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A THREE PARAMETER HYPER-POISSON DISTRIBUTION AND SOME OF ITS PROPERTIES

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1. Introduction

Bardwell and Crow (1964) considered a two parameter family of discrete distributions, namely the hyper-Poisson distribution (HPD), with probability mass function (p.m.f.)

$$\begin{aligned} p_y &= P(Y_1 = y) \\ &= \phi^{-1}(1;\lambda;\theta)\theta^y/(\lambda)_y \end{aligned} \tag{1}$$

for y = 0, 1, 2, ... , $\lambda > 0$, $\theta > 0$ and

$$\phi(a;b;z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

is the confluent hypergeometric series in which $(a)_0 = 1$ and for k = 1, 2, ... ,

$$(a)_k = a(a+1)...(a+k-1) = \Gamma(a+k)/\Gamma(a)$$

For a detailed account of confluent hypergeometric series refer Mathai and Haubold (2008) or Chapter 13 of Abramowitz (1965). The probability generating function (p.g.f.) of the HPD with p.m.f. (1) is

$$H(s) = \phi^{-1}(1;\lambda;\theta)\phi(1;\lambda;\theta s)$$

The mean and variance of the HPD are respectively $\frac{\phi(2;\lambda+1;\theta)}{\phi(1;\lambda;\theta)}\frac{\theta}{\lambda}$ and $\frac{1}{\lambda}[\frac{2}{\lambda+1}\frac{\phi(3;\lambda+2;\theta)}{\phi(1;\lambda;\theta)} - \frac{1}{\lambda}\frac{[\phi(2;\lambda+1;\theta)]^2}{[\phi(1;\lambda;\theta)]^2}]\theta^2 + \frac{\phi(2;\lambda+1;\theta)}{\phi(1;\lambda;\theta)}\frac{\theta}{\lambda}$.

When $\lambda = 1$ the HPD reduces to the Poisson distribution and when λ is a positive integer, the distribution is known as the displaced Poisson distribution of Staff (1964). Bardwell and Crow (1964) termed the distribution as sub-Poisson when $\lambda < 1$ and super-Poisson when $\lambda > 1$. Various methods of estimation of the parameters of the distribution were discussed in Bardwell and Crow (1964) and Crow and Bardwell (1965). Some queuing theory associated with hyper-Poisson arrivals has been worked out by Nisida (1962). Roohi and Ahmad (2003) attempted estimation of the parameters of the HPD using negative moments. Kemp (2002) developed a q-analogue of the distribution and Ahmad (2007) introduced and studied Conway-Maxwell hyper-Poisson distribution. Kumar and Nair[2011, 2012a] developed extended versions of the hyper-Poisson distribution and discussed some of their applications. Kumar and Nair (2012b) considered an alternative form of the HPD through the p.m.f.

$$\begin{aligned} q_y &= P(Y_2 = y) \\ &= \phi(1 + y; \lambda + y; -\theta)\ \theta^y/(\lambda)_y \end{aligned} \tag{2}$$

in which $\lambda > 0$ and $\theta > 0$. A distribution with p.m.f. (3) they named as the alternative hyper-Poisson distribution (AHPD). Clearly, when $\lambda = 1$, the AHPD reduces to the Poisson distribution. An important characteristic of the AHPD is that it is under-dispersed when $\lambda < 1$ and over-dispersed when $\lambda > 1$. The p.g.f. of AHPD is the following, for x = 0, 1, 2, ...

$$Q(s) = \phi[1; \lambda; \theta(s - 1)]$$

The mean and variance of AHPD are respectively $\frac{\theta}{\lambda}$ and $\frac{\theta}{\lambda}\left[1 + \frac{\theta}{\lambda}\frac{(\lambda-1)}{(\lambda+1)}\right]$. Bardwell and Crow (1964) considered the classical data derived from haemacytometer yeast cell counts and shown that the HPD gives a better fit to the data compared to Poisson distribution as well as two parameter Neyman Type- A distribution. Kumar and Nair (2012) considered two data sets among them the first data is on - the distribution of the epileptic seizure counts and the other data is on the distribution of corn borers in a field experiment, and shown that in both cases the AHPD gives better fits compared to the HPD. But there may be situations where both the HPD and the AHPD are not so suitable, but an analogous model will be more appropriate. For example the data on distribution of the counts of red mites on apple lives given in Tables 1 and 2 or data on the distribution of epileptic seizure counts given in Tables 3 and 4 of this paper. Hence, through this paper we develop a three parameter class of distribution as a generalization of both the HPD and AHPD. Such a generalized model opens up more flexibility in modeling situations where both the HPD and the AHPD are not giving better fits. This new class of distribution, we termed as "the alpha generalized hyper-Poisson distribution"(or in short "the AGHPD"). In section 2 we give the definition of the AGHPD and derive its p.g.f., expression for factorial moments, raw moments, mean, variance, and recursion formulae for its probabilities, raw moments and factorial moments. In section 3 we discuss the estimation of the parameters of the AGHPD by the method of moments and the method of maximum likelihood, and in section 4 we have considered certain real life data applications. Section 5 contains a generalized likelihood ratio test for testing the significance of the additional parameter of the model and section 6 contains a simulation study for comparing the performance of the estimators obtained in the paper.

2. The Alpha Generalized Hyper-Poisson Distribution

In this section we present the definition of the AGHPD and obtain some of its important properties.

Definition 1. *A non-negative integer valued random variable Y is said to follow the alpha generalized hyper-Poisson distribution (or in short the AGHPD) if its probability mass function (p.m.f.) $g_y = P(Y = y)$ is*

$$g_y = \frac{\phi(1+y;\lambda+y;\alpha)}{\phi(1;\lambda;\theta+\alpha)}\frac{\theta^y}{(\lambda)_y} \tag{3}$$

for $y = 0, 1, 2, \ldots, \lambda > 0$, $\alpha \in R = (-\infty, \infty)$ and $\theta > 0$.

Note that, when $\lambda = 1$, the AGHPD reduces to the Poisson distribution with parameter θ, when $\alpha = 0$ the p.m.f. given in (5) reduces to the p.m.f. p_y of the HPD as given in (1) and when $\alpha = -\theta$, the p.m.f. (5) reduces to the p.m.f. q_y of AHPD as given in (2). Now we obtain the p.g.f. of the AGHPD through the following result in the light of the series representation:

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} B(n,r) = \sum_{r=0}^{\infty}\sum_{n=0}^{r} B(n, r-n). \tag{4}$$

in which B(n,r) is any real valued function of n and r.

Proposition 2. *The p.g.f. $G(s)$ of the AGHPD with p.m.f. (5) is the following, in which $\delta = \phi^{-1}(1;\lambda;\theta+\alpha)$.*

$$G(s) = \delta\,\phi(1;\lambda;\theta s+\alpha) \tag{5}$$

Proof. By definition, the p.g.f. of the AGHPD with p.m.f. (5) is

$$\begin{aligned} G(s) &= \sum_{y=0}^{\infty} g_y s^y \\ &= \delta\sum_{y=0}^{\infty}\phi(1+y;\lambda+y;\alpha)\frac{\theta^y}{(\lambda)_y}s^y. \end{aligned} \tag{6}$$

Expand the confluent hypergeometric series in (8) to get

$$G(s) = \delta\sum_{y=0}^{\infty}\sum_{j=0}^{\infty}\frac{(1+y)_j\,\alpha^j}{(\lambda+y)_j\,j!}\frac{\theta^y}{(\lambda)_y}s^y. \tag{7}$$

Now apply (6) in (9) to obtain

$$G(s) = \delta\sum_{y=0}^{\infty}\sum_{j=0}^{y}\frac{(1+y-j)_j}{(\lambda+y-j)_j}\frac{\alpha^j}{j!}\frac{(\theta s)^{y-j}}{(\lambda)_{y-j}}. \tag{8}$$

If we apply the relation$(\lambda)_x(\lambda+x)_r = (\lambda)_{x+r}$ in (10), we get

$$G(s) = \delta \sum_{y=0}^{\infty} \sum_{j=0}^{y} \frac{(1)_y}{(1)_{y-j}(\lambda)_y} \frac{\alpha^j}{j!} (\theta s)^{y-j}$$

$$= \delta \sum_{y=0}^{\infty} \frac{1}{(\lambda)_y} (\theta s + \alpha)^y,$$

since $(1)_x = x!$ and by binomial theorem. Thus we have

$$G(s) = \delta \sum_{y=0}^{\infty} \frac{(1)_y}{(\lambda)_y} \frac{(\theta s + \alpha)^y}{y!},$$

which leads to (7).

Define the following notations, for any $\lambda > 0$ and for any integer $j \geq 1$, $\lambda^* = (1, \lambda)$ and $\lambda^* + j = (1 + j,\ \lambda + j)$. Next we have the following results.

PROPOSITION 3. *The following is a simple recursion formula for the probabilities $g_y = g_y(\lambda^*; \alpha, \theta)$ of the AGHPD with p.g.f. (7), for $y \geq 1$.*

$$g_{y+1}(\lambda^*; \alpha, \theta) = \frac{\Lambda_1}{\lambda(y+1)} [\theta g_y(\lambda^* + 1; \alpha, \theta)] \tag{9}$$

in which for j=1, 2, ... , $\Lambda_j = \delta\phi(1+j; \lambda+j; \theta+\alpha)$and δ is as given in Proposition 2.

PROOF. On differentiating (7) with respect to s, we have

$$\sum_{y=0}^{\infty} (y+1)\, g_{y+1}(\lambda^*; \alpha, \theta)\, s^y = \frac{\delta\theta}{\lambda} \phi(2; \lambda + 1; \theta s + \alpha). \tag{10}$$

Also, from (7) we have

$$\delta^* \phi(2; \lambda + 1; \theta s + \alpha) = \sum_{y=0}^{\infty} g_y(\lambda^* + 1; \alpha, \theta)\, s^y. \tag{11}$$

where $\delta^* = [\phi(2; \lambda + 1; \theta + \alpha)]^{-1}$. Relations (12) and (13) together lead to the following.

$$\sum_{y=0}^{\infty} (y+1)\, g_{y+1}(\lambda^*; \alpha, \theta)\, s^y = \frac{\Lambda_1 \theta}{\lambda} \sum_{y=0}^{\infty} g_y(\lambda^* + 1; \alpha, \theta) s^y \tag{12}$$

On equating coefficient of s^y on both sides of (14) we get (11).

For computational purpose we obtain an expression for the r^{th} raw moment of the AGHPD through the following result.

PROPOSITION 4. *For $r \geq 1$, an expression for the r^{th} raw moment $\mu_r^{\prime}$ of the AGHPD is*

$$\mu_r^{\prime} = \delta \sum_{k=0}^{r} S(r,k) \frac{(1)_k \theta^k}{(\lambda)_k} F_1(1+k, -, -; \lambda+k; \alpha, \theta), \tag{13}$$

in which S(r,k) is the Stirling number of second kind (c.f.Riordan, 1968) and

$$F_1(a,b,b';c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}} \frac{x^n y^n}{m!n!} \tag{14}$$

is the Horn-Appel function.

PROOF. For $r \geq 1$, the r^{th} raw moment $\mu_r^{\prime}$ of a random variable Y with p.m.f. (5) is

$$\begin{aligned} \mu_r^{\prime} &= \sum_{y=0}^{\infty} y^r \frac{\phi(1+y;\lambda+y;\alpha)}{\phi(1;\lambda;\theta+\alpha)} \frac{\theta^y}{(\lambda)_y} \\ &= \sum_{y=0}^{\infty} \sum_{k=0}^{r} \frac{S(r,k)y!}{(y-k)!} \frac{\phi(1+y;\lambda+y;\alpha)}{\phi(1;\lambda;\theta+\alpha)} \frac{\theta^y}{(\lambda)_y}, \end{aligned}$$

if we apply the equation(1.54) of (Johnson et.al, pp. 12). Next on expanding the confluent hypergeometric function in the numerator to get

$$\begin{aligned} \mu_r^{\prime} &= \sum_{k=0}^{r} \frac{S(r,k)}{\phi(1;\lambda;\theta+\alpha)} \sum_{y=0}^{\infty} \sum_{x=0}^{\infty} \frac{(1+y)_x \, \alpha^x y! \theta^y}{(\lambda+y)_x \, x!(y-k)!(\lambda)_y} \\ &= \sum_{k=0}^{r} \frac{S(r,k)(1)_k \theta^k}{\phi(1;\lambda;\theta+\alpha)(\lambda)_k} \sum_{x=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1+k)_{x+m} \, \alpha^x \theta^m}{(\lambda+k)_{x+m} \, x!m!} \end{aligned} \tag{15}$$

since $(\lambda)_x(\lambda+x)_r = (\lambda)_{x+r}$. Now (15) implies (13) in the light of (14)
Now we obtain the mean and variance of the AGHPD through the following result.

PROPOSITION 5. *Mean and variance of the AGHPD are $Mean = \frac{\Lambda_1}{\lambda}\theta$ and $Variance = \frac{\theta^2}{\lambda}[\frac{2}{\lambda+1}\Lambda_2 - \frac{1}{\lambda}\Lambda_1^2] + \frac{\theta}{\lambda}\Lambda_1$, in which Λ_1 and Λ_2 are as given in Proposition 3.*

The proof is straight forward and hence omitted.

PROPOSITION 6. *The following is a recursion formula for the raw moments $\mu_n(\lambda^*)$ of the AGHPD , for $n \geq 0$.*

$$\mu_{n+1}(\lambda^*) = \frac{\Lambda_1\,\theta}{\lambda}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\mu_{n-k}(\lambda^*+1) \tag{16}$$

PROOF. The characteristic function $\phi_Y(t)$ of the AGHPD with p.g.f. (5) has the following series representation. For $t \in R$ and $i = \sqrt{-1}$,

$$\begin{aligned} \phi_Y(t) &= G(e^{it}) \\ &= \delta\ \phi(1;\lambda;\theta e^{it}+\alpha) \\ &= \sum_{n=0}^{\infty}\mu_n(\lambda^*)\frac{(it)^n}{n!} \end{aligned} \tag{17}$$

On differentiating (19) with respect to t, we have

$$\frac{\delta}{\lambda}\,\theta e^{it}\,\phi(2;\lambda+1;\theta e^{it}+\alpha) = \sum_{n=1}^{\infty}\mu_n(\lambda^*)\frac{(it)^{n-1}}{(n-1)!} \tag{18}$$

By using (19) we get the following from (20)

$$\begin{aligned} \sum_{n=0}^{\infty}\mu_{n+1}(\lambda^*)\frac{(it)^n}{n!} &= \frac{\Lambda_1\theta}{\lambda}\sum_{n=0}^{\infty}\mu_n(\lambda^*+1)\frac{(it)^n}{n!}e^{it} \\ &= \frac{\Lambda_1\,\theta}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mu_n(\lambda^*+1)\frac{(it)^{n+k}}{n!\,k!}, \end{aligned} \tag{19}$$

by the expansion of the exponential term e^{it}. Now apply (6) in (21) to obtain the following.

$$\sum_{n=0}^{\infty}\mu_{n+1}(\lambda^*)\frac{(it)^n}{n!} = \frac{\Lambda_1\theta}{\lambda}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\mu_{n-k}(\lambda^*+1)\frac{(it)^n}{n!} \tag{20}$$

On equating the coefficients of $\frac{(it)^n}{n!}$ on both sides of (22) we get (18).

PROPOSITION 7. *The following is a recursion formula for the factorial moments $\mu_{[n]}(\lambda^*)$ of the AGHPD for $n \geq 1$, in which $\mu_{[0]}(\lambda^*) = 1$.*

$$\mu_{[n+1]}(\lambda^*) = \frac{\Lambda_1\theta}{\lambda}\mu_{[n]}(\lambda^*+1) \tag{21}$$

PROOF. The factorial moment generating function $F_Y(s)$ of the AGHPD with p.g.f. (7) has the following series representation.

$$F_Y(s) = G(1+s) = \delta\phi[1;\lambda;\theta(1+s)+\alpha] = \sum_{n=0}^{\infty}\mu_{[n]}(\lambda^*+1)\frac{s^n}{n!} \tag{22}$$

Differentiate (24) with respect to s to obtain

$$\frac{\delta}{\lambda}\theta\phi[2;\lambda+1;\theta(1+s)+\alpha]=\sum_{n=1}^{\infty}\mu_{[n]}(\lambda^*)\frac{s^{n-1}}{(n-1)!}. \tag{23}$$

By using (24) we get the following from (25).

$$\sum_{n=0}^{\infty}\mu_{[n+1]}(\lambda^*)\frac{s^n}{n!}=\frac{\Lambda_1\,\theta}{\lambda}\sum_{n=0}^{\infty}\mu_{[n]}(\lambda^*+1)\frac{s^n}{n!} \tag{24}$$

Now on equating the coefficients of $\frac{s^n}{n!}$ on both sides of (26) we get (23).

3. Estimation

Here we consider the estimation of the parameters α, λ and θ of the AGHPD by the method of moments and the method of maximum likelihood.

3.1. Method of moments

In the method moments, the first three raw moments of the AGHPD are equated corresponding sample raw moments τ_1, τ_2 and τ_3 respectively. Thus we obtain the following system of equations, in which Λ_1, Λ_2 and Λ_3 are as defined in Proposition 3.

$$\frac{\theta\,\Lambda_1}{\lambda}=\tau_1 \tag{25}$$

$$\frac{2\,\Lambda_2\theta^2}{\lambda(\lambda+1)}+\frac{\Lambda_1\theta}{\lambda}=\tau_2 \tag{26}$$

and

$$\frac{6\,\Lambda_3\theta^3}{\lambda(\lambda+1)(\lambda+2)}+\frac{6\Lambda_2\theta^2}{\lambda(\lambda+1)}+\frac{\Lambda_1\theta}{\lambda}=\tau_3 \tag{27}$$

Now the moment estimators of α, λ and θ of the AGHPD are obtained by solving the non-linear equations (27), (28), and (29).

3.2. Method of maximum likelihood

Let a(y) be the observed frequency of y events and k be the highest value of y observed. Then the log likelihood function of the sample is

$$\log L=\sum_{y=0}^{k}a(y)\log g_y$$

Let $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ denote the maximum likelihood estimates of the parameters α, λ and θ of the AGHPD . Now $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ are obtained by solving the normal equations (30), (31) and (32) as given below, in which for j=1, 2, ... and any $\eta \in R$, $\Delta_j(\eta) = \phi(1+j; \lambda+j; \eta)$, $\Psi(\beta) = [\Gamma(\beta)]^{-1}\left[\frac{d}{d\beta}\Gamma(\beta)\right]$, for $\beta > 0$ and Λ_j is as defined in Result 2.4.

$$\frac{\partial \log L}{\partial \alpha} = 0,$$

equivalently

$$\sum_{y=0}^{k} a(y) \frac{1}{\Delta_y(\alpha)} \left\{ \frac{1+y}{\lambda+y} \Delta_{1+y}(\alpha) - \frac{1}{\lambda} \Delta_y(\alpha) \delta \Lambda_1 \right\} = 0, \tag{28}$$

$$\frac{\partial \log L}{\partial \lambda} = 0,$$

equivalently

$$\sum_{y=0}^{k} a(y) \left\{ \sum_{x=0}^{\infty} \Delta_y^{-1}(\theta) \frac{(1+y)_x \theta^x}{(\lambda+y)_x x!} [\psi(\lambda+y) - \psi(\lambda+y+x)] - \right.$$
$$\left. \delta \sum_{x=0}^{\infty} \frac{(\alpha+\theta)^x}{(\lambda)_x} [\psi(\lambda) - \psi(\lambda+x)] + \psi(\lambda) - \psi(\lambda+y) \right\} = 0, \tag{29}$$

and

$$\frac{\partial \log L}{\partial \theta} = 0,$$

equivalently

$$\sum_{y=0}^{k} a(y) \frac{1}{\theta^y} \left\{ y\theta^{y-1} - \frac{\theta^y}{\lambda} \Lambda_1 \right\} = 0. \tag{30}$$

4. Real life data application

In this section we have considered two real life data applications for illustrating the usefulness of the AGHPD.

The first data set given in the first two columns of Tables 1 or 2 describes the number of European red mites on apple leaves, taken from Garman (1951). On july 18, 1951, 25 leaves were selected at random, from each of the six trees and counted the number of adult females on each leaf is considered here in this data set. The second data set given in the first two column of Tables 3 or 4 describes the epileptic seizure counts. Thirteen patients with intractable epilepsy controlled by anti-convulsant drugs were observed for times between three months and five years. Information about the number of daily seizures was recorded. The data given in Tables 3 or 4 corresponding to the daily seizure counts of a particular patient for 351 days. We have computed both the moment estimators and the maximum likelihood estimators of the parameters of the models- the HPD, the AHPD and the AGHPD along with their corresponding standard errors. Further, we obtain the expected frequencies in each case and computed the corresponding chi-square values and Kolmogorov-Smirnoff distance values along with respective P-values. The results obtained in respect of data set 1 are presented in Table 1 and Table 2 and the results corresponding to second data set are given in Table 3 and Table 4. From the calculated values of the chi-square statistic, Kolmogorov-Smirnoff distance measure and P-values, one can observe that the AGHPD gives better fits to both the data sets compared to the existing models.

5. Testing of hypothesis

Note that when $\alpha = 0$ the AGHPD reduces to the HPD as per definition 2.1. So, in order to check the validity of the AGHPD model compared to the HPD model, in this section we discuss the testing of the significance of the additional parameter α by using generalized likelihood ratio test as given below. Here the null hypothesis is $H_0 : \alpha = 0$ against the alternative hypothesis $H_1 : \alpha \neq 0$ and the test statistic is

$$-2log\Lambda = 2[logL(\hat{\Omega}; x) - logL(\hat{\Omega}^*; x)] \tag{31}$$

where $\hat{\Omega}$ is the maximum likelihood estimate of $\Omega = \lambda,\ \theta,\ \alpha)$ with no restriction, and $\hat{\Omega}^*$ is the maximum likelihood estimate of Ω when $\alpha = 0$. The test statistic $-2log\Lambda$ given in (33) is asymptotically distributed as χ^2 with one degree of freedom. [for details see Rao (1973)]. We have computed the values of $logL(\hat{\Omega}; x), logL(\hat{\Omega}^*; x)$ and the test statistic for the AGHPD are presented in Table 5. Since the critical value for the test with significance level =0.05 and degree of freedom one is 3.84, The null hypothesis is rejected in both the cases.

TABLE 1
Observed distribution of the counts of Red Mites on Apple Leaves [P. Garman, 1951] and expected frequencies computed using HPD, AHPD and AGHPD using the method of moments.

		Expected frequency by method of moments		
count	observed 'f'	HPD	AHPD	AGHPD
0	70	60.32	68.14	68.09
1	38	42.80	38.04	38.12
2	17	25.24	20.82	20.85
3	10	12.73	11.17	11.16
4	9	5.61	5.87	5.85
5	3	2.19	3.02	3.00
6	2	0.77	1.52	1.51
7	1	0.25	0.72	0.75
8	0	0.07	0.35	0.36
Total	150	150	150	150
Degrees of freedom		3	3	2
Estimated value of parameters and standard error within brackets		$\hat{\lambda} = 4.911$ (2.096×10^{-5}) $\hat{\theta} = 3.485$ (3.172×10^{-5})	$\hat{\lambda} = 11.822$ (2.186×10^{-6}) $\hat{\theta} = 13.56$ (4.264×10^{-6})	$\hat{\alpha} = 70.759$ (8.106×10^{-8}) $\hat{\lambda} = 147.467$ (7.819×10^{-8}) $\hat{\theta} = 43.925$ (9.917×10^{-9})
χ^2 - value		4.346	5.212	2.912
P - value based on χ^2 - value		0.226	0.157	0.533
Kolmogorov-Smirnoff distance value		0.088	0.044	0.022
P value based on Kolmogorov-Smirnoff distance value		0.654	0.748	0.930

TABLE 2
Observed distribution of the counts of Red Mites on Apple Leaves [P. Garman, 1951] and expected frequencies computed using HPD, AHPD and AGHPD using the method of maximum likelihood.

		Expected frequency by method of m.l.e		
count	observed 'f'	HPD	AHPD	AGHPD
0	70	68.20	63.55	68.30
1	38	38.05	39.86	38.00
2	17	20.79	23.19	20.74
3	10	11.13	12.47	11.11
4	9	5.85	6.18	5.8
5	3	3.01	2.83	3.01
6	2	1.56	1.19	1.53
7	1	0.71	0.47	0.76
8	0	0.21	0.17	0.37
Total	150	150	150	150
Degrees of freedom		3	3	2
Estimated value of parameters and standard error within brackets		$\hat{\lambda} = 47.818$ (2.181×10^{-3}) $\hat{\theta} = 26.67$ (3.261×10^{-4})	$\hat{\lambda} = 3.805$ (2.262×10^{-4}) $\hat{\theta} = 4.443$ (4.116×10^{-5})	$\hat{\alpha} = -4.246$ (3.096×10^{-6}) $\hat{\lambda} = 43.065$ (2.185×10^{-6}) $\hat{\theta} = 26.226$ (4.026×10^{-7})
χ^2 - value		3.412	4.62	2.412
P - value based on χ^2 - value		0.332	0.461	0.543
Kolmogorov-Smirnoff distance value		0.050	0.040	0.021
P value based on Kolmogorov-Smirnoff distance value		0.687	0.787	0.942

TABLE 3
Observed distribution of epileptic seizure counts [Albert, 1991; Hand et al., 1994, p. 133] and expected frequencies computed using HPD, AHPD and AGHPD using the method of moments.

		Expected frequency by method of moments		
count	observed 'f'	HPD	AHPD	AGHPD
0	126	126	122.19	124.01
1	80	86.74	87.28	85.25
2	59	56.73	58.86	57.78
3	42	35.35	37.23	37.49
4	24	21.02	21.99	22.67
5	8	11.96	12.09	12.58
6	5	6.52	6.19	6.36
7	4	3.42	2.96	3.92
8	3	1.72	1.32	1.23
Total	351	351	351	351
Degrees of freedom		4	4	3
Estimated value of parameters and standard error within brackets		$\hat{\lambda} = 19.07$ (4.184×10^{-4}) $\hat{\theta} = 13.12$ (5.134×10^{-5})	$\hat{\lambda} = 3.528$ (4.552×10^{-6}) $\hat{\theta} = 5.448$ (3.022×10^{-6})	$\hat{\alpha} = -6.24$ (3.831×10^{-13}) $\hat{\lambda} = 1.713$ (3.963×10^{-12}) $\hat{\theta} = 4.022$ (7.431×10^{-14})
χ^2 - value		18.093	6.653	2.60
P - value based on χ^2 - value		0.013	0.155	0.457
Kolmogorov-Smirnoff distance value		0.031	0.033	0.009
P value based on Kolmogorov-Smirnoff distance value		0.714	0.682	0.972

TABLE 4
Observed distribution of epileptic seizure counts [Albert, 1991; Hand et al. , 1994, p. 133] and expected frequencies computed using HPD, AHPD and AGHPD using the method of maximum likelihood.

		Expected frequency by method of m.l.e		
count	observed 'f'	HPD	AHPD	AGHPD
0	126	98.48	112.41	121.59
1	80	99.06	89.45	88.43
2	59	73.96	64.19	59.22
3	42	43.89	41.03	36.86
4	24	21.63	23.31	21.48
5	8	9.10	11.81	11.78
6	5	3.35	5.37	6.12
7	4	1.09	2.21	3.02
8	3	0.32	0.83	1.42
Total	351	351	351	351
Degrees of freedom		3	4	3
Estimated value of parameters and standard error within brackets		$\hat{\lambda} = 2.876$ (2.787×10^{-6}) $\hat{\theta} = 2.896$ (5.076×10^{-4})	$\hat{\lambda} = 2.404$ (4.292×10^{-7}) $\hat{\theta} = 3.776$ (4.356×10^{-5})	$\hat{\alpha} = 12.789$ (2.731×10^{-13}) $\hat{\lambda} = 28.717$ (2.849×10^{-12}) $\hat{\theta} = 12.435$ (6.451×10^{-14})
χ^2- value		4.12	4.22	2.449
P - value based on χ^2 - value		0.127	0.377	0.513
Kolmogorov-Smirnoff distance value		0.077	0.04	0.012
P value based on Kolmogorov-Smirnoff distance value		0.124	0.570	0.857

TABLE 5
Calculated values of the test statistic in case of generalized likelihood ratio test.

	log L($\hat{\Omega}$;x)	logL($\hat{\Omega}^*$;x)	Test Statistic
1	29.231	-97.098	252.658
2	-264.375	-290.274	651.789

TABLE 6
Bias and standard errors of each of the parameters of the simulated data sets corresponding to (i) $\alpha = 0.95$, $\lambda = 1.25$ and $\theta = 1.25$ (under dispersed case) (ii) $\alpha = 1.25$, $\lambda = 1.75$ and $\theta = 1.75$ (under dispersed case) (iii) $\alpha = 1.50$, $\lambda = 0.41$ and $\theta = 0.25$ (over dispersed case).

		method of moments			method of m.l.e		
Data sets	sample sizes	α	λ	θ	α	λ	θ
	500	0.743	0.813	0.646	0.622	0.787	0.539
		(0.451)	(0.358)	(0.351)	(0.343)	(0.249)	(0.249)
(i)	1000	0.401	0.370	0.360	0.370	0.260	0.211
		(0.331)	(0.258)	(0.265)	(0.321)	(0.233)	(0.210)
	5000	0.120	0.141	0.120	0.110	0.131	0.110
		(0.007)	(0.005)	(0.003)	(0.005)	(0.003)	(0.001)
	500	0.631	0.796	0.681	0.522	0.687	0.551
		(0.417)	(0.320)	(0.445)	(0.406)	(0.30)	(0.434)
(ii)	1000	0.339	0.428	0.436	0.327	0.329	0.415
		(0.316)	(0.231)	(0.364)	(0.237)	(0.202)	(0.261)
	5000	0.124	0.147	0.150	0.119	0.138	0.110
		(0.004)	(0.002)	(0.005)	(0.003)	(0.001)	(0.003)
	500	0.580	0.680	0.671	0.520	0.630	0.570
		(0.435)	(0.414)	(0.346)	(0.324)	(0.404)	(0.302)
(iii)	1000	0.340	0.313	0.330	0.310	0.310	0.240
		(0.234)	(0.316)	(0.217)	(0.204)	(0.302)	(0.208)
	5000	0.191	0.180	0.182	0.161	0.120	0.141
		(0.0041)	(0.0014)	(0.004)	(0.003)	(0.0011)	(0.002)

6. Simulation

In this section we carried out a simulation study of the AGHPD variate for the comparison of the estimation procedures discussed in section 3 of this paper.
We have simulated the AGHPD variates for the following sets of values of its parameters for samples sizes n = 500, 1000 and 5000.
(i) $\alpha = 0.95;\lambda = 1.25;\theta = 1.25$ (under dispersed case)
(ii) $\alpha = 1.25;\lambda = 1.75;\theta = 1.75$ (under dispersed case) and
(iii)$\alpha = 1.50;\lambda = 0.41;\theta = 0.25$ (over dispersed case)

For each of these simulated samples, we have fitted the AGHPD and obtain the estimates of the parameters by both method of moments and method of maximum likelihood, computed the bias and standard errors in each case and listed in Table 6. From Table 6 it can observe that both bias and standard errors in respect of each parameters are in decreasing order as the sample size increases and likelihood estimators have less bias and standard errors compared to moment estimators.

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REFERENCES

M. ABRAMOWITZ, I.A. STEGUN (1965). *Hand book of Mathematical Functions.* Dover, New York.

M. AHMAD (2007). *A short note on Conway-Maxwell-hyper Poisson distribution.* Pakistan Journal of Statistics , 23, pp. 135-137.

P.S. ALBERT (1991). *A two-state Markov mixture model for a time series of epileptic seizure Counts.* Biometrics, 47, pp. 1371-1381.

G.E. BARDWELL, E.L. CROW (1964). *A two parameter family of hyper-Poisson distributions.* Journal of American statistical Association, 59, pp. 133-141.

E.L. CROW, G.E. BARDWELL. (1965). *Estimation of the parameters of the hyper-Poisson distributions.* Classical and Contagious Discrete Distributions. G. P. Patil (editor), pp. 127-140, Pergamon Press, Oxford.

P. GARMAN (1951). *Original data on European red mite on apple leaves.* Connecticut.

D.J.HAND, F. DALY, A.D. LUNN, K.J. MCCONWAY ,E. OSTROWSKI (1994). *A Hand Book of Small Data Sets.* Chapman and Hall, London.

N.L. JOHNSON, A.W. KEMP, S.KOTZ (2005). *Univariate Discrete Distributions.* Wiley, New York.

C.D KEMP, A.W. KEMP (1965). *Some properties of the Hermite distribution.* Biometrika, 52, pp. 381-394.

C.D. KEMP (2002). *q-analogues of the hyper-Poisson distribution.* Journal of Statistical Planning and Inference, 101, pp. 179-183.

C.S. KUMAR, B.U. NAIR (2011). *Modified version of hyper-Poisson distribution and its applications.* Journal of Statistics and Applications, 6, pp. 23-34.

C.S. KUMAR, B.U. NAIR (2012a). *An extended version of hyper-Poisson distribution and some of its applications.* Journal of Applied Statistical Sciences, 19, pp. 81-88.

C.S. KUMAR, B.U. NAIR (2012b). *An alternative hyper-Poisson distribution.* 'Statistica, 72(3), pp. 357-369.

A.M. MATHAI, H.J. HAUBOLD (2008). *Special Functions for Applied Scientists.* Springer, New York.

T. NISIDA (1962). *On the multiple exponential channel queuing system with hyper-Poisson arrivals*. Journal of the Operations Research Society, 5, pp. 57-66.

C.R. RAO (1973). *Linear Statistical Inference and its Applications (second edition)*. Johnwiley, New York.

J. RIORDAN (1968). *Combintorial identities*. Wiley, New York.

A. ROOHI, M. AHMAD (2003a). *Estimation of the parameter of hyper-Poisson distribution using negative moments*. Pakistan Journal of Statistics, 19, pp. 99-105.

A. ROOHI, M. AHMAD (2003b). *Inverse ascending factorial moments of the hyper-Poisson probability distribution*. Pakistan Journal of Statistics, 19, pp. 273-280.

P.J. STAFF (1964). *The displaced Poisson distribution*. Australian Journal of Statistics, 6, pp. 12-20.

G.M. STIRRETT, G. BEALL, M. TIMONIOA (1937). *A field experiment on the control of the European corn borer*. Pyrausta nubilalis Hubn. by Beauveria bassiana Vuill.Scient. Agri 17, pp. 587-591.

SUMMARY

A three parameter hyper-Poisson distribution and some of its properties

A new class of distribution is introduced here as a generalization of the well-known hyper-Poisson distribution of Bardwell and Crow (J. Amer. Statist. Associ., 1964) and alternative hyper-Poisson distribution of Kumar and Nair (Statistica, 2012), and derive some of its important aspects such as mean, variance, expressions for its raw moments, factorial moments, probability generating function and recursion formulae for its probabilities, raw moments and factorial moments. The estimation of the parameters of the distribution by various methods are considered and illustrated using some real life data sets. Further, a test procedure is suggested for testing the significance of the additional parameter and a simulation study is carried out for comparing the performance of the estimators.

Keywords: Confluent hypergeometric series; Displaced Poisson distribution; Factorial moments; Hermite distribution; Probability generating function