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THE WEAK PARETO LAW AND REGULAR VARIATION IN THE TAILS (*)

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1. INTRODUCTION AND SUMMARY

The strong Pareto law requires that for a distribution function $F_{(x_0,\alpha)}(x)$,

$$\frac{x^{-\alpha}}{1-F(x)}=1 \qquad (x \geq x_0 > 1, \alpha > 0). \tag{1}$$

It was first suggested by Pareto (1896) as a "universal law" for income distributions and is being discussed in e.g. Bhattacharya (1963) or Arnold et al. (1986); it immediately leads to the Pareto distributions $F(x)=1-(x_0/x)^{\alpha}$. The weak Pareto law by Mandelbrot (1960) only requires that

$$\lim_{x\to\infty}\frac{x^{-\alpha}}{1-F(x)}=1. \tag{2}$$

Almost all popular income distributions obey the weak Pareto law.

Other versions of the weak Pareto law were introduced by Kakwani (1980) and Esteban (1986). Below we establish relationship among these versions of the weak Pareto law and discuss its implications for regular variation of tail probabilities.

2. VARIOUS VERSIONS OF THE WEAK PARETO LAW

By definition, a function $f:\mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at infinity with index ρ (in short: $f \in RV_\rho$) if

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$$\lim_{t\to\infty} \frac{f(tx)}{f(t)} = x^{\rho}. \tag{3}$$

Trivially, $x^{-\alpha} \in RV_{-\alpha}$. Assuming that the Mandelbrot weak Pareto law (2) holds, we have

$$\lim_{t\to\infty} \frac{(tx)^{-\alpha}}{1-F(tx)} \cdot \frac{1-F(t)}{t^{-\alpha}} = \lim_{t\to\infty} \frac{x^{-\alpha}}{\frac{1-F(tx)}{1-F(t)}} = 1. \tag{4}$$

Therefore,

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}, \text{ and } 1-F(x) \in RV_{-\alpha}.$$

A distribution which obeys the weak Pareto Law is thus regularly varying at infinity.

However, it need not hold, as is sometimes claimed (e.g. Merkies and Steyn, 1993, Theorem 1) that $1-F(x) \in RV_{-\alpha}$ implies the Mandelbrot weak Pareto law. Take

$$1-F(x) = \frac{x^{-\alpha}}{\ln(x)}. \tag{5}$$

Distributions of this type are characterized by an asymptotically constant slope in the Pareto diagram. Then $x^{-\alpha}/\ln(x) \in RV_{-\alpha}$, but

$$\frac{x^{-\alpha}}{1-F(x)} = \ln(x) \to \infty \text{ as } x \to \infty. \tag{6}$$

Another generalization of the Pareto law, first explored by Kakwani (1980), is

$$\lim_{t\to\infty} \frac{x \cdot f(x)}{1-F(x)} = \alpha > 0. \tag{7}$$

We refer to this as the Kakwani weak Pareto law (KWPL). From Karamata's (1930) theorem, the relationship (7) is equivalent to $f' \in RV_{-\alpha-1}$. See also Bingham et al. (1987).

Yet another version of the weak Pareto law is the requirement, discussed by Esteban (1986), that

$$\lim_{x\to\infty}\frac{f(x)+x\cdot f'(x)}{-f(x)}=\alpha>0. \tag{8}$$

We call this the Esteban weak Pareto law (EWPL). Contrary to what is sometime claimed (see e.g. Merkies and Steyn 1993), it is not weaker than the Kakwani weak Pareto law. The reason is that, if the limit in (8) exists, it must be equal to the limit in (7), as (8) is obtained from (7) by taking derivatives in the numerator and denominator. However, the limit need not exist.

If it exists, one can again invoke Karamata's theorem to show that then $f'(x)\in RV_{-\alpha-2}$. We therefore have the following chain of implications:

$$\begin{array}{lll} & \Rightarrow & MWPL \\ PL & & \\ & \Rightarrow & EWPL \quad \Leftrightarrow f'\in RV_{-\alpha-2} \Rightarrow KWPL \Leftrightarrow 1-F\in RV_{-\alpha} \Leftrightarrow f\in RV_{-\alpha-1} \end{array}$$

3. SOME EXAMPLES

For each of the implications above, we give an example of an economic income distribution which satisfies the weaker law but not the stronger one.

(i) $MWPL \not\Rightarrow PL$:
Take the Lomax-distribution, where

$$1-F(x)=\left[1+\left(\frac{x-x_0}{\sigma}\right)\right]^{-\alpha} \qquad (x\geq x_0,\alpha>0)$$

or the log-logistic distribution, where

$$1-F(x)=\left[1+\left(\frac{x-x_0}{\sigma}\right)^{\frac{1}{\gamma}}\right]^{-1} \qquad (x\geq x_0,\gamma>0).$$

or the Dagum model, where

$$F(x)=\alpha+\frac{1-\alpha}{(1+\lambda x^{-\delta})^{\beta}} \qquad (\lambda>0)$$

(see e.g. Dagum 1977, 1983, 2001). It is easily seen that all distributions obey the Mandelbrot weak Pareto law, but not the Pareto law.

(ii) $EWPL \nRightarrow PL$:

Take the log-Pareto distribution discussed in Ziebach (2000), where

$$f(x) = \frac{k(\alpha \ln x + \beta)}{x^{\alpha+1}(\ln x)^{\beta+1}} \qquad (x \geq x_0 > 0, \alpha > 0, \beta \geq -\alpha \ln x_0) \tag{9}$$

And where $k = x_0^{\alpha}(\ln x_0)^{\beta}$. It is straightforwardly checked that

$$\lim_{x\to\infty}\left(1 + \frac{x \cdot f'(x)}{f(x)}\right) = -\alpha,$$

so the distributions obey the Esteban weak Pareto law. However, from

$$\lim_{x\to\infty} \frac{x^{-\alpha}}{1-F(x)} = \lim_{x\to\infty} \frac{x^{-\alpha}}{\dfrac{x_0^{\alpha}(\ln x_0)^{\beta}}{x^{\alpha}(\ln x)^{\beta}}} = \lim_{x\to\infty} k(\ln x)^{\beta} = \infty, \tag{10}$$

it is also obvious that it does not obey the Mandelbrot weak Pareto law, and therefore, a fortiori, the Pareto law.

(iii) $KWPL \nRightarrow EWPL$

Take the example from Merkies and Steyn (1993) where $F(x) = 1 - \exp(-\varphi(x))$ with $\varphi(x) > 0$ and nondescending, $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Setting

$$\varphi'(x) = \frac{\alpha}{x} + \frac{1+\sin(x)}{x^2}, \tag{11}$$

It is straightforwardly checked that

$$\frac{x \cdot f(x)}{1-F(x)} = x \cdot \varphi'(x) = \alpha + \frac{1+\sin(x)}{x} \to \alpha \tag{12}$$

as $x \to \infty$, so the distribution follows the Kakwani weak Pareto law. However, the limit in (8) does not exist. Therefore, the distribution does not obey the Esteban weak Pareto law.

(iv) $EWPL \nRightarrow MWPL$

Take once more the log-Pareto distribution from (9). We have already shown above that it obeys the Esteban weak Pareto law, but not the Mandelbrot weak Pareto law.

(v) $KWPL \not\Rightarrow MWPL$

This can likewise be shown by invoking the log-Pareto distribution from (9), which does not follow the Mandelbrot weak Pareto law.

From

$$\lim_{x\to\infty}\frac{x\cdot f(x)}{1-F(x)}=\lim_{x\to\infty}\frac{\alpha\ln x+\beta}{\ln x}=\alpha, \tag{13}$$

it is however obvious that it obeys the Kakwani weak Pareto law (this also follows from our general chain of implication above).

(vi) $MWPL \not\Rightarrow KWPL$

Let, for large x

$$1-F(x)=(x+\sin x)x^{-(1+\alpha)} \qquad (\alpha\geq 1). \tag{14}$$

Then

$$\frac{x^{-\alpha}}{1-F(x)} \quad \to \quad 1,$$

so the Mandelbrot weak Pareto law obtains. However,

$$\lim_{x\to\infty}\frac{x\cdot f(x)}{1-F(x)}=\lim_{x\to\infty}\left((1+\alpha)+\frac{1+\cos x}{1+\dfrac{\sin x}{x}}\right) \tag{15}$$

does not exist, so the Kakwani weak Pareto law does not hold.

(vii) $MWPL \not\Rightarrow EWPL$

Take once more the distribution defined by (14), which obeys the Mandelbrot weak Pareto law. However, it does not obey the Kakwani weak Pareto law and therefore, a fortiori, the Esteban weak Pareto law.

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RIASSUNTO

Legge debole di Pareto e code a variazione regolare

Nel lavoro si mostra come la legge debole di Pareto, utilizzata per spiegare il comportamento nelle code della distribuzione dei redditi, implichi che la probabilità nelle code decada in modo regolare pur non valendo l’implicazione inversa. Si stabiliscono anche le implicazioni tra diverse versioni della legge debole di Pareto.

SUMMARY

The weak Pareto law and regular variation in the tails

We show that the weak Pareto law, as used to characterize the tail behaviour of income distributions, implies regularly varying tail probabilities, but that the reverse implication does not hold. We also establish implications among other versions of the weak Pareto law.