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ON A LESS CUMBERSOME METHOD OF ESTIMATION OF PARAMETERS OF TYPE III GENERALIZED LOGISTIC DISTRIBUTION BY ORDER STATISTICS

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1. Introduction

The simplicity of the logistic distribution and its importance as a growth curve have made it one of the most important statistical models. The shape of the logistic distribution (similar to that of the normal distribution) makes it simpler and also profitable on suitable occasions to choose it as a model instead of the normal distribution. Pearl and Reed (1920, 1924), Schultz (1930) and Oliver (1982) applied the logistic model as a growth model in human populations and in the study of the populations of some biological organisms. Some applications of logistic functions in bioassy problems were discussed by Berkson(1944) and Wilson and Worcester (1943). Other applications and significant developments concerning the logistic distribution can be found in the book by Balakrishnan (1992). Balakrishnan and Leung (1988) defined three types of generalized logistic distributions by compounding logistic distribution with some other well known models and named them as Type I, Type II and Type III generalized logistic distributions. Type III generalized logistic distribution was earlier derived by Gumbel (1944). In this article, our main interest is to deal with estimation problems of Type III generalized logistic distribution.

A random variable X is said to follow a Type III generalized logistic distribution (Type III GLD)with parameters μ, σ and a if its pdf is given by (see, Blakrishnan and Lee 1998)

$$g(x;a,\mu,\sigma)=\frac{1}{\sigma\beta(a,a)}\left[\frac{e^{\left(-\frac{x-\mu}{\sigma}\right)}}{\left[1+e^{\left(-\frac{x-\mu}{\sigma}\right)}\right]^2}\right]^a$$

$$-\infty<x<\infty,-\infty<\mu<\infty,\sigma>0,a>0, \qquad (1)$$

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The standard form of Type III generalized logistic distribution defined in(1) is obtained by putting $\mu = 0$ and $\sigma = 1$ and we may write $g_0(y)$ to denote this pdf. Then we have

$$g_0(y) = \frac{1}{\beta(a,a)} \left[\frac{e^{-y}}{[1+e^{-y}]^2} \right]^a, -\infty < y < \infty, a > 0. \tag{2}$$

For more details relating to the above family of distributions see also, Gumbel (1944) and Davidson (1980).

It is well known that if H(x) is an absolutely continuous cumulative distribution function (cdf) then the function F(x)defined in terms of an incomplete beta integral as

$$F(x) = \frac{1}{\beta(\gamma,\delta)} \int_0^{H(x)} t^{\gamma-1}(1-t)^{\delta-1}dt, \quad \gamma > 0, \delta > 0, \tag{3}$$

is also a cdf. The probability density function (pdf) corresponding to the cdf F(x) is given by

$$f(x) = \frac{1}{\beta(\gamma,\delta)}(H(x))^{\gamma-1}(1-H(x))^{\delta-1}h(x), \quad \gamma > 0, \delta > 0. \tag{4}$$

If we put $H(x) = \frac{1}{1+e^{-\frac{(x-\mu)}{\sigma}}}$ and $h(x) = \frac{1}{\sigma}\frac{e^{-\frac{(x-\mu)}{\sigma}}}{\left[1+e^{-\frac{(x-\mu)}{\sigma}}\right]^2}$ in (4), then the resulting pdf is known as beta-logistic distribution. If further we put $\gamma = \delta = a$ then the resulting distribution is known as Type III GLD. Hence one may call the Type III GLD also as symmetric beta-logistic distribution.

Maximum likelihood method of estimation of the location and scale parameters of a) normal distribution b) Laplace distribution and c) Cauchy distribution are extensively discussed in the available literature. For more details see Johnson et al.(1994, 1995). Likewise based on a random sample of size n drawn from Type III GLD with pdf given in (1), the ML equations $\frac{\partial l}{\partial a} = 0, \frac{\partial l}{\partial \mu} = 0$ and $\frac{\partial l}{\partial \sigma} = 0$ are respectively given by

$$n\left[2\psi(a) - \psi(2a)\right] + \sum_{i=1}^{n}\left[\frac{x_i-\mu}{\sigma}\right] + 2\sum_{i=1}^{n} log\left(1 + exp\left(\frac{x_i-\mu}{\sigma}\right)\right) = 0 \tag{5}$$

$$a\left[1 + 2\sum_{i=1}^{n}\frac{exp\left(\frac{x_i-\mu}{\sigma}\right)}{1+exp\left(\frac{x_i-\mu}{\sigma}\right)}\right] = 0 \tag{6}$$

$$n\sigma + 2a\mu\sum_{i=1}^{n}\frac{exp\left(\frac{x_i-\mu}{\sigma}\right)}{1+exp\left(\frac{x_i-\mu}{\sigma}\right)} + a\sum_{i=1}^{n}(x_i-\mu) = 0. \tag{7}$$

Then one may, use Newton-Raphson method of solving for obtaining the MLE's $\tilde{a}$, $\tilde{\mu}$ and $\tilde{\sigma}$ of a, μ and σ involved in (1).

One may observe several practical situations in which Type III GLD becomes the most appropriate model. For example: we consider a real data set originally

TABLE 1
Parameter estimates and K-S statistics for single fibres data set of 10 mm

Distribution	Normal	Laplace	Cauchy	Type III GLD
Maximum				$\tilde{\alpha}$=4.680850
Likelihood	$\tilde{\mu}$=3.058830	$\tilde{\mu}$=2.977	$\tilde{\mu}$=2.98711	$\tilde{\mu}$=3.047060
estimators	$\tilde{\sigma}$=0.616034	$\tilde{\sigma}$=0.502833	$\tilde{\sigma}$=0.403347	$\tilde{\sigma}$=0.892756
K-S Statistics	0.098459	0.120187	0.12442	0.09700

reported by Badar and Priest (1982). For illustrative purpose we reproduce below the above mentioned data on single fibers 10 mm in gauge lengths with sample size 63.

Data Set : 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

For the above data using MLE method we have fitted each of the following distributions: 1) normal distribution 2) Laplace distribution 3) Cauchy distribution and 4) Type III GLD. The estimated parameters of the fitted distributions and the Kolmogorov Smirnov goodness-of-fit statistic values are given in the following table 1.

From the table we observe that for the above data, the Type III generalized logistic distribution is the most appropriate model. Thus we conclude that the usual symmetric models such as normal, Laplace, Cauchy etc become less effective models to study certain real life situations. But however Type III GLD enters in such situations as the most appropriate symmetric model.

Maximum likelihood method of estimation ends up with some limitations. When the sample size is small in many cases MLE is not even unbiased. Though one may obtain the asymptotic variance of the MLE's, the exact variance for the small sample cases are generally not explicitly available. Thus for small sample situation Lloyd's (1952) best linear unbiased estimation of the location and scale parameters of a distribution by order statistics is considered as a very good method of estimation. However when the sample size is large or moderately large the requirement of obtaining the means, variances and co-variances of the order statistics of an equivalent sample size arising from the standard form of the given distribution makes the Lloyd's method of estimation very difficult. In such situations Thomas and Sreekumar (2004, 2008) proposed a method of estimation by U-statistics using best linear unbiased estimators of the location and scale parameters of a distribution with an appropriate small sample size as kernels. Some works associated with U-statistics of this nature are seen in the available literature. For more details see, Sreekumar and Thomas (2006, 2007, 2008) and Thomas and Baiju (2012).

Hence the main objective of this paper is to determine the best linear unbiased estimators based on small sample sizes of the location and scale parameters of Type III GLD for some known values of shape parameter a, and to use them to generate appropriate U-statistics for estimating those parameters for any sample size. We have further illustrated by a real life example about the surprising nature of these U-statistics in terms of their performances when compared with the corresponding maximum likelihood estimators which are not available explicitly.

2. Moments of Type III Generalized Logistic Distribution

In this section we first proves the following theorem which establishes the existence of all moments of Type III GLD.

Theorem 1. *All moments of integer orders of Type III GLD defined by (2) exist.*

Proof. The pdf of the standard form of Type III GLD is defined in (2). If Y is a random variable with pdf $g_0(y)$ then

$$\begin{aligned} E(Y^k) &= \int_{-\infty}^{\infty} y^k g_0(y)dy \\ &= \int_{-\infty}^{0} y^k g_0(y)dy + \int_{0}^{\infty} y^k g_0(y)dy \\ &= I_1 + I_2. \end{aligned}$$

where $I_1 = \int_{-\infty}^{0} y^k g_0(y)dy$ and $I_2 = \int_{0}^{\infty} y^k g_0(y)dy$.
Clearly we have

$$\begin{aligned} \frac{e^{-y}}{(1+e^{-y})^2} &\leq e^{-y}, \quad y > 0, a > 0, \\ \left[\frac{1}{(1+e^{-y})^2}\right]^a &\leq 1, \quad y > 0, a > 0. \end{aligned}$$

Therefore

$$\int_{0}^{\infty} y^k \left[\frac{e^{-y}}{[1+e^{-y}]^2}\right]^a dy \leq \int_{0}^{\infty} y^k e^{-ay} dy. \tag{8}$$

Since $y^k e^{-ay}$ is integrable for any positive integer k and $a > 0$ we assert that the left side integral of (8) is finite. This proves that $I_2 < \infty$. The proof of the existence of the integral I_1 is similar and hence is omitted. This proves the theorem.

It may be noted that if the parent distribution admits the moments of order k, then the moments of all order statistics of the same order exist as such.

TABLE 2

Expected values $\alpha_{r:m}$ of order statistics arising from standard Type III GLD for m=2(1)9 and a=1.5(0.5)2.5 and for a=4.68085.

m	r	$\alpha_{r:m}$ for $r = \left[\frac{m+1}{2}\right] + 1$ to m			
		a=1.5	a=2	a=2.5	a=4.68085
2	2	0.759449	0.633333	0.553590	0.387458
3	3	1.139174	0.950000	0.830385	0.581187
4	3	0.385723	0.324459	0.285133	0.201613
	4	1.390324	1.158514	1.012136	0.707712
5	4	0.642872	1.312951	0.475222	0.336022
	5	1.577187	0.540076	1.146365	0.800634
6	4	0.258636	0.218210	0.192110	0.136297
	5	0.834990	0.702042	0.616778	0.435885
	6	1.725626	1.435133	1.252282	0.873584
7	5	0.452613	0.381867	0.336193	0.238519
	6	0.987941	0.830112	0.729012	0.514831
	7	1.848574	1.535969	1.339494	0.933376
8	5	0.194554	0.164393	0.144861	0.102947
	6	0.607448	0.512352	0.450992	0.319863
	7	1.114772	0.966032	0.821686	0.579821
	8	1.953402	1.621675	1.413464	0.983888
9	6	0.350197	0.295907	0.260750	0.185304
	7	0.736074	0.620575	0.546113	0.387142
	8	1.222972	1.026163	0.900421	0.634872
	9	2.044706	1.696113	1.477596	1.027513

Other values of $\alpha_{r:m}$ are obtained by the formula $\alpha_{r:m} = -\alpha_{m-r+1:m}$ and $\alpha_{p+1:2p+1} = 0$ for p=1,2,3,4.

Let $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$ be the order statistics of a random sample of size m arising from Type III GLD defined in (1). Define $Y_{r:m} = \frac{X_{r:m}-\mu}{\sigma}$, $r = 1, 2, \ldots, m$. Then $(Y_{1:m}, Y_{2:m}, \cdots, Y_{m:m})$ are distributed as the order statistics of a random sample of size m drawn from the Type III GLD(a,0,1) with pdf given by (2).Let

$$E(Y_{r:m}) = \alpha_{r:m}, \quad 1 \leq r \leq m,$$

$$Var(Y_{r:m}) = v_{r,r:m} \quad 1 \leq r \leq m,$$

$$Cov(Y_{r:m}, Y_{s:m}) = v_{r,s:m}, 2 \leq r < s \leq m.$$

Balakrishnan and Lee (1998) have presented a reparametrized model of Type III GLD, which has a standard normal distribution as the limiting distribution as $a \to \infty$. They studied the order statistics and moments from this reparametrized distribution and tabulated the means, variances and co-variances of order statistics for only one sample size m=20 and for shape parameter values a=0.5(0.5)3(1)6(2)12. In this work we have independently evaluated the means of all order statistics arising from Type III GLD defined in (2) for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085 by mathcad software and are presented in table 2. The variances and co-variances of all order statistics of a sample of size m arising from (2) for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085 also have been computed using mathcad software and are given in table 3. In table 2 and table 3 we have included entries for $a = 4.68085$, as we require it to deal with the case of illustrating the proposed method of estimation in this paper for a real life example.

TABLE 3

Variances and covariances $v_{r,s:n}$ of order statistics arising from standard Type III GLD for $1 \leq r \leq s \leq n$, n=2(1)9 and a=1.5(0.5) 2.5, 4.68085

n	r	s	a=1.5	a=2	a=2.5	a=4.68085
2	1	1	1.29284	0.88876	0.67425	0.32601
	1	2	0.57676	0.40111	0.30646	0.15012
3	1	1	1.12480	0.76277	0.57368	0.27277
	1	2	0.49502	0.34499	0.26393	0.12962
	1	3	0.30769	0.21253	0.16168	0.07853
	2	2	0.76377	0.53907	0.41571	0.20730
4	1	1	1.04244	0.69851	0.52130	0.24411
	1	2	0.45375	0.31408	0.23927	0.11656
	1	3	0.28651	0.19928	0.15225	0.07457
	1	4	0.20271	0.13882	0.10503	0.05049
	2	2	0.61499	0.43380	0.33441	0.16666
	2	3	0.39856	0.28189	0.21764	0.10877
5	1	1	0.99260	0.65859	0.48833	0.22571
	1	2	0.42838	0.29422	0.22304	0.10764
	1	3	0.27179	0.18855	0.14382	0.07020
	1	4	0.19614	0.13572	0.10335	0.05029
	1	5	0.14886	0.10114	0.07613	0.03622
	2	2	0.54342	0.38118	0.29286	0.14503
	2	3	0.35143	0.24867	0.19205	0.09603
	2	4	0.25642	0.18082	0.13935	0.06940
	3	3	0.47437	0.33727	0.26124	0.13134
6	1	1	0.95878	0.63095	0.46528	0.21268
	1	2	0.41098	0.28019	0.21142	0.10111
	1	3	0.26115	0.18031	0.13711	0.06653
	1	4	0.18997	0.13149	0.10014	0.04873
	1	5	0.14709	0.10118	0.07677	0.03709
	1	6	0.11675	0.07873	0.05898	0.02780
	2	2	0.50067	0.34891	0.26700	0.13125
	2	3	0.32303	0.22778	0.17555	0.08744
	2	4	0.23701	0.16745	0.12920	0.06448
	2	5	0.18453	0.12953	0.09954	0.04931
	3	3	0.40748	0.28966	0.22434	0.11276
	3	4	0.30218	0.21503	0.16663	0.08385
7	1	1	0.93411	0.61047	0.44807	0.20284
	1	2	0.39817	0.26965	0.20260	0.09609
	1	3	0.25308	0.17383	0.13175	0.06352
	1	4	0.18483	0.12761	0.09702	0.04705
	1	5	0.14435	0.09955	0.07563	0.03663
	1	6	0.11675	0.07986	0.06037	0.02897
	1	7	0.09559	0.06404	0.04777	0.02234
	2	2	0.47192	0.32679	0.24909	0.12155
	2	3	0.30381	0.21325	0.16389	0.08121
	2	4	0.22346	0.15760	0.12147	0.06050
	2	5	0.17533	0.12346	0.09507	0.04727
	2	6	0.14225	0.09935	0.07612	0.03748
	3	3	0.36782	0.26070	0.20156	0.10099
	3	4	0.27279	0.19414	0.01505	0.07572
	3	5	0.21518	0.15286	0.11834	0.05942
	4	4	0.34329	0.24494	0.19012	0.09595

Continued...

n	r	s	a=1.5	a=2	a=2.5	a=4.68085
8	1	1	0.91520	0.59455	0.43462	0.19507
	1	2	0.38829	0.26138	0.19563	0.09207
	1	3	0.24672	0.16860	0.12737	0.06102
	1	4	0.18058	0.12424	0.09424	0.04549
	1	5	0.14169	0.09763	0.07412	0.03584
	1	6	0.11560	0.07939	0.06015	0.02898
	1	7	0.09633	0.06554	0.04938	0.02354
	1	8	0.08068	0.05374	0.03994	0.01854
	2	2	0.45112	0.31052	0.23581	0.11426
	2	3	0.28983	0.20244	0.15512	0.07643
	2	4	0.21342	0.15004	0.11542	0.05728
	2	5	0.16810	0.11833	0.09110	0.04527
	2	6	0.13751	0.09646	0.07411	0.03669
	2	7	0.11482	0.07979	0.06095	0.02984
	3	3	0.34132	0.24097	0.18587	0.09272
	3	4	0.25304	0.17972	0.13912	0.06986
	3	5	0.20019	0.14232	0.11023	0.05540
	3	6	0.16427	0.11359	0.08992	0.04501
	4	4	0.30544	0.21791	0.16914	0.08535
	4	5	0.24298	0.17344	0.13465	0.06799
9	1	1	0.90017	0.58176	0.42374	0.18875
	1	2	0.38039	0.25468	0.18995	0.08878
	1	3	0.24157	0.16428	0.12372	0.05891
	1	4	0.17703	0.12133	0.09181	0.04411
	1	5	0.13929	0.09577	0.07261	0.03500
	1	6	0.11419	0.07846	0.05946	0.02865
	1	7	0.09596	0.06563	0.04960	0.02377
	1	8	0.08173	0.05533	0.04155	0.01968
	1	9	0.06967	0.04616	0.03419	0.01577
	2	2	0.43526	0.29796	0.22550	0.10856
	2	3	0.27914	0.19404	0.14824	0.07264
	2	4	0.20564	0.14404	0.11056	0.05463
	2	5	0.16235	0.11406	0.08770	0.04348
	2	6	0.13340	0.09365	0.07197	0.03565
	2	7	0.11230	0.07846	0.06013	0.02963
	2	8	0.09577	0.06624	0.05044	0.02456
	3	3	0.32222	0.22653	0.17428	0.08653
	3	4	0.23876	0.16907	0.13064	0.06538
	3	5	0.18919	0.13434	0.10397	0.05219
	3	6	0.15586	0.11057	0.08552	0.04288
	3	7	0.13146	0.09281	0.07157	0.03570
	4	4	0.28025	0.19958	0.15474	0.07794
	4	5	0.22306	0.15923	0.12363	0.06243
	4	6	0.15434	0.13143	0.10197	0.05142
	5	5	0.26880	0.19218	0.14935	0.07555

3. Best Linear Unbiased Estimation of Location and Scale Parameters of Type III GLD Using Order Statistics

In the available literature, estimation of location and scale parameters of Type III GLD is seen discussed only for a sample of size n=20 assuming that the shape parameter is given (see, Balakerishnan and Lee 1998). Hence we devote this section to estimate the location parameter μ and scale parameter σ of $g(x; a, \mu, \sigma)$ by order statistics for given values of a and for all sample sizes $n \leq 9$.
Let $\underline{\alpha} = (\alpha_{1:m}, \alpha_{2:m}, \cdots, \alpha_{m:m})'$ and $V = ((v_{r,s:m}))$ be the vector of means and dispersion matrix of the vector of order statistics of a random sample of size m drawn from $g_0(y)$. In section 2, we have already tabulated the means involved in $\underline{\alpha}$, variances and co-variances involved in V for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085. Since $g_0(y)$ is symmetric about zero, the Lloyd's (1952) BLUE's of μ and σ involved in (1) and their variances assume a reduced form and are given by

(see David and Nagaraja (2003), P-189, see also Thomas (1990))

$$\hat{\mu_m} = \frac{\underline{1}'V^{-1}\underline{X}}{(\underline{1}'V^{-1}\underline{1})}, \qquad Var(\hat{\mu_m}) = \frac{\sigma^2}{(\underline{1}'V^{-1}\underline{1})} \tag{9}$$

and

$$\hat{\sigma_m} = \frac{\underline{\alpha}'V^{-1}\underline{X}}{(\underline{\alpha}'V^{-1}\underline{\alpha})}, \qquad Var(\hat{\sigma_m}) = \frac{\sigma^2}{(\underline{\alpha}'V^{-1}\underline{\alpha})}. \tag{10}$$

It is clear that BLUE's of μ and σ are linear functions of order statistics $X_{1:m}, X_{2:m}, \ldots, X_{m:m}$. Consequently one can also write $\hat{\mu}_m$ and $\hat{\sigma}_m$ as

$$\hat{\mu}_m = \sum_{r=1}^{m} c_{r:m} X_{r:m}, \quad \hat{\sigma}_m = \sum_{r=1}^{m} d_{r:m} X_{r:m}, \tag{11}$$

where $c_{r:m}$ and $d_{r:m}$, are constants independent of μ and σ such that $c_{r:m} = c_{m-r+1:m}$ and $d_{r:m} = -d_{m-r+1:m}$, $r = 1, 2, \cdots, [n/2]$ where [.] is the usual greatest integer function. We have computed the coefficients $c_{r:m}$ of $X_{r:m}$ in $\hat{\mu}_m$ and the value of $\sigma^{-2}Var(\hat{\mu_m})$ and are given in table 4 for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085. Similarly the values of the coefficients $d_{r:m}$ of $X_{r:m}$ in $\hat{\sigma}_m$ and the value of $\sigma^{-2}Var(\hat{\sigma_m})$ are computed and are given in table 5 for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085.

4. Estimation of the Parameters of Type III GLD Using U-Statistics

The BLUE's of the location and scale parameters of a distribution by order statistics of a random sample of size m requires the evaluation of all means, variances and co-variances of order statistics of an equivalent sample size arising from the standard form of the original distribution. This makes the method unfriendly to applied statisticians. However if one obtain the BLUE's of μ and σ by order statistics for a small or moderate sample size m and use it as kernel of degree m to construct appropriate U-statistics to estimate μ and σ, then these U-statistics would be highly useful as they estimate the parameters explicitly. Moreover these estimators are highly preferred as they possess the optimal properties of BLUE's as well as those of U-statistics. It may be noted that the U-statistics obtained in this method are distributed asymptotically normal and hence those U-statistics can be even used for testing of hypothesis problems on the location or scale parameters involved in Type III GLD for large sample sizes. U-statistics was first introduced by Hoeffding (1948) and is considered as one of the top 20 breakthroughs of twentieth century in statistics (for details see, Sen (1990)). Hence in this section we estimate the parameters μ and σ of Type III GLD using U-statistics based on best linear functions of order statistics as kernels, when the shape parameter a is known.

Let the BLUE of μ as given in (9) be represented as

$$h_1(X_1, X_2, \ldots, X_m) = c_{1:m}X_{1:m} + c_{2:m}X_{2:m} + \ldots + c_{m:m}X_{m:m} \tag{12}$$

and that of σ as given in (10) be represented as

$$h_2(X_1, X_2, \ldots, X_m) = d_{1:m}X_{1:m} + d_{2:m}X_{2:m} + \cdots + d_{m:m}X_{m:m}, \tag{13}$$

TABLE 4

Coefficients of $c_{r:m}$ of order statistics $X_{r:m}$ in the BLUE $\hat{\mu}_m = \sum_{r=1}^{m} c_{r:m} X_{r:m}$ of μ and $\sigma^{-2} Var(\hat{\mu}_m)$

m	r	a=1.5		a=2		a=2.5		a=4.68085	
		$c_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$c_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$c_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$c_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$
2	1	0.500000	0.934802	0.500000	0.644934	0.500000	0.490358	0.500000	0.238068
3	1	0.274250		0.288179		0.296996		0.313960	
	2	0.451500	0.616359	0.423642	0.427210	0.406008	0.325555	0.372080	0.158526
4	1	0.324405		0.192218		0.203028		0.224472	
	2	0.324405	0.458785	0.307782	0.318953	0.296972	0.243434	0.275528	0.118790
5	1	0.123488		0.139722		0.150556		0.172661	
	2	0.240857		0.233445		0.228008		0.216020	
	3	0.271310	0.365116	0.253666	0.254346	0.242871	0.194330	0.222638	0.094969
6	1	0.092351		0.107420		0.117695		0.139179	
	2	0.186029		0.184243		0.182191		0.176336	
	3	0.221620	0.303119	0.208337	0.211457	0.200114	0.161684	0.184485	0.079103
7	1	0.072129		0.085900		0.095473		0.115919	
	2	0.148446		0.150016		0.150083		0.148189	
	3	0.182792		0.173745		0.167905		0.156481	
	4	0.193264	0.259078	0.180680	0.180925	0.173078	0.138418	0.158823	0.067776
8	1	0.058175		0.070720		0.079589		0.098909	
	2	0.121586		0.125165		0.126552		0.127269	
	3	0.153049		0.147322		0.143408		0.135295	
	4	0.167190	0.226192	0.156793	0.158087	0.150451	0.120999	0.138527	0.059285
9	1	0.048096		0.059351		0.067758		0.085979	
	2	0.101700		0.090048		0.108711		0.111157	
	3	0.130051		0.169160		0.124337		0.118798	
	4	0.145166		0.111548		0.132009		0.122364	
	5	0.149975	0.200703	0.139787	0.139747	0.134371	0.107470	0.123404	0.052685

Other values of $c_{r:m}$ are obtained from $c_{r:m} = c_{m-r+1:m}$

TABLE 5

Coefficients of $d_{r:m}$ of order statistics $X_{r:m}$ in $\hat{\sigma}_m = \sum_{r=1}^{m} d_{r:m} X_{r:m}$ and $\sigma^{-2} Var(\hat{\sigma}_m)$

m	r	a=1.5		a=2		a=2.5		a=4.68085	
		$d_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$d_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$d_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$	$d_{r:m}$	$\frac{Var(\hat{\mu}_m)}{\sigma^2}$
2	1	-0.658372	0.620773	-0.789474	0.607870	-0.903196	0.600060	-1.290462	0.585807
3	1	-0.438915		-0.526316		-0.602130		0.313960	
	2	0.000000	0.314829	0.000000	0.304842	0.000000	0.298752	0.372080	0.287523
4	1	-0.324797		-0.393037		-0.452230		-0.653163	
	2	-0.125548	0.211272	-0.137649	0.203520	-0.148289	0.198740	-0.187233	0.189812
5	1	-0.256351		-0.312977		-0.362187		-0.529005	
	2	-0.148843		-0.164723		-0.178453		-0.227546	
	3	0.000000	0.159056	0.000000	0.152768	0.000000	0.148865	0.000000	0.141497
6	1	-0.211068		-0.259663		-0.302027		-0.445659	
	2	-0.146939		-0.164360		-0.179152		-0.230894	
	3	-0.050587	0.127549	-0.054815	0.122280	-0.058720	0.118988	-0.073633	0.112734
7	1	-0.179002		-0.221650		-0.258961		-0.385639	
	2	-0.138274		-0.156219		-0.171267		-0.222948	
	3	-0.071794		-0.078227		-0.084079		-0.105961	
	4	0.000000	0.106469	0.000000	0.101939	0.000000	0.099095	0.000000	0.093662
8	1	-0.155162		-0.193201		-0.226600		-0.340242	
	2	-0.128214		-0.146176		-0.161119		-0.211662	
	3	-0.080165		-0.087923		-0.094861		-0.120283	
	4	-0.027142	0.091372	-0.029307	0.087401	-0.031335	0.084902	-0.039234	0.080092
9	1	-0.136777		-0.171521		-0.201396		-0.304676	
	2	-0.118466		-0.143400		-0.150820		-0.199707	
	3	-0.082499		-0.072875		-0.098671		-0.126051	
	4	-0.042050		-0.056454		-0.048827		-0.061266	
	5	0.000000	0.080025	0.000000	0.076683	0.000000	0.074261	0.000000	0.069956

Other values of $d_{r:m}$ are obtained from $d_{r:m} = -d_{m-r+1:m}$

where $c_{1:m}, c_{2:m}, \ldots, c_{m:m}$ and $d_{1:m}, d_{2:m}, \ldots, d_{m:m}$ are constants. Now from Thomas and Sreekumar(2008) for a random sample of size n $(n > m)$ drawn from (1) the U-statistic for estimating μ using the kernel(12)is given by

$$U_{1:n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} c_{i+1:m} \right] X_{r:n} \tag{14}$$

and the U-statistic for estimating σ using the kernel (13) is given by

$$U_{2:n}^{(m)} = \frac{1}{\binom{n}{m}} \sum_{r=1}^{n} \left[\sum_{i=0}^{m-1} \binom{n-r}{m-1-i} \binom{r-1}{i} d_{i+1:m} \right] X_{r:n}, \tag{15}$$

where we define $\binom{r-1}{i} = 0$ for $i \geq r$ and $\binom{n-r}{m-1-i} = 0$ for $n - r < m - 1 - i$.
If we write
$\xi_c^{(m)} = Cov[h_1(X_1, \cdots, X_c, X_{c+1}, \cdots, X_m), h_1(X_1, \cdots, X_c, X_{m+1}, \cdots, X_{2m-c})]$,
as the co-variance between two $h_1(.)$ functions with exactly c common observations and
$\psi_c^{(m)}$=Cov$[h_2(X_1, \cdots, X_c, X_{c+1}, \cdots, X_m), h_2(X_1, \cdots, X_c, X_{m+1}, \cdots, X_{2m-c})]$,
as the co-variance between two $h_2(.)$ functions with exactly c common observations for $c = 1, 2, \cdots, m$, then the variances of $U_{1:n}^{(m)}$ and $U_{2:n}^{(m)}$ are given by (See, Hoeffding, 1948)

$$Var[U_{1:n}^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \xi_c^{(m)}, \tag{16}$$

$$Var[U_{2:n}^{(m)}] = \frac{1}{\binom{n}{m}} \sum_{c=1}^{m} \binom{m}{c} \binom{n-m}{m-c} \psi_c^{(m)}. \tag{17}$$

Clearly

$$\xi_m^{(m)} = Var[h_1(X_1, X_2, \cdots, X_m)] \tag{18}$$

and

$$\psi_m^{(m)} = Var[h_2(X_1, X_2, \cdots, X_m)]. \tag{19}$$

It may be noted that $\xi_m^{(m)}$ and $\psi_m^{(m)}$ can be obtained from tables 4 and 5 as $Var(\hat{\mu})$ and $Var(\hat{\sigma})$ respectively for m=2(1)9, a=1.5(0.5)2.5 and for a=4.68085. Now we evaluate the values of $\xi_c^{(m)}$ and $\psi_c^{(m)}$ for $c = 1.2, \ldots, m-1$, using the methodology developed by Thomas and Sreekumar (2008) as explained in the following steps.

Define the vectors b_{m+k} for $k = 1, 2, \cdots, m-1$ as

$$b'_{m+k} = \left[\frac{\sum_{i=0}^{m-1} \binom{m+k-1}{m-1-i}\binom{o}{i} c_{i+1:m}}{\binom{m+k}{m}}, \frac{\sum_{i=0}^{m-1} \binom{m+k-2}{m-1-i}\binom{1}{i} c_{i+1:m}}{\binom{m+k}{m}}, \right.$$

$$\left. \cdots, \frac{\sum_{i=0}^{m-1} \binom{0}{m-1-i}\binom{m+k-1}{i} c_{i+1:m}}{\binom{m+k}{m}} \right] \tag{20}$$

and define $w_k = \binom{m+k}{m}(b'_{m+k} V_{m+k} b_{m+k})\sigma^2 - \xi_m^{(m)}, k = 1, 2, \cdots, m-1$ where V_{m+k} is the variance co-variance matrix of the vector of order statistics of a random sample of size m+k arising from $g_0(y)$ and $\xi_m^{(m)}$ is defined in (18). Define the matrix

$$H = \begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{m}{m-1}\binom{1}{1} \\ 0 & 0 & \cdots & \binom{m}{m-2}\binom{2}{2} & \binom{m}{m-1}\binom{2}{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \binom{m}{1}\binom{m-1}{m-1} & \binom{m}{2}\binom{m-1}{m-2} & \cdots & \binom{m}{m-2}\binom{m-1}{2} & \binom{m}{m-1}\binom{m-1}{1} \end{bmatrix} \times \begin{bmatrix} \xi_1^{(m)} \\ \xi_2^{(m)} \\ \vdots \\ \xi_{m-1}^{(m)} \end{bmatrix} \tag{21}$$

and the vector $w = (w_1, w_2, \cdots, w_{m-1})'$.
Then the components $\xi_c^{(m)}$, $c = 1, 2, \cdots, m-1$ involved in (16)are solved from the following equations

$$\left(\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)}\right)' = H^{-1}W. \tag{22}$$

Similarly, the values of $\psi_c^{(m)}$, $c = 1, 2, \cdots, m-1$ can be obtained as

$$\left(\psi_1^{(m)}, \psi_2^{(m)}, \ldots, \psi_{m-1}^{(m)}\right)' = H^{-1}Z, \tag{23}$$

where $Z' = (z_1, z_2, \ldots, z_{m-1})$ with $z_k = \binom{m+k}{m}(g'_{m+k} V_{m+k} g_{m+k})\sigma^2 - \psi_m^{(m)}$ and g_{m+k} is obtained from (20) just by replacing each $c_{i:m}$ by $d_{i:m}$, $i = 1, 2, \ldots, m$. Once we obtain the values of $\xi_c^{(m)}, \psi_c^{(m)}$, $c = 1, 2, \ldots, m-1$ from (22) and (23) respectively, then the exact variances of U- statistics for estimating μ and σ based on any sample of size n can be obtained using (16) and (17) without any further direct evaluation of moments of order statistics.

The main advantage of this method is that if one uses the BLUE based on a sample of size m as the kernel, then the evaluation of variances and co-variances of order statistics of samples of sizes up to 2m-1 arising from (2) alone are necessary to obtain the explicit expressions for the variances of the U-statistics $U_{1:n}^{(m)}$ and

TABLE 6
Value of $\xi_c^{(m)}$ *for m=2(1)5 and* $c = 1, 2, \cdots, m$.

m	c	a=1.5	a=2	a=2.5	a=4.68085
2	1	0.467401	0.322469	0.245179	0.119034
	2	0.934802	0.644934	0.490358	0.238068
3	1	0.203680	0.141683	0.108160	0.052791
	2	0.408140	0.283670	0.216471	0.105605
	3	0.616359	0.427210	0.325555	0.158526
4	1	0.113345	0.079172	0.060577	0.029649
	2	0.227260	0.158578	0.121264	0.059330
	3	0.342220	0.238427	0.182776	0.089034
	4	0.458785	0.318953	0.243434	0.118790
5	1	0.072265	0.039859	0.038726	0.018974
	2	0.144524	0.101110	0.077412	0.037941
	3	0.217430	0.151901	0.116236	0.056927
	4	0.290908	0.202970	0.155202	0.075937
	5	0.365116	0.254346	0.194330	0.094969

$U_{2:n}^{(m)}$ for any sample size, however large it may be. For example if for a given values of a we use $\hat{\mu}$ and $\hat{\sigma}$ as given in (11) for m=4, then with the evaluation of moments of order statistics arising from the standard Type III GLD for sample sizes up to 7, one can obtain the explicit form of appropriate U-statistic estimators for μ and σ and their variances for any sample of size n, however large it may be. Using the values of variances and co-variances of order statistics, and the coefficients of BLUEs of μ and σ given in section 3, we have obtained the values of $\xi_c^{(m)}$ and$\psi_c^{(m)}$ for $c = 1, 2, \ldots, m-1$, m=2,3,4,5 a=1.5(0.5)2.5 and for a=4.68085 and are given in table 6 and table 7. For practising statisticians these tables will be helpful to determine the variance of the U-statistics estimators.

It is unrealistic to assume always that the shape parameter a involved in Type III GLD is known. It is known from Balakrishnan and Lee (1998, p.130) that for a Type III GLD random variable X with pdf (1)

$$\begin{aligned} \beta_2 &= \frac{E(X-\mu)^4}{(Var(X))^2}, \\ &= 3 + \frac{\psi'''(a)}{2(\psi'(a))^2}, \end{aligned} \tag{24}$$

where $\psi(a)$ is the well known digammma function, $\psi'(a)$ and $\psi'''(a)$ are the first and third derivatives of ψ with respect to a. Since the expression for β_2 as given in(24) is free of the location parameterμ and scale parameter σ of the Type III GLD, using a sample $X_1, X_2, \cdots, X_n$ drawn from (1) we estimate a as the solution of

$$3 + \frac{\psi'''(a)}{2(\psi'(a))^2} = \frac{\sum\limits_{i=1}^{n}(X_i - \overline{X_n})^4}{(\sum\limits_{i=1}^{n}(X_i - \overline{X_n})^2)^2}. \tag{25}$$

TABLE 7
Value of $\psi_c^{(m)}$ for m=2(1)5 and $c = 1, 2, \cdots, m$.

m	c	a=1.5	a=2	a=2.5	a=4.68085
2	1	0.161857	0.153328	0.148126	0.138381
	2	0.620773	0.607870	0.600060	0.585807
3	1	0.071950	0.068145	0.065821	0.061508
	2	0.176880	0.169759	0.165404	0.157344
	3	0.314829	0.304842	0.298752	0.287523
4	1	0.040450	0.038324	0.037020	0.034552
	2	0.089100	0.085023	0.082489	0.077730
	3	0.146046	0.140085	0.136395	0.129455
	4	0.211272	0.203520	0.198740	0.189812
5	1	0.025785	0.023838	0.023631	0.021945
	2	0.054689	0.052070	0.050424	0.047229
	3	0.086489	0.082626	0.080208	0.075680
	4	0.121260	0.116180	0.113015	0.106941
	5	0.159056	0.152768	0.148865	0.141497

TABLE 8
U-statistics estimates and K-S statistics for single carbon fibres data set of 10mm

Type III generalised logistic distribution	Kernel sizes			
	m=2	m=3	m=4	m=5
U-statistics	$U_{1:63}^{(2)} = 3.00241$	$U_{1:63}^{(3)} = 3.00043$	$U_{1:63}^{(4)} = 2.998937$	$U_{1:63}^{(5)} = 2.98915$
	$U_{2:63}^{(2)} = 0.80785$	$U_{2:63}^{(3)} = 0.80785$	$U_{2:63}^{(4)} = 0.808316$	$U_{2:63}^{(5)} = 0.78255$
K-S Statistics	0.091401	0.090196	0.089172	0.089512

One may use Mathematica software for obtaining the solution $\hat{\hat{a}}$ of a from(25).

Since for the real data set describing the gauge length of single fibres 10 mm considered in section 1, Type III GLD provides the best fit, we once again use the data set to illustrate the U- statistics estimation of the location and scale parameters of the distribution. For the data we have computed the sample kurtosis as

$$\frac{\sum_{i=1}^{n}(X_i - \overline{X_n})^4}{(\sum_{i=1}^{n}(X_i - \overline{X_n})^2)^2} = 3.234852.$$

On using this value in the right sidee of (25) and solving for a using mathematica software we obtain $\hat{\hat{a}} = 4.68085$. Thus by taking 4.68085 as the known value of the parameter a we have used the method explained in this paper to obtain U-statistics estimators $U_{1:63}^{(2)}, U_{1:63}^{(3)}, U_{1:63}^{(4)}, U_{1:63}^{(5)}$ for μ and $U_{2:63}^{(2)} = U_{2:63}^{(3)}, U_{2:63}^{(4)}, U_{2:63}^{(5)}$ for σ together with K-S statistic values and are given in table 8.

From this table it is to be noted that the K-S statistic value computed based on U-statistics estimators for each of kernels of degree 2,3,4 and 5 have values less than that computed based on MLE method of estimation(see, table 1 for details).

5. Relation Between U-Statistics Estimators and Other Standard Estimators

Though, the U-statistics estimators developed by Thomas and Sreekumaar (2004, 2008) seems to be independent of other known estimators, their estimators for some choices of m turns out to be the same of some already known standard estimators like unbiased estimators of σ based on Gini's mean difference and the unbiased estimator of μ namely the sample mean $\overline{X}$. Further we observe that $U_{2:n}^{(2)} = U_{2:n}^{(3)}$. In this section we discuss about these inter relationship between U-statistics and with some other estimators.

If X_1 and X_2 are two independent observations drawn from the distribution with pdf $g(x; a, \mu, \sigma)$, then $Y_i = \frac{X_i - \mu}{\sigma}$, i=1,2 are distributed as two independent and identically distributed random variables arising from $g_0(y)$. Clearly

$$\Delta_F = E|X_1 - X_2| = \sigma E(Y_{2:2} - Y_{1:2}) = \sigma(\alpha_{2:2} - \alpha_{1:2}), \tag{26}$$

is defined as the Gini's mean difference of the Type III GLD. Then an unbiased estimate of σ based on the Gini's mean difference of the sample is given by,(see Samuel and Thomas (2003))

$$\sigma^* = \frac{2}{n(n-1)(\alpha_{2:2} - \alpha_{1:2})} \sum_{i=1}^{[n/2]} (n - 2i + 1) R_{i:n}, \tag{27}$$

where $R_{i:n} = X_{n-i+1:n} - X_{i:n}$ and [.] represents the usual greatest integer function. Clearly σ^* is an unbiased estimator of σ. Now we prove the following theorem which describes the inter relationship between σ^*, $U_{2:n}^{(2)}$ and $U_{2:n}^{(3)}$ and that between $\overline{X_n}$ and $U_{1:n}^{(2)}$

Theorem 2. *Let $X_{1:n}, X_{2:n}, \ldots, X_{n;n}$ be the order statistics of a random sample of size n arising from an absolutely continuous distribution having pdf $g(x; a, \mu, \sigma)$ (as defined in (1)) with location parameter μ, scale parameter σ and for given value of the shape parameter a. Then $U_{2:n}^{(2)} = \sigma^*$ where σ^* is as defined in (27). As $g(x; a, \mu, \sigma)$ is symmetric about μ, we have $U_{2:n}^{(2)} = U_{2:n}^{(3)}$ and $U_{1:n}^{(2)} = \overline{X}_n$, where $\overline{X}_n$ is the sample mean.*

Proof. Using (10), one can derive the BLUE $\hat{\sigma}$ based on order statistics of a random sample of size 2 drawn from $g(x; a, \mu, \sigma)$ as

$$\hat{\sigma}_2 = (\alpha_{2:2} - \alpha_{1:2})^{-1} R_{1:2}, \tag{28}$$

which is same as the kernel used for constructing the U-statistic (27). This proves that $U_{2:n}^{(2)} = \sigma^*$. Since $g(x; a, \mu, \sigma)$ is symmetric about μ, its standard form $g_0(y)$ is symmetric about zero, and hence from (28) we further have

$$\hat{\sigma}_2 = (2\alpha_{2:2})^{-1} R_{1:2}. \tag{29}$$

Also using the symmetric property of $g(x; a, \mu, \sigma)$ and using (10) for n=3 and simplifying we get

$$\hat{\sigma}_3 = (2\alpha_{3:3})^{-1} R_{1:3}. \tag{30}$$

Using $\hat{\sigma}_2 = (2\alpha_{2:2})^{-1}R_{1:2}$ as the kernel we obtain

$$U_{2:n}^{(2)} = \frac{-1}{2\alpha_{2:2}\binom{n}{2}}\sum_{i=1}^{n}(n-2i+1)X_{i:n}. \tag{31}$$

If $\hat{\sigma}_3 = (2\alpha_{3:3})^{-1}R_{1:3}$ is used as a kernel of degree 3 we obtain,

$$\begin{aligned} U_{2:n}^{(3)} &= \frac{-1}{2\alpha_{3:3}\binom{n}{3}}\sum_{i=1}^{n}\left[\binom{n-r}{2}-\binom{r-1}{2}\right]X_{i:n} \\ &= \frac{-1}{4\alpha_{3:3}\binom{n}{3}}\sum_{i=1}^{n}[(n-2)(n-2i+1)]X_{i:n} \\ &= \frac{-3}{2\alpha_{3:3}n(n-1)}\sum_{i=1}^{n}(n-2i+1)X_{i:n} \\ &= \frac{3\alpha_{2:2}}{2\alpha_{2:2}}U_{2:n}^{(2)}. \end{aligned} \tag{32}$$

From David and Nagaraga (2003, p.49), we have

$$3\alpha_{2:2} = 2\alpha_{3:3}. \tag{33}$$

Using the above relation in (32), we get $U_{2:n}^{(2)} = U_{2:n}^{(3)}$.

Using (9), and using the symmetric property of $g_0(y)$ we can obtain the BLUE μ based on random sample of size 2 as

$$\hat{\mu}_2 = 0.5(X_{1:2}+X_{2:2}) = \overline{X_2}. \tag{34}$$

Hence the corresponding U-statistic generated from $\overline{X}_2$ is

$$U_{1:n}^{(2)} = \frac{1}{n}\sum_{r=1}^{n}X_{r:n} = \overline{X_n} \tag{35}$$

and thus the theorem is proved.

6. Comparison of the U-statistic Estimators with Some Standard Unbiased Estimators

To compare the efficiency of our U-statistic estimator for μ, we take the usual moment estimator $\overline{X}_n$ of μ, given by $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n}X_{i:n}$. Clearly $\overline{X}_n$ is the sample mean and is unbiased for μ.

To compare the efficiency of U-statistic estimator for σ, we take an unbiased estimator of σ based on Gini's mean difference namely σ^* given in (27). In order to compare the efficiency of the U-statistic estimator of $U_{2:n}^{(4)}$ and $U_{2:n}^{(5)}$ with σ^*, we require the expression for $Var(\sigma^*)$. If X_1, X_2 and X_3 are independent random variables with common pdf (1), then considering σ^* as a U- Statistic estimator based on kernel of degree 2, from theorem 5.1 we have

$$Var(U_{2:n}^{(2)}) = \frac{2(n-2)\psi_1^{(2)} + \psi_2^{(2)}}{\binom{n}{2}}, \tag{36}$$

Using (10) and (23) for m=2 and simplifying we obtain

$$\psi_1^{(2)} = \left[\frac{(v_{1,1:3} - v_{1,3:3})}{3\alpha_{1:2}^2} - \frac{(v_{1,1:2} - v_{1,2:2})}{4\alpha_{1:2}^2}\right]\sigma^2 \ and \ \psi_2^{(2)} = \left[\frac{(v_{1,1:2} - v_{1,2:2})}{2\alpha_{1:2}^2}\right]\sigma^2. \tag{37}$$

From(36) and (37), we obtain the variance of the unbiased estimator σ^* $(= U_{2:n}^{(2)})$ as

$$Var(\sigma^*) = \frac{1}{\binom{n}{2}}\left[\frac{2(n-2)(v_{1,1:3} - v_{1,3:3})}{3\alpha_{1:2}^2} - \frac{(n-3)(v_{1,1:2} - v_{1,2:2})}{2\alpha_{1:2}^2}\right]\sigma^2. \tag{38}$$

Using the values of $\xi_c^{(m)}$ and $\psi_c^{(m)}$ given in table 6 and 7, we have obtained the variances of $U_{1:n}^{(m)}$ and $U_{2:n}^{(m)}$ for n=5 (5) 20 (10) 40 (20) 100; m=2(1)5, a=1.5(0.5)2.5 and for a=4.68085 and are given tables 11 and 9. In theorem 5.1, we have $Var(U_{1:n}^{(2)}) = Var(\overline{X_n})$ and $Var(U_{2:n}^{(2)}) = Var(U_{2:n}^{(3)}) = Var(\sigma^*)$, and hence in table 11 and table 9 we have given the values of $Var(\overline{X_n})$ instead of $Var(U_{1:n}^{(2)})$ and given the values of $Var(\sigma^*)$ instead of $Var(U_{2:n}^{(2)})$ or $Var(U_{2:n}^{(3)})$.

From tables we observe that the variances of all U-statistics reduces drastically as the sample size n increases. We also observe that the variance of U-statistics decrease as the shape parameter a of Type III GLD increases. Tandem to this we observe that as the shape parameter a of the Type III GLD increases the distribution tends to be one with relatively shorter tail.

For comparing the estimators in $U_{1:n}^{(m)}$, we have computed the efficiency $e\left(U_{1:n}^{(m)}|\overline{X}_n\right)$ of $U_{1:n}^{(m)}$ relative to $\overline{X}_n = U_{1:n}^{(2)}$ for m=3(1)5 and n=5(5) 20 (10) 40 (20) 100 and are given in table 11. It is observed that $e\left(U_{1:n}^{(m)}|\overline{X}_n\right)$ is greater than unity in all cases for m=3,4,5. For comparing the estimators in $U_{2:n}^{(m)}$ we have computed the efficiency $e\left(U_{2:n}^{(m)}|\sigma^*\right)$ of $U_{2:n}^{(m)}$ relative to $\sigma^* = U_{2:n}^{(2)} = U_{2:n}^{(3)}$ is calculated for m=4(1)5 and n=5(5) 20 (10) 40 (20) 100 and are given in table 9. It is observed that $e\left(U_{2:n}^{(m)}|\sigma^*\right)$ is also greater than unity in all cases for m=4,5.

The asymptotic relative efficiency $E_{1,m}\left(U_{1:n}^{(m)}|\overline{X}_n\right)$ of $U_{1:n}^{(m)}$ relative to $\overline{X}_n = U_{1:n}^{(2)}$ and $E_{2,m}\left(U_{2:n}^{(m)}|\sigma^*\right)$ of $U_{2:n}^{(m)}$ relative to $\sigma^* = U_{2:n}^{(2)} = U_{2:n}^{(3)}$ are given by

$$E_{1,m}\left(U_{1:n}^{(m)}|\overline{X}_n\right) = \lim_{n\to\infty}\left[\frac{Var(\overline{X}_n)}{Var(U_{1:n}^{(m)})}\right] = \frac{4\xi_1^{(2)}}{m^2\xi_1^{(m)}}$$

and

$$E_{2,m}\left(U_{2:n}^{(m)}|\sigma^*\right) = \lim_{n\to\infty}\left[\frac{Var(\sigma^*)}{Var(U_{2:n}^{(m)})}\right] = \frac{4\psi_1^{(2)}}{m^2\psi_1^{(m)}}.$$

We have evaluated the above asymptotic relative efficiencies $E_{1,m}$ for m=3(1)5, $E_{2,m}$ for m=4,5, a=1.5(0.5)2.5, for a=4.68085 and are given in table 10.

TABLE 9
Variances of $U_{2:n}^{(m)}$ and relative efficiency for $e\left(U_{2:n}^{(m)}/\sigma^*\right)$ for m=4(1)5.

a	n	$\frac{Var(\sigma^*)}{\sigma^2}$	$\frac{Var(U_{2:n}^{(4)})}{\sigma^2}$	$\frac{Var(U_{2:n}^{(5)})}{\sigma^2}$	$e\left(U_{2:n}^{(4)}/\sigma^*\right)$	$e\left(U_{2:n}^{(5)}/\sigma^*\right)$
1.5	5	0.159191	0.159091	0.159056	1.00070	1.00085
	10	0.071344	0.071292	0.071242	1.00073	1.00143
	15	0.045991	0.045961	0.045902	1.00065	1.00194
	20	0.033935	0.033915	0.033855	1.00059	1.00236
	30	0.022264	0.022252	0.022199	1.00054	1.00293
	40	0.016567	0.016559	0.016513	1.00048	1.00327
	60	0.010958	0.010953	0.010919	1.00046	1.00357
	80	0.008187	0.008183	0.008156	1.00049	1.00380
	100	0.006534	0.006532	0.006510	1.00031	1.00369
2	5	0.152784	0.152773	0.152768	1.00007	1.00010
	10	0.068025	0.068020	0.067948	1.00007	1.00113
	15	0.043756	0.043756	0.043509	1.00007	1.00568
	20	0.032251	0.032251	0.031942	1.00008	1.00967
	30	0.021136	0.021134	0.020827	1.00009	1.01484
	40	0.015719	0.015717	0.015442	1.00013	1.01794
	60	0.010392	0.010390	0.010175	1.00019	1.02133
	80	0.007762	0.007760	0.007586	1.00026	1.02320
	100	0.006194	0.006190	0.006048	1.00065	1.02414
2.5	5	0.148882	0.148864	0.148865	1.00012	1.00011
	10	0.066002	0.065990	0.065985	1.00018	1.00026
	15	0.042394	0.042384	0.042362	1.00024	1.00076
	20	0.031224	0.031217	0.031189	1.00022	1.00112
	30	0.020449	0.020443	0.020415	1.00029	1.00167
	40	0.015202	0.015198	0.015173	1.00029	1.00191
	60	0.010047	0.010044	0.010024	1.00030	1.00229
	80	0.007502	0.007500	0.007485	1.00030	1.00227
	100	0.005986	0.005984	0.005972	1.00033	1.00234
4.68085	5	0.141609	0.141530	0.141497	1.00056	1.00079
	10	0.062220	0.062170	0.062121	1.00080	1.00159
	15	0.039845	0.039810	0.039714	1.00088	1.00330
	20	0.029303	0.029270	0.029173	1.00113	1.00446
	30	0.019161	0.019140	0.019051	1.00110	1.00577
	40	0.014234	0.014220	0.014142	1.00098	1.00651
	60	0.009400	0.009390	0.009332	1.00106	1.00729
	80	0.007017	0.007010	0.006963	1.00100	1.00776
	100	0.005598	0.005590	0.005554	1.00143	1.00792

TABLE 10
Asymptotic relative efficiency $U_{1:n}^{(m)}$ and $U_{2:n}^{(m)}$ as estimators of μ and σ

a	1.5	2	2.5	4.68085
$E_{1,3}\left(U_{1:n}^{(3)}\vert\overline{X}_n\right)$	1.019903	1.01155	1.00747	1.002141
$E_{1,4}\left(U_{1:n}^{(4)}\vert\overline{X}_n\right)$	1.030925	1.01825	1.01185	1.003693
$E_{1,5}\left(U_{1:n}^{(5)}\vert\overline{X}_n\right)$	1.03486	1.29444	1.01298	1.003765
$E_{2,4}\left(U_{2:n}^{(4)}\vert\sigma^*\right)$	1.00035	1.00021	1.00031	1.001252
$E_{2,5}\left(U_{2:n}^{(5)}\vert\sigma^*\right)$	1.00435	1.02913	1.00293	1.008930

TABLE 11

Variances of $U_{1:n}^{(m)}$ and relative efficiency for $e\left(U_{1:n}^{(m)}/\overline{X}_n\right)$ for m=3(1)5

a	n	$\frac{Var(\overline{X}_n)}{\sigma^2}$	$\frac{Var(U_{1:n}^{(3)})}{\sigma^2}$	$\frac{Var(U_{1:n}^{(4)})}{\sigma^2}$	$\frac{Var(U_{1:n}^{(5)})}{\sigma^2}$	$e\left(U_{1:n}^{(3)}/\overline{X}_n\right)$	$e\left(U_{1:n}^{(4)}/\overline{X}_n\right)$	$e\left(U_{1:n}^{(5)}/\overline{X}_n\right)$
1.5	5	0.373921	0.367624	0.365533	0.365116	1.01713	1.02295	1.02412
	10	0.186960	0.183493	0.181872	0.181110	1.01889	1.02798	1.03230
	15	0.124640	0.122281	0.121114	0.120566	1.01929	1.02911	1.03379
	20	0.093480	0.091696	0.090791	0.090382	1.01946	1.02962	1.03428
	30	0.062320	0.061121	0.060500	0.060235	1.01962	1.03008	1.03461
	40	0.046740	0.045837	0.045365	0.045171	1.01970	1.03031	1.03473
	60	0.031160	0.030556	0.030237	0.030112	1.01977	1.03053	1.03480
	80	0.023370	0.022916	0.022676	0.022584	1.01980	1.03061	1.03480
	100	0.018696	0.018333	0.018139	0.018067	1.01980	1.03071	1.03480
2	5	0.257975	0.255428	0.254532	0.254346	1.00997	1.01353	1.01427
	10	0.128988	0.127586	0.126890	0.125501	1.01099	1.01653	1.02778
	15	0.085992	0.085039	0.084538	0.080567	1.01121	1.01720	1.06734
	20	0.064494	0.063773	0.063385	0.058505	1.01131	1.01750	1.10237
	30	0.042996	0.042511	0.042245	0.037387	1.01141	1.01778	1.15003
	40	0.032247	0.031882	0.031680	0.027344	1.01145	1.01790	1.17931
	60	0.021498	0.021254	0.021117	0.017726	1.01148	1.01804	1.21279
	80	0.016123	0.015940	0.015837	0.013095	1.01148	1.01806	1.23123
	100	0.012899	0.012752	0.012669	0.010379	1.01153	1.01815	1.24280
2.5	5	0.196143	0.194886	0.194428	0.194330	1.00645	1.00882	1.00933
	10	0.098072	0.097379	0.097027	0.096854	1.00712	1.01077	1.01258
	15	0.065381	0.064910	0.064657	0.064541	1.00726	1.01120	1.01301
	20	0.049036	0.048680	0.048484	0.048401	1.00731	1.01139	1.01312
	30	0.032691	0.032451	0.032317	0.032267	1.00740	1.01157	1.01314
	40	0.024518	0.024338	0.024236	0.024200	1.00740	1.01164	1.01314
	60	0.016345	0.016225	0.016156	0.016134	1.00740	1.01170	1.01308
	80	0.012259	0.012168	0.012117	0.012101	1.00740	1.01172	1.01306
	100	0.009807	0.009735	0.009693	0.009682	1.00740	1.01176	1.01291
4.68085	5	0.095227	0.095053	0.094985	0.094969	1.00183	1.00255	1.00272
	10	0.047614	0.047517	0.047463	0.047439	1.00204	1.00318	1.00369
	15	0.031742	0.031677	0.031636	0.031622	1.00205	1.00335	1.00379
	20	0.023807	0.023757	0.023725	0.023717	1.00210	1.00346	1.00379
	30	0.015871	0.015838	0.015815	0.015811	1.00210	1.00354	1.00379
	40	0.011903	0.011878	0.011861	0.011858	1.00210	1.00354	1.00379
	60	0.007936	0.007919	0.007907	0.007906	1.00215	1.00367	1.00379
	80	0.005952	0.005939	0.005930	0.005929	1.00215	1.00367	1.00381
	100	0.004761	0.004751	0.004744	0.004743	1.00215	1.00367	1.00381

7. Conclusion

Thus we conclude that when a better suitable model than the well-known symmetric models such as: normal, double exponential, Cauchy and so on is required one may search it from the Type III GLD family of distributions. Also if one chooses a Lloyds BLUE of location and scale parameters of Type III GLD with sample size m as a kernel, then with the knowledge of the means, variances and co-variances of order statistics of random sample of sizes between m and 2m-1 arising from the standard form of the distribution, one can use the results of this paper to estimate effectively the parameters and derive their variances for any sample size (say even for n=1000 or more) by U-statistics without any further direct evaluation of moments of order statistics.

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Summary

In this work we have derived appropriate U-statistics from a sample of any size exceeding a specified integer to estimate the location and scale parameters of Type III generalized logistic distribution without the knowledge or by evaluation of the means, variances and co-variances of order statistics of an equivalent sample size arising from the corresponding standard form of distribution. The exact variances and the asymptotic variances of the estimators have been obtained. The efficiency of the obtained estimators relative to some of the standard estimators have been also obtained. An illustration describing the betterness of U-statistics estimation method over the classical maximum-likelihood method is also given.

Keywords: Best linear unbiased estimators based on order statistics; Beta-Cauchy distribution; Gini's mean difference; Moment estimators; U-statistics.