	0.1307837	5.211601	0.06776987	2.245029	0.120462	6.220359	0.07074178
	16	2.210637	0.08508801	5.207937	0.06635705	2.216674	0.09082911	6.214638	0.06917726
	20	2.205179	0.0815763	5.207617	0.06114342	2.210007	0.08497755	6.193346	0.05463395
30	12	2.218481	0.1220195	5.214065	0.07356288	2.236031	0.1122036	6.218901	0.07189593
	15	2.244198	0.09963355	5.210096	0.06802988	2.228203	0.1016959	6.208113	0.06483563
	24	2.195392	0.07115373	5.209478	0.06758128	2.19641	0.07234936	6.202962	0.06024173
	30	2.189607	0.0605371	5.208539	0.06393413	2.176261	0.05928837	6.204176	0.05940488
50	20	2.206203	0.07857486	5.238891	0.09442493	2.191333	0.0670037	6.216795	0.07098946
	25	2.203551	0.07345039	5.226853	0.08115761	2.184433	0.06142801	6.214154	0.06977389
	40	2.159913	0.04543208	5.209465	0.06696698	2.15157	0.03682177	6.21612	0.06889521
	50	2.165206	0.04367941	5.203225	0.06356012	2.140585	0.03277235	6.210628	0.06807916

Here, EV=expected value; MSE=mean square error.

TABLE 2

The MSEs, $\hat{\lambda}_{ML}, \hat{\theta}_{ML}, Md\hat{T}SF, \hat{R}_{ML}(t)$ *and* $\hat{h}_{ML}(t)$ *with N=10000,* $\lambda = 2, \theta = 5, t = 1.8, R(t) = 0.12177, h(t) = 1.8664, MdTSF = 0.9928$

		$\hat{\lambda}_{ML}$				$\hat{\theta}_{ML}$				$\hat{R}_{ML}(t)$ and $\hat{h}_{ML}(t)$				$\hat{M}dTSF$	
n	m	EV	MSE	λ_{Lc}	λ_{Uc}	EV	MSE	θ_{Lc}	θ_{Uc}	EV	$R_{MSE}(t)$	EV	$h_{MSE}(t)$	EV	$MdTSF_{MSE}$
20	8	2.516772	0.4515471	1.112505	3.921038	6.043283	1.385528	0.7784289	11.30814	0.0741134	0.003924923	2.425185	0.530386	0.3286175	0.6284103
	10	2.434744	0.3225734	1.210062	3.659426	5.999493	1.325861	0.9982802	11.00071	0.08001229	0.003149592	2.336998	0.380646	0.889639	0.02626591
	16	2.338223	0.1788687	1.39232	3.284126	5.968235	1.267764	1.452574	10.4839	0.08853164	0.00220595	2.231794	0.2138113	0.93350628	0.01307016
	20	2.316086	0.1657632	1.46048	3.171693	5.962575	1.240088	1.63123	10.29392	0.09182791	0.002009896	2.206689	0.1975083	0.94065402	0.01050037
30	12	2.401921	0.2557954	1.313118	3.490725	6.009535	1.346643	1.706662	10.31241	0.08231192	0.002811488	2.301634	0.304476	0.90694587	0.01827684
	15	2.353267	0.2055738	1.390486	3.316048	5.92027	1.167238	1.863624	9.976917	0.086872	0.002454561	2.248781	0.2458692	0.93138544	0.01318454
	24	2.297906	0.1380174	1.539019	3.056793	5.963833	1.237799	2.262185	9.66548	0.0931578	0.001757219	2.187035	0.1645452	0.94887726	0.00867817
	30	2.25518	0.1017315	1.577246	2.933115	5.964612	1.225471	2.438641	9.490584	0.09925675	0.001351311	2.138516	0.1207596	0.96033982	0.006650774
50	20	2.302762	0.1415643	1.502238	3.103285	5.963766	1.234209	3.079802	8.847729	0.09187583	0.00179172	2.193088	0.1688115	0.93891037	0.01026392
	25	2.27523	0.1234666	1.561433	2.989027	5.955072	1.218825	2.749572	9.160572	0.09651679	0.00155424	2.161273	0.1467117	0.94556656	0.008999505
	40	2.231225	0.08292604	1.662612	2.799837	5.942028	1.197306	2.594613	9.289444	0.1023322	0.001081027	2.111676	0.09759213	0.96651341	0.005665663
	50	2.204684	0.0624741	1.692183	2.717186	5.893823	1.09636	3.203957	8.583689	0.1047418	0.0008482828	2.082988	0.07317595	0.97424549	0.004061965

TABLE 3

The MSEs, $\hat{\lambda}_{ML}, \hat{\theta}_{ML}, Md\hat{T}SF, \hat{R}_{ML}(t)$ and $\hat{h}_{ML}(t)$ with N=10000, $\lambda = 2, \theta = 6, t = 1.8, R(t) = 0.1490, h(t) = 1.84053, MdTSF = 1.080$

		$\hat{\lambda}_{ML}$				$\hat{\theta}_{ML}$				$\hat{R}_{ML}(t)$ and $\hat{h}_{ML}(t)$				$\hat{M}dTSF$	
n	m	EV	MSE	λ_{L^c}	λ_{U^c}	EV	MSE	θ_{L^c}	θ_{U^c}	EV	$R_{MSE}(t)$	EV	$h_{MSE}(t)$	EV	$MdTSF_{MSE}$
20	8	2.44921	0.34048	1.12241	3.77600	7.00744	1.34651	1.66921	12.34568	0.09335	0.00510	2.33702	0.41789	0.55754	0.00055
	12	2.37823	0.24576	1.21768	3.53878	6.98553	1.28297	1.11919	12.85188	0.10184	0.00401	2.25743	0.30303	0.57445	0.00051
	16	2.32126	0.15674	1.40911	3.23340	6.97512	1.27222	0.82919	13.12105	0.10718	0.00308	2.19465	0.19519	0.58913	0.00048
	20	2.26969	0.12342	1.45618	3.08320	6.97101	1.25968	1.88605	12.05596	0.11581	0.00241	2.13520	0.15305	0.60152	0.00046
30	12	2.33507	0.19187	1.31376	3.35637	6.99414	1.32647	2.83962	11.14865	0.10704	0.00334	2.20905	0.23714	0.58540	0.00049
	15	2.32770	0.17555	1.41198	3.24342	7.00376	1.32504	2.63630	11.37122	0.10718	0.00323	2.20144	0.21792	0.58700	0.00049
	24	2.25934	0.10217	1.53592	2.98276	6.99823	1.31990	1.96458	12.03187	0.11656	0.00212	2.12368	0.12652	0.60517	0.00045
	30	2.22374	0.07862	1.57600	2.87148	6.98875	1.31094	2.19541	11.78209	0.12268	0.00165	2.08244	0.09669	0.61459	0.00043
50	20	2.29358	0.13113	1.52637	3.06079	6.98082	1.29346	3.05647	10.90517	0.11079	0.00262	2.16347	0.16298	0.59552	0.00047
	25	2.25073	0.09316	1.56827	2.93320	6.95276	1.20540	3.21393	10.69159	0.11689	0.00202	2.11487	0.11575	0.60609	0.00045
	40	2.20912	0.06619	1.66167	2.75657	6.92932	1.18913	3.75441	10.10423	0.12355	0.00145	2.06723	0.08152	0.61653	0.00043
	50	2.18611	0.05159	1.69380	2.67843	6.91783	1.16204	3.56315	10.27250	0.12772	0.00109	2.04033	0.06259	0.62334	0.00042

TABLE 4
PT-II CBR samples under different censoring scheme ($S_{n:m}$) for fixed $n = 23$,$p = 0.5$

Scheme	i	1	2	3	4	5	6	7	8	9	10	11	12
$S_{23:9}$	$r_{i:9:23}$	2	2	2	0	0	2	2	2	2			
	$x_{i:9:23}$	17.88	41.52	48.4	54.12	55.56	67.8	68.88	98.64	127.92			
$S_{23:9}$	$r_{i:9:23}$	0	0	0	0	0	0	0	0	14			
	$x_{i:9:23}$	17.88	28.92	33	41.52	42.12	45.6	48.4	51.84	51.96			
$S_{23:12}$	$r_{i:12:23}$	1	1	1	1	1	0	0	0	0	2	2	2
	$x_{i:12:23}$	98.64	127.92	17.88	33	42.12	48.4	51.96	55.56	67.8	68.64	68.64	68.88
$S_{23:12}$	$r_{i:12:23}$	0	0	0	0	0	0	0	0	0	0	0	11
	$x_{i:12:23}$	17.88	28.92	33	41.52	42.12	45.6	48.4	51.84	51.96	54.12	55.56	67.8
$S_{23:18}$	$r_{i:18:23}$	1	1	0	0	0	0	0	0	0	0	0	0
	$x_{i:18:23}$	17.88	33	42.12	45.6	48.4	51.84	51.96	54.12	55.56	67.8	68.64	68.64
$S_{23:18}$	$r_{i:18:23}$	0	0	0	0	0	0	0	0	0	0	0	0
	$x_{i:18:23}$	17.88	28.92	33	41.52	42.12	45.6	48.4	51.84	51.96	54.12	55.56	67.8
		0	0	0	1	1	1						
		68.88	84.12	93.12	98.64	105.84	128.04						
		0	0	0	0	0	5						
		68.64	68.64	68.88	84.12	93.12	98.64						

TABLE 5
The LSEs, $\hat{\lambda}_{LS}$, $\hat{\theta}_{LS}$ and MLEs $\hat{\lambda}_{ML}$, $\hat{\theta}_{ML}$, $\hat{MdTSF}$, $\hat{R}(t)$, $\hat{h}(t)$ and 95% confidence intervals (CI) for λ and θ with various censoring schemes.

Scheme			95%CI ofλ				95%CI of θ			t=72.22	
$S_{m:n}$	$\hat{\lambda}_{LS}$	$\hat{\lambda}_{ML}$	λ_{L^c}	λ_{U^c}	$\hat{\theta}_{LS}$	$\hat{\theta}_{ML}$	θ_{L^c}	θ_{U^c}	$Md\hat{T}SF_{ML}$	$\hat{R}(t)_{ML}$	$\hat{h}(t)_{ML}$
$S_{9:23}$	0.0311	0.0206	0.0084	0.0328	8.3512	5.0000	0.6139	9.3862	96.5497	0.1540	0.0497
$S_{9:23}$	0.0356	0.0448	0.0229	0.0667	7.0001	10.1288	0.6727	19.5849	59.8729	0.4150	0.0269
$S_{12:23}$	0.0378	0.0239	0.0117	0.0361	7.0044	5.0000	0.8872	9.1128	83.1250	0.3667	0.0298
$S_{12:23}$	0.0341	0.0376	0.0207	0.0546	7.0001	7.9278	1.3387	14.5168	64.7733	0.4481	0.0250
$S_{18:23}$	0.0384	0.0345	0.0219	0.0472	7.0000	7.8575	2.0622	13.6527	70.3525	0.3530	0.0307
$S_{18:23}$	0.0356	0.0359	0.0224	0.0493	6.9995	7.3285	1.9028	12.7542	65.7969	0.4150	0.0269
$S_{23:23}$	0.0374	0.0358	0.0237	0.0480	6.9929	7.3241	2.2388	12.4094	65.8346	0.3750	0.0292

TABLE 6

PT-II CBR samples under different censoring scheme ($S_{n:m}$) for fixed $n = 31, p = 0.5$

Scheme	i	1	2	3	4	5	6	7	8	9	10	11	12	13
$S_{31:9}$	$r_{i:9:31}$	2	2	4	4	4	4	0	0	2				
	$x_{i:9:31}$	18.83	23.03	24.321	26.77	31.11	34.76	37.09	39.58	44.045				
$S_{31:9}$	$r_{i:9:31}$	0	0	0	0	0	0	0	0	23				
	$x_{i:9:31}$	18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52				
$S_{31:12}$	$r_{i:12:31}$	1	1	1	1	1	0	0	1	1	4	4	4	
	$x_{i:12:31}$	18.83	21.657	23.23	24.321	25.52	26.69	26.77	26.78	27.67	31.11	34.76	37.09	
$S_{31:12}$	$r_{i:12:31}$	0	0	0	0	0	0	0	0	0	0	0	19	
	$x_{i:12:31}$	18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77	
$S_{31:18}$	$r_{i:18:31}$	1	0	0	0	0	0	0	0	0	0	0	0	0
	$x_{i:18:31}$	18.83	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77	26.78	27.05
		0	0	4	4	4								
		27.67	29.9	31.11	34.76	37.09								
$S_{31:18}$	$r_{i:18:31}$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$x_{i:18:31}$	18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77	26.78
		0	0	0	0	13								
		27.05	27.67	29.9	31.11	33.2								
$S_{31:18}$	$r_{i:18:31}$	0	0	0	0	0	0	0	0	0	0	0	0	0
	$x_{i:18:31}$	18.83	20.8	21.657	23.03	23.23	24.05	24.321	25.5	25.52	25.8	26.69	26.77	26.78
		0	0	0	0	0	0	0	0	0	0	0	0	0
		27.05	27.67	29.9	31.11	33.2	33.73	33.76	33.89	34.76	35.75	35.91	36.98	37.08
		0	0	0	0	0								
		37.09	39.58	44.045	45.29	45.381								

TABLE 7

The LSEs, $\hat{\lambda}_{LS}$, $\hat{\theta}_{LS}$ and MLEs $\hat{\lambda}_{ML}$, $\hat{\theta}_{ML}$, $\hat{MdTSF}$, $\hat{R}(t)$, $\hat{h}(t)$ and 95% confidence intervals (CI) for λ and θ with various censoring schemes.

Scheme			95%CI ofλ				95%CI of θ			t=30	
$S_{m:n}$	$\hat{\lambda}_{LS}$	$\hat{\lambda}_{ML}$	λ_{L^c}	λ_{U^c}	$\hat{\theta}_{LS}$	$\hat{\theta}_{ML}$	θ_{L^c}	θ_{U^c}	$Md\hat{T}SF_{ML}$	$\hat{R}(t)_{ML}$	$\hat{h}(t)_{ML}$
$S_{9:31}$	0.09994	0.08981	0.04760	0.13202	20.00000	24.03873	0.00120	52.93034	39.48511	0.60140	0.06093
$S_{9:31}$	0.10238	0.19047	0.09493	0.28602	20.00000	23.05342	0.00320	53.75884	28.74743	0.57391	0.06485
$S_{12:31}$	0.10236	0.10973	0.06178	0.15769	21.40992	31.14815	0.00450	71.28121	34.67729	0.59908	0.06261
$S_{12:31}$	0.09811	0.19632	0.11248	0.28016	19.91707	49.00043	0.00560	59.30896	28.49229	0.62059	0.05813
$S_{18:31}$	0.12844	0.13788	0.08712	0.18863	32.50901	51.60121	0.00100	68.15195	31.26041	0.46279	0.09264
$S_{18:31}$	0.11494	0.14517	0.09183	0.19852	22.56790	57.50334	0.00030	76.05151	30.43498	0.47997	0.08143
$S_{31:31}$	0.12666	0.16703	0.12067	0.21339	19.99992	97.19058	0.00018	103.04340	29.59390	0.33226	0.10280

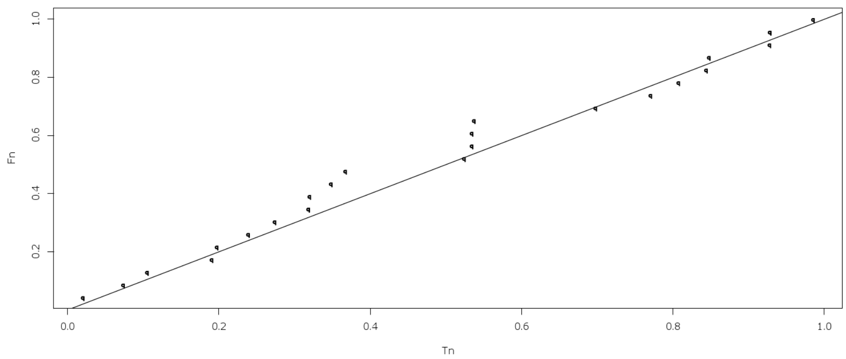

Figure 4 – PP plot for endurance of deep groove ball bearings data.

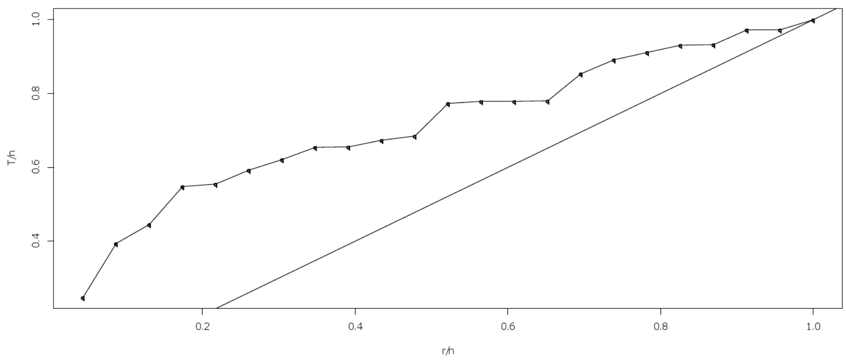

Figure 5 – TTT plot for endurance of deep groove ball bearings data.

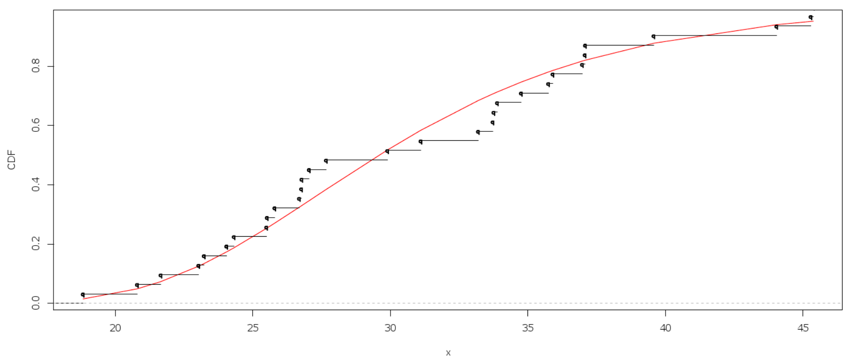

Figure 6 – CDF plots for glass airplane window data.

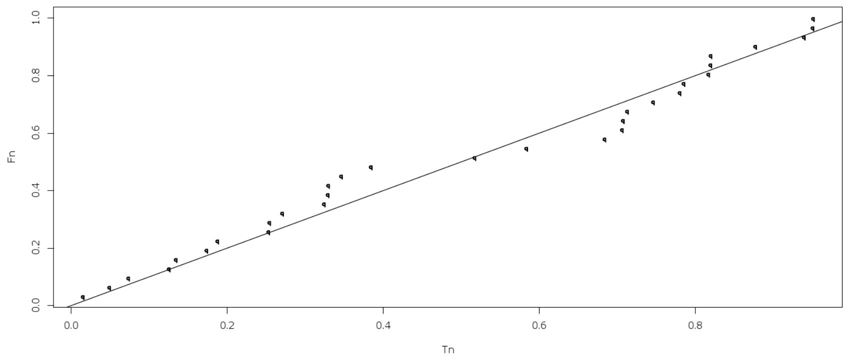

Figure 7 – PP plot for glass airplane window data.

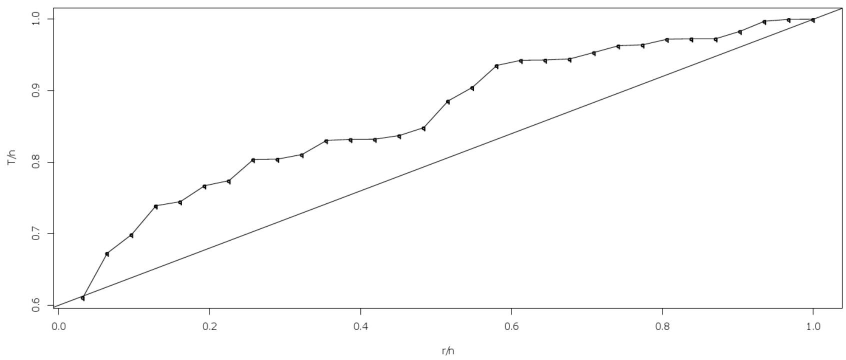

Figure 8 – TTT plot for glass airplane window data.

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Summary

In this paper, a poissoin-exponential distribution(PED) is considered as a lifetime model. Its statistical characteristics and important distributional properties are discussed by Louzada-Neto et al. [13]. The method of Maximum likelihood estimation and least square estimation of parameters involved along with reliability and failure rate functions is also studied here. In view of cost and time constraints, Progressive type-II censored data with binomial removals (PT-II CBRs) have been used. Finally, two real data examples are given to show the practical applications of the paper.